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Existence and Regularity of Solution for Strongly Nonlinear $p(x)$-Elliptic Equation with Measure Data

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Abstract. The first part of this paper is devoted to study the existence of solution for nonlinear $p(x)$ elliptic problem $A(u)=\mu$ in Ω, $u=0$ on $\partial\Omega$, with a right-hand side measure, where Ω is a bounded open set of $\mathbb{R}^N$, $N\geqslant 2$ and $A(u)=-\mathrm{div}(a(x,u,\nabla u))$ is a Leray-Lions operator defined from $W_0^{1,p(x)}(\Omega)$ in to its dual $W^{-1,p'(x)}(\Omega)$. However the second part concerns the existence solution, of the following setting nonlinear elliptic problems $A(u)+g(x,u,\nabla u)=\mu$ in Ω, $u=0$ on $\partial\Omega$. We will give some regularity results for these solutions.

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1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, ($N\geqslant 2$). In this paper, we deal with the following Direchlet problem:

$$\begin{cases} Au+g(x,u,\nabla u)=\mu, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}$}$$

with non-standard p-structure which involves a variable growth exponent $p(\cdot)$.

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The principal part of the above equation is the operator A defined by

$$Au=-\operatorname{div}(a(x,u,\nabla u)),$$

which satisfies the classical Leray-Lions conditions with some variable exponent $p(\cdot)$. However the non linearity g is supposed to satisfy the sign condition and some natural growth with respect to ∇u, but no growth with u is supposed. The second member μ considered in this paper is not regular that is a Radon measure.

The study of partial differential equation involving $p(x)$ growth conditions has received specific attention in recent decades. This is a consequence of the fact that such equations can be used to model phenomena which arise in mathematical physics. Electro rheological fluids and elastic mechanics are two examples of physical fields which benefit from such kinds of studies.

Problem $(\mathcal{P})$ is studied in [1] where $a=a(x,\nabla u)$ which satisfying the classical Leray-Lions conditions with some constant exponent p such that: $2-\frac{1}{N}<p<N$. The solution obtained in [1] admits the following regularity $u\in W_0^{1,q}(\Omega)$ for all:

$$1\leqslant q<\frac{N(p-1)}{N-1}.$$

The approach used by the authors in [1] is to approximate the measure μ by a sequence (f_n) in $W^{-1,p'(x)}\cap L^1(\Omega)$ which converge to μ. The limiting process hinges of the proof of the almost pointwise convergence of the sequence $(\bigtriangledown u_n)$, where u_n is the weak solution of the problem $(\mathcal{P})$ with $\mu=f_n$.

When trying to relax the coefficients of a, that is $a(\cdot)$ have not a polynomial growths, then the problem $(\mathcal{P})$ is formulated is the general setting of Orlicz spaces generated by an N-function M which appears in the non classical growths of $a(\cdot)$. In this case, we found the work [2] which treat the study of a problem $(\mathcal{P})$ in Orlicz-Sobolev spaces. The solutions obtained in [2] and [3] belongs to $T_0^{1,M}(\Omega)\cap W_0^1L_B(\Omega)$ for any $B\in P_M$, where $B\in P_M$ is a special class of N-functions. For others works, we refer the reader [4–8] and [9].

In the recent years, variable exponent Sobolev spaces have attracted an increasing amount attention, the impulse, for this mainly comes from there physical applications, such in image processing (underline the borders, eliminate the noise) and electro-rheological fluids.

In the framework of variable exponent Sobolev spaces, we list the works [6,7,10,11] and others, where the second member μ of the problem $(\mathcal{P})$ is taking as an element of $L^1(\Omega)$ or is a measure which admits the composition $\mu=f-\operatorname{div}(F)$, with $f\in L^1(\Omega)$ and $F\in\prod L^{p(x)}(\Omega)$.

Our purpose in this paper is to study the existence of solution of the problem $(\mathcal{P})$, in the case where the datum μ is a finite Radon measure and in the case of variable exponent,