Existence Theorem for a Class of Nonlinear Fourth-order Schrödinger-Kirchhoff-Type Equations

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Abstract. This paper is concerned with the existence of nontrivial solutions for the following fourth-order equations of Kirchhoff type

\[
\begin{align*}
\Delta^2 u - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + \lambda V(x) u &= f(x, u), \quad x \in \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N),
\end{align*}
\]

where \(a, b\) are positive constants, \(\lambda \geq 1\) is a parameter, and the nonlinearity \(f\) is either superlinear or sublinear at infinity in \(u\). With the help of the variational methods, we obtain the existence and multiplicity results in the working spaces.

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1 Introduction

In this paper, we consider the following nonlinear Schrödinger-Kirchhoff-type problem:

\[
\begin{align*}
\Delta^2 u - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + \lambda V(x) u &= f(x, u), \quad x \in \mathbb{R}^N, \\
u &\in H^2(\mathbb{R}^N),
\end{align*}
\]

where \(\Delta^2\) is the biharmonic operator and \(\nabla u\) denotes the spatial gradient of \(u\). Moreover, \(a, b\) are positive constants, \(\lambda \geq 1\) is a parameter, and the potential \(V(x)\) satisfies the following conditions:

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\((V_1)\) \(V(x) \in C(R^N, \mathbb{R})\) and \(\inf_{x \in \mathbb{R}^N} V(x) \geq A > 0\), for some \(A\);

\((V_2)\) there exists \(B > 0\) such that the set \(\{x \in \mathbb{R}^N : V(x) \leq B\}\) is nonempty and \(\text{meas}\{x \in \mathbb{R}^N : V(x) \leq B\} < +\infty\), where “meas” means the Lebesgue measure in \(\mathbb{R}^N\).

Problem \((P_\lambda)\) is called nonlocal because of the presence of the term \(-\left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right)\), which implies that equation \((P_\lambda)\) is no longer a point-wise identity. In the recent years, there are many papers about Kirchhoff-type problems without the term of biharmonic operator. On the bounded domain, positive solutions are investigated by authors such as Ma and Rivera [1] and Alves et al. [2]. Sign-changing solutions have been obtained by Zhang and Perera [3] and Mao and Zhang [4]. If the problem is set on \(\mathbb{R}^N\), some interesting results can be found in [5–7] and the references there in.

It is well-known that the following fourth-order elliptic equation of Kirchhoff type:

\[
\begin{cases}
\Delta^2 u - \left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u), & x \in \Omega, \\
u = 0, \nabla u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

is related to the stationary analogue of the equation of Kirchhoff type:

\[
u_{tt} - \Delta^2 u - \left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u), \quad x \in \Omega,
\]

which is regarded as a good approximation for describing nonlinear vibrations of beams or plates (see [8, 9]). Recently, Ma [10, 11] considered the fourth-order equation

\[
\begin{cases}
\quad u^{(4)} - M \left(\int_0^1 |u'(x)|^2 \, dx\right) u'' = h(x) f(x, u), & 0 \leq x \leq 1, \\
u (0) = u (1) = 0, & u'' (0) = u'' (1) = 0,
\end{cases}
\]

and obtain the multiplicity of solutions.

Later on, Wang and An [12] get the existence of nontrivial solution of a fourth-order elliptic equation

\[
\begin{cases}
\Delta^2 u - M \left(\int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u), & x \in \Omega, \\
u = 0, \quad \Delta u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

under some conditions on the function \(M(t)\) and \(f\) with Mountain Pass theorem. Moreover, Wang et al. also consider the existence of the nontrivial solutions for \((1.1)\) with a parameter \(\lambda\) (see [13]), where \(M = a + bt\) and \(b \geq 0\).

We note that problem \((P_\lambda)\) with \(a = 1, b = 0\), reduces fourth-order elliptic equations

\[
\begin{cases}
\Delta^2 u - \Delta u + \lambda V(x) u = f(x, u), & x \in \mathbb{R}^N, \\
u \in H^2(\mathbb{R}^N).
\end{cases}
\]