J. Partial Differential Equations
Vol. 2, No. 4(1989), 67—82

UNSTEADY FLOW OF POWER LAW FLUIDS IN POROUS MEDIUM WITH DOUBLE POROSITY[1]

Bai Donghua
(Sichuan University)

(Received Feb. 16, 1988; revised Sept. 22, 1988)

1. Introduction

In this paper, we consider the system of nonlinear partial differential equations of the form

$$a_i \frac{\partial u_i}{\partial t} = k_i((u_{ix})^m)_x + (-1)^{i-1}(f(u_2) - f(u_1)), \quad i = 1,2, m > 1 \quad (1.1)$$

A system of type (1.1) arises in the unsteady flow of Non-Newtonian fluids in porous medium consisting of two components. According to the argument as in the works [1, 2], the system (1.1) can be obtained from the modified Darcy's law in the absence of the yield stress in one dimensional space. In this case the modified Darcy's law gives the relation between the velocity and pressure of the fluid as following

$$v = \left(-\frac{k}{\mu}\frac{\partial p}{\partial x}\right)^{\frac{1}{n}}, \qquad 0 < n < 1$$

We are interested in the following initial-boundary value problem related to system (1.1):

$$(I) \qquad \begin{aligned} & a_i \frac{\partial u_i}{\partial t} = k_i((u_{ix})^m)_x + (-1)^{i-1}(f(u_2) - f(u_1)), \quad \text{in } H \\ & u_i(x,0) = u_{i0}(x) \qquad x \geq 0 \\ & u_i|_{x=0} = \underline{u} \qquad t > 0 \qquad (i = 1,2) \end{aligned}$$

where $H = (0, +\infty) \times (0, +\infty)$, $a_i, k_i (i=1,2)$, m and $\underline{u}$ are the real numbers with $a_i > 0$, $k_i > 0$ and $m > 1$, the function $f \in C^1(R)$ and the functions $u_{i0}(x) (i=1,2)$ satisfy the hypotheses: $u_{i0}(x)$ are non-decreasing and Lipschitz continuous functions on $\overline{R}_+$ and

$$\lim_{x \to +\infty} u_{i0}(x) = \bar{u} = \text{const. with } \underline{u} < \bar{u} < +\infty \quad (1.2)$$

Definition 1.1 *A pair of functions* $\{u_1, u_2\}$ *will be called a weak solution of Problem*

① The project supported by National Natural Science Foundation of China.

(I) *if it satisfies*

(i) $u_i(x,t)$ *are bounded and continuous on* $\overline{H}$ *and* $u_i(0,t)=\underline{u}$, $\underline{u}\le u_i(x,t)\le\bar{u}$ $(i=1,2)$;

(ii) $u_{ix}(x,t)$ *are nonnegative and bounded on* $\overline{H}$;

(iii) $u_{it}\in L^2_{loc}(H)$;

(iv) $$\int_{t_0}^{t_1}\int_{x_0}^{x_1}\{a_i u_i \varphi_{it}-k_i(u_{ix})^m\varphi_{ix}+(-1)^{i-1}(f(u_2)-f(u_1))\varphi_i\}dxdt$$

$$-\int_{x_0}^{x_1}a_i u_i\varphi_i dx\Big|_{t_0}^{t_1}=0 \qquad (i=1,2) \tag{1.3}$$

for any $0\le x_0<x_1<+\infty$, $0\le t_0<t_1<+\infty$ *and for all* $\varphi_i\in \dot{W}^{1,1}((x_0,x_1)\times(t_0,t_1))$ $(i=1,2)$.

In Section 2, we shall show the uniqueness of the weak solution of Problem (I). In Section 3, we shall establish some estimates for the solutions of the approximate problem (I^{ε}_{n}). Those estimates enable us to prove the existence of Problem I in Section 4.

2. Uniqueness

Lemma 2.1 *Let* $f\in C^1(R)$ *and* f' *be nonnegative and nondecreasing function. Then for any real numbers* a_i *and* b_i $(i=1,2)$ *the following inequality holds*

$$J(a_1,b_1;a_2,b_2)$$
$$=[(f(a_2)-f(b_2))-(f(a_1)-(f(b_1))][(a_2-b_2)^+-(a_1-b_1)^+]$$
$$\ge 0$$

where $g^+=\max\{g,0\}$.

Proof If $(a_1-b_1)^+=0$ or $(a_2-b_2)^+=0$, for example, $(a_1-b_1)^+=0$ and $(a_2-b_2)^+\ge 0$, namely, $a_2\ge b_2$ and $a_1\le b_1$. By the intermediate value theorem and $f'\ge 0$, we have

$$J=[f'(\theta_2)(a_2-b_2)-f'(\theta_1)(a_1-b_1)](a_2-b_2)\ge 0$$

Hence we only need to prove this lemma in the case when $(a_i-b_i)^+>0$, i. e., $a_i>b_i$ $(i=1,2)$. Since $J(a_1,b_1;a_2,b_2)=J(a_2,b_2;a_1,b_1)$, we can assume $a_2\ge a_1$.

Case 1 $b_2\le b_1$. We have $a_2\ge a_1>b_1\ge b_2$. Therefore, by $f'\ge 0$,

$$J=[f(a_2)-f(a_1)+f(b_1)-f(b_2)][(a_2-a_1+b_1-b_2)]\ge 0$$

Case 2 $b_2\ge b_1$ and $a_2-b_2\ge a_1-b_1$. We have

$$J=\left[(a_2-b_2)\int_0^1 f'(b_2+\theta(a_2-b_2))d\theta-(a_1-b_1)\int_0^1 f'(b_1+\theta(a_1-b_1))d\theta\right]$$
$$\cdot(a_2-b_2-a_1+b_1)$$