J. Partial Differential Equations
Vol. 2, No. 1 (1989), 16—30

GLOP OLUTIONS FOR A COUPLED KDV SYSTEM

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(Received May 31, 1987; revised August 5, 1987)

1. Introduction

The coupled KdV system

$$\begin{cases} u_t - a(u_{xxx} + 6uu_x) - 2bvv_x = 0, \\ v_t + v_{xxx} + cuv_x = 0 \end{cases} \tag{1. 1}$$

arises in physics[1, 2], which describes the interaction of two long waves with different dispersion relations. It has been proved that system (1. 1) has two- and three-soliton solutions if there is a special relation between the dispersion relations of the two long waves.

In the present work we shall show existence and uniqueness of global solutions satisfying the periodic initial-value conditions

$$\begin{cases} U(x+2D,\ t) = U(x,\ t) \\ U(x,\ 0) = U_0(x) \end{cases} \tag{1. 2}$$

or the initial value condition

$$U(x,\ 0) = U_0(x) \tag{1. 3}$$

for the coupled system (1. 1) in the domain $Q_T^* = \{|x| < \infty, 0 \leq t \leq T\}$, where $T > 0$, $U(x, t) = (u(x, t), v(x, t))$, $U_0(x) = (u_0(x), v_0(x))$.

We shall obtain the solution to the periodic problem (1. 1), (1. 2) as a limit of solutions to the perturbed system

$$\begin{cases} u_t = -\varepsilon u_{xxxx} + a(u_{xxx} + 6uu_x) + 2bvv_x \\ v_t = -\varepsilon v_{xxxx} - v_{xxx} - cuv_x \end{cases} \tag{1. 4}$$

with periodic condition (1. 2). The difficult part of our development, as in all previous work on the KdV and its generalizations, is in obtaining a priori estimates for the norms of solutions to the perturbed problem. In the final section we will also state theorems for the initial-value problem (1. 1), (1. 3) analogous to the periodic initial-value problem (1. 1), (1. 2).

2. The Existence Theorem for the Perturbed Problem

Let us consider the periodic initial-value problem (1. 4), (1. 2). To solve the problem we linearize system (1. 4) and obtain

Lemma 1 *Let $U_0 \in H^2(-D, D)$ and $f \in L_2(Q_T)$ be periodic with respect to x with period $2D$, where $f=(f_1, f_2)$, $Q_T=\{(x,t): -D<x<D, 0\leq t\leq T\}$, then the linear parabolic system*

$$\begin{cases} u_t = -\varepsilon u_{xxxx} + a u_{xxx} + f_1 \\ v_t = -\varepsilon v_{xxxx} - v_{xxx} + f_2 \end{cases} \tag{2. 1}$$

with the periodic initial-value condition (1. 2) has one and only one solution $U(x, t)$ and

$$\| U \|_{L_\infty(0,T;H^2(-D,D))} + \| U \|_{W_2^{(4,1)}(Q_T)} \leq C_1 (\| U_0 \|_{H^2(-D,D)} + \| f \|_{L_2(Q_T)}) \tag{2. 2}$$

where C_1 is a constant.

Proof From the theory on parabolic partial differential equation, we can obtain the existence of solutions to the periodic initial-value problem (2. 1), (1. 2).

In order to get the estimation, we take the inner product of (2. 1) and U, then integrate the resultant relation over rectangular domain Q_t, we have

$$\| U(\cdot, t) \|^2_{L_2(-D,D)} + 2 \| U_{xx} \|^2_{L_2(Q_t)} \leq \| U \|^2_{L_2(Q_t)} + \| f \|^2_{L_2(Q_t)} + \| U_0 \|^2_{L_2(-D,D)}$$

By the Gronwall inequality, there is

$$\| U \|^2_{L_\infty(0,T;L_2(-D,D))} \leq e^T (\| U_0 \|^2_{L_2(-D,D)} + \| f \|^2_{L_2(Q_T)})$$

Then taking the inner product of system (2. 1) and vector U and integrating the resultant relation over Q_t, we obtain the expression

$$\| U_{xx}(\cdot, t) \|^2_{L_2(-D,D)} + \varepsilon \| U_{xxxx} \|^2_{L_2(Q_t)} \leq \| U_{0xx} \|^2_{L_2(-D,D)} + \frac{1}{\varepsilon} \| f \|^2_{L_2(Q_t)}$$

from which we have

$$\| U_{xx} \|_{L_\infty(0,T:L_2(-D,D))} + \varepsilon \| U_{xxxx} \|^2_{L_2(Q_t)} \leq \| U_{0xx} \|^2_{L_2(-D,D)} + \frac{1}{\varepsilon} \| f \|^2_{L_2(Q_t)}$$

Besides, using system (2. 1) and the above results, we can also get the estimation for $\| U_t \|_{L_2(Q_t)}$. So the inequality (2. 2) holds, which ensures the uniqueness of solution.

Corollary *Let $D_x^k D_t^h f(x,t) \in L_2(Q_t)$, $U_0 \in H^{(k+4h+2)}(-D, D)$ for $h \geq 0$ and $k \geq 0$, then for the solution U to the problem (2. 1), (1. 2), we have*

$$D_x^k D_t^h U \in L_\infty(0, T; H^2(-D, D)) \cap W_2^{(4,1)}(Q_T)$$

and the inequality analogous to the inequality (2. 2) holds.

Using lemma 1, we can show the following result:

Lemma 2 *Let $a+1>0$, $bc>0$, $U_0(x) \in H^2(-D, D)$ be periodic with period $2D$. Then*