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THRESHOLD PHENOMENA FOR THE INITIAL VALUE PROBLEM OF NERVE FIBRE BUNDLES

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1. Introduction

In recent years many results regarding the conduction of nerve impulses along an axon have been got. In particular, attention was paid to the threshold phenomena (see [1]−[3] and the cited papers wherein). In this paper we are interested in more complicated situation. Namely, we consider the transmission of nerve impulses along the nerve fibre bundles in which the interaction may take place between neighbouring excited axons.

We consider the following model for a finite number N of neighbouring axons (refer to [4]). Imagine that the axons are running parallel to each other and we locate each axon in cross section by a vertex of a graph.

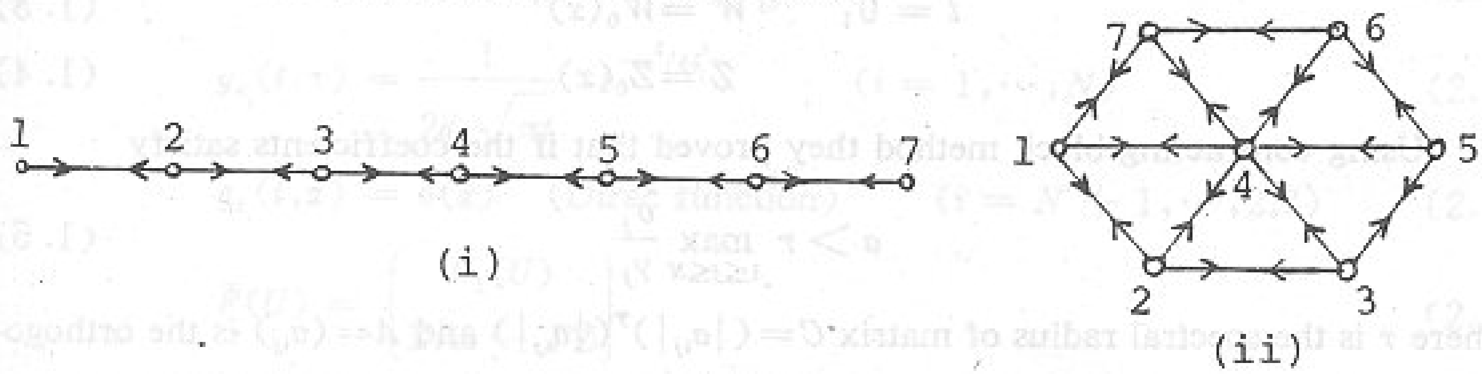

Fig. 1

where "∘" represents the axon and the straight line between neighbouring axons expresses the interactions each other. Throughout this paper we make the following basic assumption:

Axons which are next to each interact and non-adjacent axons do not interact.

If we number each fibre according to its position as the vertex of a graph we can write down a coupled system of reaction diffusion equations in the form

$$W_t = (\Lambda I - aB)W_{xx} + H(W, Z) \tag{1.1}$$

$$Z_t = \Sigma W - \Gamma Z \tag{1.2}$$

where W, Z are N-dimensional vectors and W is the N-tuple of membrane potentials, Λ

and a constants satisfying $A \gg a > 0$, I is the $N \times N$ identity matrix, H is the N-tuple of ionic membrane currents for each fibre, $H = F(W) - Z$, $F(W) = (f(W_1), \cdots f(W_N))^T$. f is of the usual cubic form $f(x) = x(1-x)(x-a)\ (a>0)$, $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_N)$, $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_N)$, where $\sigma_i, \gamma_i > 0\ (i=1, \cdots, N)$, $B = (b_{ij})$ the adjacency matrix for the graph defined as follows.

If $\mathscr{K}$ is a graph on N vertices $V_1, \cdots, V_N$, we define the adjacency matrix

$$b_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \text{ and there is an edge in } \mathscr{K} \\ & \text{connecting } V_i, V_j \\ 0 & \text{otherwise} \end{cases}$$

e. g. In the configurations i), ii) mentioned above we have

$$B = \begin{bmatrix} 0100\cdots0 \\ 1010\cdots0 \\ 0101\cdots0 \\ \cdots\cdots\cdots \\ 0000\cdots1 \\ 0000\cdots0 \end{bmatrix}, \quad B = \begin{bmatrix} 0101001 \\ 1011000 \\ 0101100 \\ 1110111 \\ 0011010 \\ 0001101 \\ 1001010 \end{bmatrix}$$

In [4] P. Grendrod and B. D. Sleeman studied the Cauchy problem of (1. 1), (1. 2)

$$t = 0: \quad W = W_0(x) \tag{1. 3}$$

$$Z = Z_0(x) \tag{1. 4}$$

Using contracting block method they proved that if the coefficients satisfy

$$a > r \max_{1 \le j \le N} \frac{\sigma_j}{\gamma_j} \tag{1. 5}$$

where r is the spectral radius of matrix $C = (|a_{ij}|)^T (|a_{ij}|)$ and $A = (a_{ij})$ is the orthogonal transformation matrix of $M = \Lambda I - aB$, then problem (1. 1)—(1. 4) admits a unique global solution and the solution decays exponentially to zero provided that initial values are small. This corresponds to the biological fact that a minimum stimulus is needed to trigger nerve fibre bundles; smaller stimuli lead to no signal transmitted down the axons. But assumption (1. 5) is an extra hypothesis. For example, in case (i) of Fig. 1 we need $a \ge \frac{2}{(N+1)} \mathrm{ctg}^2 \frac{\pi}{2(N+1)} \max_{1 \le j \le N} \frac{\sigma_j}{\gamma_j}$, in case (ii) $a > 5.323 \max_{1 \le j \le N} \frac{\sigma_j}{\gamma_j}$, where a is the constant in the cubic function f.

In this paper we adopt a different approach, namely, the energy method to obtain the similar threshold result without any restrictions on the coefficients.