J. Partial Differential Equations
Vol. 2, No. 4(1989), 89—95

A UNIQUENESS THEOREM FOR A CLASS OF DEGENERATE QUASILINEAR PARABOLIC EQUATIONS OF FOURTH ORDER

Yin Jingxue
(Dept. of Math., Jilin University)

(Received July 21, 1988)

1. Introduction

This paper is a continuation of the recent paper [1], where the first boundary value problem

$$\frac{\partial u}{\partial t} + D^4 A(u) = f, \quad \text{in} \quad Q_T = (0,T) \times (0,1) \tag{1.1}$$

$$A(u)(t,0) = A(u)(t,1) = DA(u)(t,0) = DA(u)(t,1) = 0 \tag{1.2}$$

$$u(0,x) = u_0(x) \tag{1.3}$$

is considered with

$$A(u) = \int_0^u a(s)ds \tag{1.4}$$

and $a(s)$ is a nonnegative, smooth function, $D = \frac{\partial}{\partial x}$. In particular, the uniqueness was proved for generalized solutions in the class

$$X = \{u; A(u) \in L^\infty(0,T;H_0^2(I)), \frac{\partial}{\partial t}A(u) \in L^2(Q_T), u \in L^\infty(Q_T)\}$$

Our interest is in extending the uniqueness result to generalized solutions of (1.1)—(1.3) in the sense of the following.

Definition *By a generalized solution of the problem* (1.1)—(1.3), *we mean a function* $u \in L^2(Q_T)$ *with* $A(u) \in L^2(Q_T)$ *satisfying the following integral equality*:

$$-\int_0^1 u_0(x)\varphi(0,x)dx - \iint_{Q_T} u\frac{\partial \varphi}{\partial t}dtdx + \iint_{Q_T} A(u)D^4\varphi dtdx = \iint_{Q_T} f\varphi dtdx \tag{1.5}$$

for any $\varphi \in C^\infty(\overline{Q}_T)$ *with* $\varphi(t,0) = \varphi(t,1) = D\varphi(t,0) = D\varphi(t,1) = \varphi(T,x) = 0$.

The main result is the following Theorem.

Theorem *Let* $u_0 \in L^2(I)$, $f \in L^2(Q_T)$, *let* u_1, u_2 *be generalized solutions of the problem* (1.1)—(1.3). *Then*

$$u_1(t,x) = u_2(t,x), \quad \text{a. e.} \quad \text{in} \quad Q_T$$

The method used to establish the uniqueness result is based on the idea of changing (1. 1), (1. 2) into an ordinary differential equality in $L^2(I)$ by a family of operators

$$T_\lambda : L^2(I) \to H_0^2(I) \cap H^4(I)$$

which is defined by the two point boundary value problem for ordinary differential equations

$$D^4(T_\lambda g) + \lambda(T_\lambda g) = g, \quad (\lambda > 0) \tag{1. 6}$$

$$(T_\lambda g)(0) = (T_\lambda g)(1) = D(T_\lambda g)(0) = D(T_\lambda g)(1) = 0 \tag{1. 7}$$

2. The Proof of the Theorem

Set $w = u_1 - u_2$, $v = A(u_1) - A(u_2)$. Then by the definition,

$$-\iint_{Q_T} w\frac{\partial\varphi}{\partial t}dtdx + \iint_{Q_T} vD^4\varphi dtdx = 0 \tag{2. 1}$$

holds for any $\varphi \in C^\infty(\overline{Q}_T)$ with $\varphi(t,0) = \varphi(t,1) = D\varphi(t,0) = D\varphi(t,1) = \varphi(T,x) = 0$, and hence, by an approximate process, for any $\varphi \in H^{2,1}(Q_T)$ with

$$\gamma\varphi(t,0) = \gamma\varphi(t,1) = \gamma D\varphi(t,0) = \gamma D\varphi(t,1) = \gamma\varphi(T,x) = 0$$

where $H^{2,1}(Q_T)$ denote the space $\{u; u \in L^2(Q_T), D^2 u \in L^2(Q_T), \frac{\partial u}{\partial t} \in L^2(Q_T)\}$, γ is a trace operator.

Now we consider the two point boundary value problem (1. 6)—(1. 7). It is easily seen that if $g \in L^2(I)$, then $T_\lambda g$ is uniquely determined by g and the following estimates hold:

$$\lambda\int_I (T_\lambda g)^2 dx, \int_I (D^2 T_\lambda g)^2 dx, \int_I (D^4 T_\lambda g)^2 dx \le \int_I g^2 dx \tag{2. 2}$$

If g depends also on the variable t and $g \in L^2(Q_T)$, then

$$\lambda\iint_{Q_T} (T_\lambda g)^2 dtdx, \iint_{Q_T} (D^2 T_\lambda g)^2 dtdx, \iint_{Q_T} (D^4 T_\lambda g)^2 dtdx \le \iint_{Q_T} g^2 dtdx \tag{2. 3}$$

We also have the following properties for T_λ,

$$\int_I (T_\lambda f)g dx = \int_I f(T_\lambda g) dx, \quad \text{if} \quad f, g \in L^2(I) \tag{2. 4}$$

In fact, we get from the definition of T_λ,

$$\int_I (T_\lambda f)g dx = \int_I (T_\lambda f)(D^4(T_\lambda g) + \lambda T_\lambda g) dx$$

$$= \int_I D^2(T_\lambda f)D^2(T_\lambda g) dx + \lambda\int_I (T_\lambda f)(T_\lambda g) dx$$