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THE DIRICHLET PROBLEM FOR THE DEGENERATE MONGE-AMPERE EQUATION

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1. Introduction

In this paper, we discuss the Dirichlet problem for the degenerate Monge-Ampere equation.

Let $\Omega \subset R^n$ be a bounded smooth strictly convex domain, and let $r(x) \in C^\infty(R^n)$ be a strictly convex function. We call $r(x)$ the defining function if

$$\Omega = \{x \in R^n \mid r(x) < 0\}$$

The problem is to find a convex function $u(x) \in C^{k+2+\alpha}(\Omega)$ which satisfies the equation:

$$\begin{cases} \det(u_{ij} + \sigma_{ij}) = \psi(x, u, \nabla u) & \text{in} \quad \Omega \\ u|_{\partial\Omega} = \phi(x) \end{cases} \tag{1. 1}$$

where $\psi(x, t, p) \in C^{k+\alpha}(\Omega \times R \times R^n)$, $\psi \geq 0$, $(0 < \alpha < 1)$, $\{\sigma_{ij}\}$ is a real symmetry matrix. $\sigma_{ij}(x) \in C^{k+\alpha}(\Omega)$, $\varphi(x) \in C^{k+\alpha}(\partial\Omega)$, ($k \geq 2$ is an integer), and $u_{ij} = \partial_i\partial_j u$, $u_i = \partial_i u$, $\nabla u = (u_1, \ldots, u_n)$. In the following we use the notations: $\nabla^2 u = (u_{ij})$, $\nabla^3 u = (u_{ijk})$, $i, j, k = 1, \ldots, n$.

We say that $v(x)$ is a sub-solution of (1. 1) if $v \in C^2(\Omega)$ and satisfies:

$$\begin{cases} \det(v_{ij} + \sigma_{ij}) \geq \psi(x, v, \nabla v) & \text{in} \quad \Omega \\ v|_{\partial\Omega} = \varphi(x) \end{cases} \tag{1. 2}$$

When $\psi(x, t, p) \geq C > 0$, $\psi_t(x, t, p) \geq 0$, and the equation (1. 1) has a sub-solution, the existence and uniqueness of the solution of (1. 1) has been proved by Caffarelli, Nirenberg and Spruck in [1].

Under the above conditions, the equation is uniformly elliptic. The main contribution in [1] is to prove the global estimation of $C^{2,\alpha}$ norm of the solutions $u(x)$ of (1. 2). The crucial point in [1] is to estimate the logarithmic modulus of continuity of u_{ij} at every point x:

$$\sum_{i,j} |u_{ij}(x) - u_{ij}(y)| \leq \frac{k}{1 + |\ln|x-y||} \qquad \begin{matrix} \forall x \in \partial\Omega \\ \forall y \in \overline{\Omega} \end{matrix} \tag{1. 3}$$

In fact, combining the above inequality with the interior estimations of u_{ijk} we obtain the global estimation of the Hölder norm of u_{ij}.

In the works of Pogorelov, Cheng S. Y. and Yau S. T. the existence of generalized solution of (1. 1) was proved, only the local regularity of the solution, i. e. $u \in C^{k+\alpha}(\Omega) \cap C^0(\Omega)$ was given.

The Monge-Ampere equation (1. 1) originates from geometrical problems e. g. the Minkowski problem. When the Gauss curvature, say $\psi(x, t, p)$ of the right hand of (1. 1), is nonnegative but zero in some points, the equation (1. 1) is a degenerate 2nd order elliptic equation.

The main results of this paper are as follows:

Theorem 1. 1 *If $f(x)$, $\varphi(x)$ and $u_\varepsilon(x)$ are such that*

1) $f(x) \in C^2(\Omega)$, $f(x) \geq 0$, $0 \overline{\in} f(\partial\Omega)$

2) $\varphi(x) \in C^4(\partial\Omega)$ $(0 < \alpha < 1)$

3) $u_\varepsilon(x) \in C^4(\Omega) \cap C^2(\overline{\Omega})$ *is convex and satisfies:*

$$\begin{cases} \det((u_\varepsilon)_{ij} + \sigma_{ij}) = (f(x))^n + \varepsilon & in \quad \Omega \\ u_\varepsilon|_{\partial\Omega} = \varphi(x) \end{cases} \tag{1. 4}$$

then we have

1) $\| u_\varepsilon \|_{2,\overline{\Omega}} \leq C_0$ *for some constant C_0 depending only on* $\|\phi\|_{4,\partial\Omega}$, $\|f\|_{2,\Omega}$ *and* $(\|f\|_{0,\Omega_{d_0}})^{-1}$.

2) $\forall K', K \subset\subset K' \subset \Omega$, *there is β, $\beta = \beta(K') > 0$, such that*

$$\| u_\varepsilon \|_{2,\beta,\overline{\Omega \backslash K'}} \leq C(d(K, K'), \| u_\varepsilon \|_{2,\overline{\Omega}})$$

where $K = \{x \in \Omega \mid f(x) = 0\}$, $d_0 = d(K, \partial\Omega)$ *and* $\Omega_d = \left\{x \in \Omega \mid d(x, \partial\Omega) \leq \frac{1}{2} d_0\right\}$.

Theoerm 1. 2 *Suppose that the conditions of Theorem1. 1 hold, then there is a convex function $u(x) \in C^2(\Omega \backslash K) \cap C^{1,1}(\overline{\Omega})$ satisfying:*

$$\begin{cases} \det(u_{ij} + \sigma_{ij}) = (f(x))^n & in \quad \Omega \backslash K \\ u|_{\partial\Omega} = \varphi(x) \end{cases} \tag{1. 5}$$

All symbols used here are the same as that of[4].

2. The Estimation of C^2-Norm of the Solutions

It is difficult to estimate the solutions of (1. 1) directly when the equation is degenerate. We perturb the right hand of (1. 1) by $\varepsilon (> 0)$. The equation becomes a 2nd order non-degenerate uniformly elliptic equation. We do have the a priori estimation of the perturbed equation, but it might be dependent with $\varepsilon > 0$. To prove the existence of solution of the original equation, we hope that the a priori estimation is independent of ε.