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ON THE CLASSICAL SOLUTIONS OF DEGENERATE QUASILINEAR PARABOLIC EQUATIONS OF THE FOURTH ORDER

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1. Introduction

Degenerate quasilinear parabolic equations of second order have been investigated in a large number of papers since 1950's. However, there are only a few papers which have been devoted to degenerate quasilinear parabolic equations of higher order. Up to the present, there are two aspects of the works on the study of degenerate quasilinear parabolic equations of higher order. One aspect of the works is the study of equations which have the property of so called fixed degeneracy, see Mkrtychyan [1], for example. The more interesting cases are those in which the degeneracy depends on the unknown functions, see Soltanov [2], [3], where the author investigates some nonlinear degenerate parabolic equations by means of the study of "Nonlinear Sobolev Spaces". Recently, Bernis [4] investigates a class of higher order parabolic equations with degeneracy depending on both the unkown functions and its derivatives, the fourth order case of which is the equation

$$\frac{\partial}{\partial t}(|u|^{q-1}\mathrm{sgn}u)+D^2(|D^2u|^{p-1}\mathrm{sgn}D^2u)=f, \qquad \left(D=\frac{\partial}{\partial x}\right) \tag{1.1}$$

Some existence result of "energy solutions" was proved by energy method.

The present paper is concerned with the first boundary value problem for the equation

$$\frac{\partial u}{\partial t}+D^4u^{2m+1}+\lambda u=f \qquad (\lambda\geq 0,\ m\geq 7) \tag{1.2}$$

which can be thought of as a special case of the equation (1.1) when $\lambda=0$. Our interest here is to find the possibility of the existence of classical solutions. We prove that the classical solutions of the first boundary value problem for the equation (1.2) is locally in time existent when $\lambda\geq 0$, $f=0$ and u_0 is sufficiently smooth and satisfies some compatible conditions. In particular, if $u_0(x)\geq 0$, then $u(t,x)\geq 0$ for all t in the existing interval. In general, one can not expect the global existence of classical solutions of the first boundary value problem for the equation (1.2), due to the fact that the nonnegative "energy solutions" can not be globally existent, which was proved by Bernis [5]. But if $\lambda > 0$, u_0 and f are sufficiently small in some sense, then the first boundary value problem for the equation (1.2) has a global classical solution which decays exponentially to zero when $f = 0$ as t goes to infinity. Another interesting conclusion is that the nonnegative classical solutions are not diffusible in the existing interval $(0, T)$ when $f = 0$, precisely speaking, if $u_0(x) \geq 0$, $\operatorname{supp} u_0 \subset [\alpha, \beta]$, then $\operatorname{supp}_x u(t, \cdot) \subset [\alpha, \beta]$, for all time $t \in (0, T)$.

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2. Notations and Lemmas

Let I be the open interval $(0, 1)$ and k be an even number satisfying $8 \leq k \leq m+2$, where m is taken from the equation (1.2) which is always assumed to be a positive integer larger than 7. Introduce the following notations related to the space $H^k(I)$.

$$<u, v>_s = \int_I D^s u D^s v dx$$

$$(u, v)_s = <u, v>_s + <u, v>_0 \qquad (s = 0, 1, 2, ...k)$$

$$|u|_s = <u, u>_s^{\frac{1}{2}}$$

$$\| u \|_s = (|u|_s^2 + |u|_0^2)^{\frac{1}{2}}$$

We prove the following lemmas which will be used in the sequel.

Lemma 2.1 *Let $u, v \in H^k(I)$. Then for any positive integers p, q, $u^p \cdot v^q \in H^k(I)$ and*

$$\| u^p v^q \|_k \leq C \| u \|_k^p \| v \|_k^q \qquad (2.1)$$

Proof. Without loss of generlity, we assume $p = q = 1$. By the embedding theorem and the interpolation inequality,

$$|uv|_0^2 = \int_I u^2 v^2 dx$$

$$\leq \sup |u|^2 \int_I v^2 dx \leq C \| u \|_k^2 \| v \|_k^2$$

$$|uv|_k^2 = \int_I (D^k(uv))^2 dx$$

$$= \sum_{r,s} \binom{k}{r} \binom{k}{s} \int_I D^r u D^s u D^{k-r} v D^{k-s} v dx$$

$$\leq C \| u \|_k^2 \| v \|_k^2$$