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A HARNACK INEQUALITY FOR DEGENERATE PARABOLIC EQUATIONS①

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Abstract

In this paper we apply the Moser iteration method to degenerate parabolic divergence structure equations. Under some conditions we get a Harnack inequality for weak solutions and from it derive Hölder estimates for weak solutions of uniformly degenerate parabolic equations and the continuity of weak solutions of non-uniformly degenerate parabolic equations.

1. Introduction

A Harnack inequality and Hölder continuity for the weak solutions of strictly elliptic and parabolic divergence structure equations are due to De Giorgi[1], Nash[2] and Moser[3, 4, 5]. In the recent years, many authors atempt to extend these results to degenerate equations and some important results have been obtained by Fabes, Kenig, Serapioni[6], Chiarenza, Serapioni[7], Chanillo, wheeden[8]. In this paper, we use the well-known Moser iteration method to study the behavior of the weak solutions of degenerate equations and obtain a Harnack inequality and Hölder continuity for the solutions. Our results are the extensions of a Harnack inequality and Hölder continuity for degenerate elliptic equations in [8] and imply the one in [7].

In order to use the Moser iteration method to get a Harnack inequality for the weak solutions of degenerate equations, we need to establish weighted imbedding theorem. The weighted imbedding theorems applied to degenerate elliptic equations in [8] are derived in [9], but we can not apply directly these theorems to degenerate parabolic equations, so in Section 2 on the basis of these theorems we establish corresponding "parabolic type" weighted imbedding theorems.

In Section 3, we state precisely the Harnack inequality and sketch its proof. The proof of the technical Lemmas, Lemma 3. 3, 3. 4, are to be stated in the last section. In

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this paper the weighted imbedding theorems and the Moser iteration procedure are different from that in [7], consequently the Harnack inequality we obtain is united with the Moser's one (see [3], [4], [5]) and possess more adaptability. In Section 3, we also derive from the Harnack inequality Hölder estimates for weak solutions of uniformly degenerate parabolic equations and the continuity of weak solutions of non-uniformly degenerate parabolic equations. However, only the continuity of weak solutions of uniformly degenerate parabolic equations is obtained in [7].

2. Weighted Imbedding Theorems

Let $w(x)$ be a non-negative, locally integrable functions in R^n $(n \geq 2)$, Ω be a bounded domain in R^n. We define function spaces

$$L^p(\Omega, w) \equiv \left\{ f;\ \| f \|_{L^p(\Omega, w)} \equiv \left(\int_\Omega |f|^p w dx \right)^{1/p} < +\infty \right\} \quad (1 < p < \infty)$$

$H^1(\Omega, w)$ $(H_0^1(\Omega, w))$ is the completion of $\mathrm{Lip}(\bar{\Omega})$ ($\mathrm{Lip}_0(\Omega)$) under the norm $\| f \|_{1, \Omega, w} \equiv \left(\int_\Omega |f|^2 dx + \int_\Omega |D_x f|^2 w dx \right)^{1/2}$, where $\mathrm{Lip}(\Omega)$ ($\mathrm{Lip}_0(\Omega)$) denote space of Lipschitz continuous functions in Ω (with compact support in Ω) and we use the notation $D_x f$ for the gradient vector $(D_{x_1} f, D_{x_2} f, \ldots, D_{x_n} f)$ of function f.

$$V_2^{1,0}(\Omega \times (a, b), w) \equiv C^0([a, b]; L^2(\Omega)) \cap L^2(a, b; H^1(\Omega, w))$$

which is endowed with the natural norm:

$$\| f \|_{V_2^{1,0}(\Omega \times (a, b), w)} \equiv \max_{a \leq t \leq b} \| f(\cdot, t) \|_{L^2(\Omega)} + \left(\int_a^b \int_\Omega |D_x f|^2 w dx dt \right)^{1/2}$$

$$W_2^{1,1}(\Omega \times (a, b), w) \equiv \{ f \in L^2(a, b; H^1(\Omega, w)) : D_t f \in L^2(a, b; L^2(\Omega)) \}$$

$$\overset{\circ}{V}{}_2^{1,0}(\Omega \times (a, b), w) \equiv C^0([a, b]; L^2(\Omega)) \cap L^2(a, b; H_0^1(\Omega, w))$$

clearly, this space is the subspace of $V_2^{1,0}(\Omega \times (a, b), w)$;

$$\overset{\circ}{W}{}_2^{1,1}(\Omega) \times (a, b), w)$$
$$\equiv \{ f \in L^2(a, b; H_0^1(\Omega, w)) : D_t f \in L^2(a, b; L^2(\Omega)) \}$$

this space is the subspace of $W_2^{1,1}(\Omega \times (a, b), w)$.

Let us introduce the following notations: B denotes a ball in R^n and $B_s(x_0)$ the ball of centre x_0 and radius s; for a ball B, sB $(s > 0)$ denotes the ball which is concentric with B whose radius is s times that of B; for a bounded measurable set E in R^n or R^{n+1}, $w(E)$ denotes the ordinary Lebesgue integral of function w in E.

In this paper, the standard summation convention that repeated indices indicate summation from 1 to n is followed and the same letter C is used to denote different con-