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GLOBAL SMOOTH SOLUTIONS TO CAUCHY PROBLEM OF EQUATIONS OF ONE-DIMENSIONAL THERMOVISCOELASTICITY

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Abstract

In this paper the Cauchy problem of the equations of one-dimensional thermoviscoelasticity, including the system of compressible viscous heat-conductive fluids is considered. It is proved that the Cauchy problem admits a unique global smooth solution provided that the difference between initial data and the equilibrium state is appropriately small in the norm of $W^{1,1}\cap H^3$. Moreover, as $t\to+\infty$, the solution has the same decay rates as the Cauchy problem of the heat equations, i. e. the best decay rates, which improves the corresponding results by Kawashima and Nishida.

1. Introduction

In this paper we are concerned with the Cauchy problem of the equations of one-dimensional thermoviscoelasticity with general form which can be written in Lagrange coordinates as follows.

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u,\ \theta,\ v_x)_x = 0 \\ \left(e+\frac{1}{2}v^2\right)_t - (\sigma v)_x - q(u,\ \theta,\ \theta_x)_x = 0 \end{cases} \tag{1.1}$$

$$t=0:\ u=u^0(x),\ v=v^0(x),\ \theta=\theta^0(x) \tag{1.2}$$

where $u, v, e(u,\theta)$, $\sigma(u,\theta,v_x)$, $\theta, q(u,\theta,\theta_x)$ denote deformation gradient, velocity, internal energy, stress, temperature and heat flux, in that order. Note that u, e and θ may only take positive values [1].

To comply with the second law of thermodynamics expressed by the Clausius-Duhem inequality [1], we must have

$$\sigma(u,\ \theta,\ 0)=\psi_u(u,\ \theta),\ \eta(u,\ \theta)=-\psi_\theta(u,\ \theta) \tag{1.3}$$

where $\psi=e-\theta\eta$ is the Helmholtz free energy and $\eta=\eta(u,\theta)$ is the specific entropy. It follows from (1.3) that

$$e_u=\sigma(u,\ \theta,\ 0)-\theta\sigma_\theta(u,\ \theta,\ 0)=-\theta^2\left(\frac{\sigma(u,\ \theta,\ 0)}{\theta}\right)_\theta \tag{1.4}$$

(1.1) is a highly nonlinear hyperbolic parabolic coupled system in which the coefficients of second order derivatives v_{xx} and θ_{xx} depend on the first order derivatives v_x, θ_x. For the special form of (1.1) (1.2), Kawashima and Nishida [4] [5] proved the global existence and uniqueness of smooth solutions when the initial data is small. But it seems difficult to apply their method to highly nonlinear coupled equations (1.1). The global existence of weak solutions in $BV \cap L^1$ space for the restrictive form of (1.1) (1.2) was proved by J. U. Kim [3].

In this paper we adopt a new approach which was first introduced by authors in [6]. Combining the local existence theorem with the a priori estimates of solutions based on the L^p decay estimates of linearized problem, we prove the global existence and uniqueness of small smooth solutions to the Cauchy problem for equations of thermoviscoelasticity in very general form (1.1) and obtain the best decay rates of solutions as $t \to +\infty$, which improves, even in the special form of the equations, corresponding results by Kawashima and Nishida [4], [5].

Throughout this paper $|\cdot|_\infty$ and $\|\cdot\|$ denote the norm of L^∞ and L^2. We also use the following notations: $D = \frac{\partial}{\partial x}$, $U(x, t) = (u, v, \theta)^T$ and

$$\|U(t)\|^2 = \|u\|^2 + \|v\|^2 + \|\theta\|^2$$

C denotes a positive constant independent of U and t.

we make the following assumptions on (1.1) (1.2) which are also reasonable in mechanics:

(i) $e = e(u, \theta)$, $\sigma = \sigma(u, \theta, v_x)$, $q = q(u, \theta, \theta_x) \in C^4$

(ii) For any fixed positive constants $\bar{u} > 0$, $\bar{\theta} > 0$ there exist positive constants γ, a_0, σ_0, q_0 depending only on $\bar{u}$, $\bar{\theta}$ such that when $|u - \bar{u}|$, $|\theta - \bar{\theta}|$, $|v_x|$, $|\theta_x| \leq \gamma$,

$$\begin{gathered} e_\theta \geq a_0 > 0, \ \sigma_u \geq \sigma_0 > 0, \ \sigma_{v_x} \geq \sigma_0 > 0, \ \sigma_\theta \neq 0 \\ q_{\theta_x} \geq q_0 > 0, \ |q_u(u, \theta, \theta_x)|, \ |q_\theta(u, \theta, \theta_x)| \leq C|\theta_x| \end{gathered} \tag{1.5}$$

(iii)

$$(u^0 - \bar{u}, \ v^0, \ \theta^0 - \bar{\theta}) \in H^3(R) \cap W^{1,1}(R) \tag{1.6}$$

We now have

Main Theorem *Under the above assumptions* (i)-(iii) *there exists an appropriately small positive number* ε *such that when* $\|\{u^0 - \bar{u}, v^0, \theta^0 - \bar{\theta}\}\|_{H^3 \cap W^{1,1}} \leq \varepsilon$ *Cauchy problem* (1.1) (1.2) *admits a unique global smooth solution* (u, v, θ), $u - \bar{u} \in C([0, +\infty); H^3 \cap W^{1,1}) \cap C^1([0, +\infty); H^2)$, $(v, \theta - \bar{\theta}) \in C([0, +\infty); H^3 \cap W^{1,1}) \cap C^1([0, +\infty); H^1) \cap L^2([0, +\infty); H^4)$. *Moreover, as* $t \to +\infty$,

$$\begin{gathered} |\{u - \bar{u}, \ v, \ \theta - \bar{\theta}\}|_\infty = O((1+t)^{-\frac{1}{2}}), \quad \|\{u - \bar{u}, \ v, \ \theta - \bar{\theta}\}\| = O((1+t)^{-\frac{1}{4}}) \\ \|D\{u, \ v, \ \theta\}\|_{L^1} = O((1+t)^{-\frac{1}{2}}) \end{gathered} \tag{1.7}$$