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GLOBAL BEHAVIOR OF POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract

Let M be a complete Riemannian manifold, and let Δ be the Laplacian on M. In this paper, we study the global properties of positive solutions of

$$\Delta U + b \cdot \nabla U + hU^a = 0$$

We mainly prove some theorems of Liouville type.

1. Introduction

A. Huber [6] and S. T. Yau [8] proved the non-existence of non-constant positive harmonic function on the complete Riemannian manifold with non-negative Ricci curvature. B. Gidas and J. Spruck [5] studied the properties of positive solutions of

$$\Delta U + hU^a = 0 \tag{1.1}$$

and

$$\Delta U + b \cdot \nabla U = 0 \tag{1.2}$$

We consider the properties of positive solutions of

$$\Delta U + b \cdot \nabla U + hU^a = 0 \tag{1.3}$$

where $b \in \mathscr{X}(M)$, $h \in C^2(M)$, $a \in R^+$.

In the following two results, we improve the results of Gidas-Spruck by relaxing their conditions on the growth of $|\nabla \log(h)|$, $|b|$ and on the range of a.

Theorem A *Let M be a complete Riemannian manifold of dimension n with non-negative Ricci curvature.*

Suppose that $h \in C^2(M)$, $b \in \mathscr{X}(M)$ and $a \in R^+$ satisfy the following conditions

(1) $\forall x \in M, h(x) \geq 0, \Delta h(x) + \nabla h(x) \cdot b(x) \geq 0$

(2) *the tensor field* $C_{n,a}|b|^2 g_{ij} + \nabla_i b_j$ *is negative definite on* M, *where*

$$C_{n,a} = \frac{2}{n} + \frac{8}{n^2}\left(\frac{2}{n} - \frac{a-1}{a}\right)^{-1}$$

(3) $|b|=o(\rho)$ *as* $\rho\to\infty$, *where* $\rho=\rho(x)$ *is the geodesic distance from* x *to a fixed point* x_0,

(4) $0<\alpha<\frac{n+4}{n}$, *if* $n=2, 3$

$$0<\alpha<\frac{n}{n-2}, \quad \text{if } n\geq 4$$

Then every non-negative solution of (1. 3) *is constant.*

Theorem B *Let M be a complete Riemannian manifold with non-negative Ricci curvature, and let* $V_x(R)$ *denote the volume of* $B_x(R)=\{y\in M: \text{dist }(y, x)\leq R\}$. *Suppose that*

(1) $V_x(R)\leq C_xR^\beta$,

(2) $1<\alpha<\frac{\beta+2}{\beta}$, $\lambda\geq 0$ *is a constant.*

If $U(x)$ *is a non-negative bounded solution of*

$$\Delta U+\lambda U^\alpha=0 \tag{1. 4}$$

then $U(x)\equiv 0$.

In Theorem B we show that the range of α for non-existence of positive bounded solution of (1. 4) depends on the growth of the volume of metric ball $B_x(R)$ and does not depend on the dimension of M.

On the other hand, H. Donnelly [3] studied the existence of bounded harmonic functions on complete Riemannian manifold with non-negative Ricci curvature outside a compact set. We generalized his result and prove that

Theorem C *Let M be a n-dimensional complete Riemannian manifold, and let* M_0 *be a compact subset of M. Suppose that*

(1) *the Ricci curvature of M is non-negative outside* M_0,

(2) *the tensor field* $\frac{1}{n}|b|^2g_{ij}+\nabla_ib_j$ *is negative definite outside* M_0,

(3) $|b|=o(\rho)$ *as* $\rho\to\infty$, *where* $\rho=\rho(x)$ *is the geodesic disdtance from* x *to a fixed point* x_0.

Then the vector space of bounded solutions of (1. 2) *is finite dimensional.*

The result shows that the conjecture of S. T. Yau, if M is uniformly equivalent to $\overline{M}$, $\overline{M}$ is complete with non-negative Ricci curvature, then the vector space of bounded harmonic functions on M is finite dimensional, is true under some additional conditions.

2. The Proof of Theorem A

In this section, we mainly prove Theorem A and obtain the gradient estimates of positive solutions of (1. 3).