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A FURTHER COMMENT ON THE COERCIVENESS THEORY FOR ELLIPTIC SYSTEMS①

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In this note we exhibit a counterexample to show that quasilinear elliptic systems of the type

$$\int_\Omega (A_{ij}^{\alpha\beta}(x,u)D_\alpha u^i D_\beta v^j + b_{ij}^\alpha D_\alpha u^i v^j + c_{ij}^\beta u^i D_\beta v^j + d_{ij}u^i v^j + f_j^\beta D_\beta v^j + g^j v^j)dx = 0, \text{ for all } v \in H_0^1(\Omega; R^N) \tag{1}$$

do not satisfy, in general, the weak coerciveness condition, i. e., the Gårding's inequality. This work is the continuition of [5].

Suppose $\Omega \subset R^n$ is a bounded smooth domain, for $x \in R^n$, $u \in R^N$, and the maps $(x, u) \to A_{ij}^{\alpha\beta}(x,u)$ are real valued C^∞ functions for $\alpha, \beta = 1, \cdots, n; i, j = 1, \cdots, N (N, n > 1)$ satisfying

$$A_{ij}^{\alpha\beta}(x_0,u_0)\xi^i\xi^j\eta_\alpha\eta_\beta \geq c_0|\xi|^2|\eta|^2 \tag{2}$$

for every fixed $x_0 \in R^n$, $u_0 \in R^N$, all $\xi \in R^N$, $\eta \in R^n$, where c_0 is a positive consant, i. e., Legendre-Hadamard condition is satisfied. Let $a(u,v)$ be the quadratic form of the left hand side of (1) for $u, v \in C_0^\infty(\Omega; R^N)$ and assume that $b_{ij}^\alpha, c_{ij}^\beta, d_{ij} \in L^\infty(\Omega)$.

The problem is, under the above assumptions, if $a(\cdot,\cdot)$ is weakly coercive, i. e., if there exist $\lambda_0 > 0$, $\lambda_1 \geq 0$, such that

$$a(u,u) \geq \lambda_0 \int_\Omega |Du|^2 dx - \lambda_1 \int_\Omega |u|^2 dx \text{ for all } u \in C_0^\infty(\Omega; R^N) \tag{3}$$

This type of problems is the content of Gårding's inequality. It is known that (3) is satisfied provided $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x) \in C^0(\bar{\Omega})$ (see [3]) and is not satisfied in general if $A_{ij}^{\alpha\beta}(x) \in L^\infty(\Omega)$ (see [5]). In this note we will show that in the present situation where $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x,u) \in C^\infty(R^n \times R^N) \cap L^\infty(R^n \times R^N)$, the answer of the problem is negative.

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① We will adopt notations and conventions of [2]. In particular, we use the summation convention with Greek letters from 1 to n and Latin letters from 1 to N and $N, n > 0$.

In fact, (3) leads to the existence of weak solutions of (1) which could be considered as the steady state of general reaction-diffusion systems studied by Amann [1]:

$$\frac{\partial u}{\partial t} - \sum_{j=1}^{n} D_j(A(x,u)Du_j) = f(x,u,Du)$$

The Example

For $n, N \geq 2$, define

$$B(x,t) = \{y \in R^n, |y-x| < t\} \quad , \quad \text{for } x \in R^n, t > 0$$

and

$$D_k = B\left(p_k, \frac{1}{2^k}\right), \quad B_k = B\left(p_k, \frac{1}{2^{k+1}}\right), \quad D = B(0,4), \quad E_k = B\left(p_k, \frac{1}{2^{k+2}}\right)$$

$k = 1, 2, \cdots$, where

$$p_k = (s_k, 0, \cdots, 0)$$

$$s_k = \begin{cases} 0 & k = 0 \\ 3\left(1 - \dfrac{1}{2^k}\right) & k = 1, 2, \cdots \end{cases}$$

Choose $\xi \in C_0^\infty(R^n)$ to satisfy

$$\xi = 1 \quad \text{on} \quad B\left(0, \frac{1}{2}\right), \quad \xi = 0 \quad \text{on } R^n \backslash B(0,1), \quad 0 \leq \xi \leq 1, \quad |D\xi| \leq C$$

Define

$$f_k(x) = \frac{1}{2^{k^2}} \int_{R^n} g_k(x-y) \chi E_k(y) dy$$

with

$$g_k(x) = 2^{n(k+2)} g(2^{k+2}x), \quad g(x) = h(x)/c$$

where

$$h(x) = \begin{cases} \exp(1/(|x|^2 - 1)) & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$c = \int_{R^n} h(x) dx$$

and χE_k is the characteristic function of E_k.

It can be easily checked that $f_k(x) \in C_0^\infty(R^n)$, $\text{supp}(f_k) = B_k$, $f_k > 0$ on B_k. Now set

$$f(x) = \sum_{k=0}^{\infty} f_k(x)$$

then $f \in C_0^\infty(D)$ by our choice of f_k's.

Define a C^∞ function $F: R^1 \to R^1$, such that F is nondecreasing, $F(t) = 0$ when $t \leq 0$; and $F(t) > 0$ when $t > 0$,

$$\lim_{t \to +\infty} F(t) = K + 2$$