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THEOREM OF UPPER-LOWER SOLUTIONS FOR A CLASS OF NONLINEAR DEGENERATE PARABOLIC SYSTEMS WITHOUT QUASI-MONOTONY①

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Abstract

In this paper we generalize the comparative method of solving some non-degenerate reaction-diffusion systems to a class of nonlinear degenerate parabolic systems without quasi-monotony, and obtain the relevant theorem of upper-lower solutions.

1. Introduction

Consider the initial boundary value problem for the following nonlinear parabolic system:

$$(\mathrm{I})\qquad \begin{cases} \dfrac{\partial \beta_1(u_1)}{\partial t}=\Delta u_1+f_1(u_1, u_2) \\ \dfrac{\partial \beta_2(u_2)}{\partial t}=\Delta u_2+f_2(u_1, u_2) \\ u_1|_{\partial\Omega\times(0,T]}=h_1(x,t),\ u_2|_{\partial\Omega\times(0,T]}=h_2(x,t) \\ u_1(x,0)=u_{10}(x,0),\ u_2(x,0)=u_{20}(x) \end{cases}$$

where $\Omega\subset R^N$ is a bounded open set and $\partial\Omega$ is sufficiently smooth. When $\delta_i\le\beta_i'(s)\le M_i$, δ_i, M_i being some positive constants independent of s, problem (I) is a non-degenerate reaction-diffusion system, for which many results of theorem of upper and lower solutions have been obtained in the cases where f_i satisfy some monotone conditions [3], [5]. And this theorem has been established for f_i without quasi-monotony in [6]. In this paper we shall prove a theorem of upper and lower solutions for problem (I) in the degenerate case where f_i have no quasi-monotony.

We need the following hypotheses:

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(H1) $\beta_i(s)$ are continuous and monotone increasing in R^1, $\beta_i(0)=0$. Denote

$$\beta_i'(s)=\begin{cases}\lim\limits_{h\to 0}\sup\dfrac{\beta_i(s)-\beta_i(s-h)}{h}, & \text{if } s>0\\ \lim\limits_{h\to 0}\sup\dfrac{\beta_i(s+h)-\beta_i(s)}{h}, & \text{if } s<0\end{cases}$$

$\beta_i'(s)$ satisfy

i) $0<a_M\le\beta_i'(s)$, $\forall\ s\in[-M,M]\setminus\{0\}$

ii) $\liminf\limits_{|s|\to 0}\beta_i'(s)=\infty$

iii) There exists a closed neighborhood of the origin $[-\delta_0,\delta_0]$, such that

$$\beta_i'(s)\le\beta_i'(r),\ s\in R^1\setminus[-\delta_0,\delta_0],\ r\in[-\delta_0,\delta_0]$$

and $\beta_i'(s)$ are monotone decreasing on $(0,\delta_0]$ and monotone increasing on $[-\delta_0,0)$.

iv) There exist some constants $\gamma>0$ and $\varepsilon>0$ such that for any $z\in(-\varepsilon,\varepsilon)$ there are

$$\frac{\beta_i(s+z)-\beta_i(z)}{s}\le\gamma\beta_i'(s+z),\ \forall\ s\ne 0,\ s+z\ne 0$$

v) $\displaystyle\int_0^{+\infty}\frac{\beta_i(s)}{s}ds=+\infty$

(H2) Denote by $\varphi_i(s)$ the inverse functions of $\beta_i(s)$, and then

$$g_i(v_1,v_2)=f_i(\varphi_1(v_1),\varphi_2(v_2)),\ i=1,2$$

are locally Lipschitz continuous with respect to $v_j (j=1,2)$.

(H3) $h_i(x,0)=u_{i0}(x)$, $x\in\overline{\Omega}$; and there exist continuous nondecreasing functions $\omega_{h_i}(s)$, $\omega_{u_{i0}}(s)$, such that

$$|h_i(x_1,t_1)-h_i(x_2,t_2)|\le\omega_{h_i}(|x_1-x_2|+|t_1-t_2|^{\frac{1}{2}})$$

$$|u_{i0}(x_1)-u_{i0}(x_2)|\le\omega_{u_{i0}}(|x_1-x_2|)$$

and $h_i(x,t)\in W_2^{2,0}(Q_T)$, where $Q_T=\Omega\times[0,T]$.

Example $\beta_i(s)=|s|^{1/m_i}\operatorname{sgn} s$, $m_i>1\ (i=1,2)$ satisfy (H1); and when $f_i(p,q)$ are locally Lipschitz continuous, $g_i(v_1,v_2)$ satisfy (H2).

It is obvious that $\beta_i(s)$ satisfy i), ii), iii) and v) of (H1). We shall verify that $\beta_i(s)$ satisfy iv) of (H1). When there is no confusion, we write m instead of m_i.

Let

$$A(s,z)=\frac{\beta_i(s+z)-\beta_i(z)}{s\beta_i'(s+z)}$$

a) When $z>0$, $s+z>0$, or $s+z<0$, $z<0$