J. Partial Differential Equations
Vol. 2, *No.* 4(1989), 83—88

POSITIVE SOLUTIONS OF $\Delta u+u^{(n+2)/(n-2)}=0$ ON CONTRACTIBLE DOMAINS

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(Received April 4, 1988; revised October 11, 1988.)

Abstract

We show that there are bounded contractible domains in R^n, $n\geq 3$, on which the Dirichlet problem for the equation $\Delta u+u^{(n+2)/(n-2)}=0$ admit positive solutions. This result, combining with the well-known nonexistence result of Pohozaev, implies that geometry of the domains plays a crucial role in the solvability of the problem.

1. Introduction

Consider the Dirichlet problem

$$\Delta u+u^{p-1}=0\ ,\quad u>0\quad \text{in } \Omega,\quad u=0\quad \text{on } \partial\Omega \tag{1.1}$$

where Ω is a bounded domain in R^n, $n\geq 3$ and $p=\dfrac{2n}{n-2}$. This problem is of interest because its solvability depends on the geometry and topology of the domain Ω. It is perhaps the simplest problem with this property. Pohozaev [6] proved that if Ω is star-shaped with respect to some point, then (1.1) admits no solutions. On the other hand, as pointed out by Kazdan and Warner [5], (1.1) has solutions on a standard annulus. Recently, Coron [3] showed the problem is solvable on domains with 'sufficiently small holes' and conjectured that the same result holds for any non-contractible domains. More recently, this conjecture has been proved to be true for $n=3$ by Bahri and Coron [1]. For general $n\geq 3$, they proved if the homology group $H_d(\Omega, Z_2)\neq 0$ for some positive integer d then the problem has a solution.

Since the results of Bahri and Coron concern only the topology of Ω, the question wether there exist contractible domains on which (1.1) is solvable has become more interesting. If such domains exist, since they have trivial topology their geometry should be responsible for the existence of solutions.

In is our aim in this note to show that there do exist contractible domains Ω for which (1. 1) has solutions. The idea that leads to the construction of such domains is actually quite simple. We just observe that an annular domain can be 'perturbed' into a contractible domain as follows. For simplicity we consider only standard annuli

$$A_s = \{x \in R^n : 0 < s < |x| < 1\}$$

Let $\varepsilon > 0$ be small and define

$$C_\varepsilon = \{x = (x_1, x') \in R^1 \times R^{n-1} = R^n : 0 \le x_1 \le 1, |x'| \le \varepsilon\}$$

Then $A_s \backslash C_\varepsilon$ is a contractible domain which may be considered as a perturbation of A_s in the sense that the Sobolev quotient Q behaves similarly on $H_0^1(A_s)\backslash\{0\}$ and on $H_0^1(A_s \backslash C_\varepsilon)\backslash\{0\}$ under a level set, provided ε is sufficiently small. Therefore, we can follow essentially the same idea of Coron in [3] to show for certain values of s and ε, Q has critical points in $H_0^1(A_s \backslash C_\varepsilon)\backslash\{0\}$, which correspond to the solutions of (1. 1) with $\Omega = A_s \backslash C_\varepsilon$.

Although our methods apply for a class of more general domains, we will state and prove our result only for the above domains $A_s \backslash C_\varepsilon$. The main result is as follows.

Theorem *There exists $s_0 \in (0,1)$ and for $s \in (0, s_0)$ there exists $\varepsilon(s) > 0$ such that if $s \in (0, s_0)$ and $\varepsilon \in (0, \varepsilon(s))$ then (1. 1) has a solution for $\Omega = A_s \backslash C_\varepsilon$.*

In the next section we will recall some well-known facts and prove a few preliminary lemmas. The proof of the theorem will be given in Section 3.

2. Known Facts and Preliminary Lemmas

Let $H_0^1(\Omega)$ be the subspace of $H^1(R^n)$ consisting of functions which are supported in Ω. That means $u \in H_0^1(\Omega)$ will be considered as functions defined on R^n with $u|R^n\backslash\Omega = 0$. Since Ω is bounded we may take $\|\nabla u\|_2$ as the norm of $u \in H_0^1(\Omega)$ where $\|\cdot\|_q$ is the $L^q(R^n)$-norm and $q > 0$. Consider the Sobolev quotient:

$$Q(u) \equiv \frac{\int |\nabla u|^2 dx}{\left[\int |u|^p dx\right]^{2/p}} = \frac{\|\nabla u\|_2^2}{\|u\|_p^2}$$

where $p = 2n/(n-2)$. (Here and in the sequel integrals are taken over R^n unless otherwise specified.) We will consider Q as a functional on $H_0^1(\Omega)\backslash\{0\}$. The following facts are well known (see [2], [1]).

Let $S = S_n = \inf\{Q(u) : u \in H_0^1(\Omega)\backslash\{0\}\}$. Then S is a positive constant independent of the domain Ω. Indeed we know $S = \frac{n}{4}(n-2)\omega_n^{2/n}$ where ω_n is the volume of the unit n-sphere. S can not be achieved by $u \in H_0^1(\Omega)\backslash\{0\}$ for any bounded domain Ω. However S is achieved by a family of smooth functions on R^n defined by