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On a Lagrangian Formulation of the Incompressible Euler Equation

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Abstract. In this paper we show that the incompressible Euler equation on the Sobolev space $H^s(\mathbb{R}^n)$, $s>n/2+1$, can be expressed in Lagrangian coordinates as a geodesic equation on an infinite dimensional manifold. Moreover the Christoffel map describing the geodesic equation is real analytic. The dynamics in Lagrangian coordinates is described on the group of volume preserving diffeomorphisms, which is an analytic submanifold of the whole diffeomorphism group. Furthermore it is shown that a Sobolev class vector field integrates to a curve on the diffeomorphism group.

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1 Introduction

The initial value problem for the incompressible Euler equation in $\mathbb{R}^n$, $n\geq 2$, reads as:

$$\begin{cases} \partial_t u+(u\cdot\nabla)u=-\nabla p, \\ \operatorname{div} u=0, \\ u(0)=u_0, \end{cases} \tag{1.1}$$

where $u(t,x)=\big(u_1(t,x),\dots,u_n(t,x)\big)$ is the velocity of the fluid at time $t\in\mathbb{R}$ and position $x\in\mathbb{R}^n$, $u\cdot\nabla=\sum_{k=1}^n u_k\partial_k$ acts componentwise on u, ∇p is the gradient of the pressure $p(t,x)$, $\operatorname{div} u=\sum_{k=1}^n \partial_k u_k$ is the divergence of u and u_0 is the value of u at time $t=0$ (with assumption $\operatorname{div} u_0=0$). The system (1.1) (going back to Euler [1]) describes a fluid motion

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without friction. The first equation in (1.1) reflects the conservation of momentum. The second equation in (1.1) says that the fluid motion is incompressible, i.e. that the volume of any fluid portion remains constant during the flow.

The unknowns in (1.1) are u and p. But as we will see later one can express ∇p in terms of u. Thus the evolution of system (1.1) is completely described by u. Therefore we will speak in the sequel of the solution u instead of the solution (u,p).

Consider now a fluid motion determined by u. If one fixes a fluid particle which at time $t=0$ is located at $x\in\mathbb{R}^n$ and whose position at time $t\geq 0$ we denote by $\varphi(t,x)\in\mathbb{R}^n$, we get the following relation between u and φ

$$\partial_t\varphi(t,x)=u\big(t,\varphi(t,x)\big),$$

i.e. φ is the flow-map of the vectorfield u. The second equation in (1.1) translates to the well-known relation $\det(d\varphi)\equiv 1$, where $d\varphi$ is the Jacobian of φ – see Majda, Bertozzi [2]. In this way we get a description of system (1.1) in terms of φ. The description of (1.1) in the φ-variable is called the Lagrangian description of (1.1), whereas the description in the u-variable is called the Eulerian description of (1.1). One advantage of the Lagrangian description of (1.1) is that it leads to an ODE formulation of (1.1). This was already used in Lichtenstein [3] and Gunter [4] to get local well-posedness of (1.1).

To state the result of this paper we have to introduce some notation. For $s\in\mathbb{R}_{\geq 0}$ we denote by $H^s(\mathbb{R}^n)$ the Hilbert space of real valued functions on $\mathbb{R}^n$ of Sobolev class s and by $H^s(\mathbb{R}^n;\mathbb{R}^n)$ the vector fields on $\mathbb{R}^n$ of Sobolev class s – see Adams [5] or Inci, Topalov, Kappeler [6] for details on Sobolev spaces. We will often need the fact that for $n\geq 1$, $s>n/2$ and $0\leq s'\leq s$ multiplication

$$H^s(\mathbb{R}^n)\times H^{s'}(\mathbb{R}^n)\to H^{s'}(\mathbb{R}^n),\quad (f,g)\mapsto f\cdot g, \tag{1.2}$$

is a continuous bilinear map.

The notion of solution for (1.1) we are interested in are solutions which lie in $C^0\big([0,T];H^s(\mathbb{R}^n;\mathbb{R}^n)\big)$ for some $T>0$ and $s>n/2+1$. This is the space of continuous curves on $[0,T]$ with values in $H^s(\mathbb{R}^n;\mathbb{R}^n)$. To be precise we say that $u,\nabla p\in C^0\big([0,T];H^s(\mathbb{R}^n;\mathbb{R}^n)\big)$ is a solution to (1.1) if

$$u(t)=u_0+\int_0^t -(u(\tau)\cdot\nabla)u(\tau)-\nabla p(\tau)\,\mathrm{d}\tau,\quad \forall 0\leq t\leq T, \tag{1.3}$$

and $\operatorname{div}u(t)=0$ for all $0\leq t\leq T$ holds. As $s-1>n/2$ we know by the Banach algebra property of $H^{s-1}(\mathbb{R}^n)$ that the integrand in (1.3) lies in $C^0\big([0,T];H^{s-1}(\mathbb{R}^n;\mathbb{R}^n)\big)$. Due to the Sobolev imbedding and the fact $s>n/2+1$ the solutions considered here are C^1 (in the x-variable slightly better than C^1) and are thus solutions for which the derivatives appearing in (1.1) are classical derivatives.

The discussion above shows that in this paper the state-space of (1.1) in the Eulerian description is $H^s(\mathbb{R}^n;\mathbb{R}^n)$, $s>n/2+1$. By the divergence-free condition for u one