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Sharp Conditions on Global Existence and Non-Global Existence of Solutions of Cauchy Problem for 1D Generalized Boussinesq Equations

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Abstract. This paper consider the Cauchy problem for a class of 1D generalized Boussinesq equations $u_{tt}-u_{xx}-u_{xxtt}+u_{xxxx}+u_{xxxxtt}=f(u)_{xx}$. By utilizing the potential well method and giving some conditions on $f(u)$, we obtain the invariance of some sets and obtain the threshold result of global existence and nonexistence of solutions.

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1 Introduction

In this paper, our main purpose is to study the Cauchy problem for the generalized Boussinesq equations

$$u_{tt}-u_{xx}-u_{xxtt}+u_{xxxx}+u_{xxxxtt}=f(u)_{xx}, \qquad x\in\mathbb{R},\ t>0, \tag{1.1}$$
$$u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x), \qquad x\in\mathbb{R}. \tag{1.2}$$

Here we give some assumptions on $f(u)$ as follows

$$(\mathrm{H}_1) \qquad \begin{cases} f(u)=\pm|u|^p, & p>4 \text{ and } p\neq 2k,\ k=3,4,\cdots, \\ \text{or } f(u)=-|u|^{p-1}u, & p>4 \text{ and } p\neq 2k+1,\ k=2,3,\cdots, \end{cases}$$

$$(\mathrm{H}_2) \qquad f(u)=\pm u^{2k}, \qquad \text{or } f(u)=-u^{2k+1}u, \qquad k=1,2,\cdots.$$

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In 1872, J. Boussinesq [1] proposed the classical Boussinesq equation

$$u_{tt}=-\gamma u_{xxxx}+u_{xx}+(u^2)_{xx}. \tag{1.3}$$

This equation describes the propagation of small amplitude long waves on the surface of shallow water and gives a scientific explanation of the existence to solitary waves. The following nonlinear partial differential equation

$$u_{tt}+u_{xxxx}=a(u_x^2)_x, \tag{1.4}$$

was derived in the study of weakly nonlinear analysis of elasto-plastic-microstructure models for longitudinal motion of elasto-plastic bar [2], where a is a constant.

Instead of the term u_{xxxx}, Eq. (1.3) became the famous improved Boussinesq equation (the IBq equation)

$$u_{tt}-u_{xx}-u_{xxtt}=(u^2)_{xx}, \tag{1.5}$$

which describes the propagation of long waves on shallow water as well. Makhankov [3] pointed out that the IBq equation

$$u_{tt}-\Delta u-\Delta u_{tt}=\Delta(u^2) \tag{1.6}$$

can be given by starting with the exact hydro-dynamical set of equations in plasma, and a modification of the IBq equation analogous to the modified Korteweg-de Vries equation yields

$$u_{tt}-\Delta u-\Delta u_{tt}=\Delta(u^3). \tag{1.7}$$

Eq. (1.7) is the so-called IMBq (modified IBq) equation. Wang and Chen [4,5] gave the local and global solution and the solution which blows up in finite time. Further, they considered the Cauchy problem of the multidimensional generalized IMBq equation

$$u_{tt}-\Delta u-\Delta u_{tt}=\Delta f(u), \tag{1.8}$$

and obtained the golbal existence of small amplitude solution.

Schneider [6] investigated the following nonlinear wave equation

$$u_{tt}-u_{xx}-u_{xxtt}-\mu u_{xxxx}+u_{xxxxtt}=(u^2)_{xx}, \tag{1.9}$$

which describes the water wave problem with surface tension. The model can also be formally derived from the two-dimensional water wave problem. The Eq. (1.9) is called "bad" Boussinesq equation as $\mu>0$ and "good" Boussineq equation as $\mu<0$. The classical Boussinesq equation can be extended to a more natural model [7]

$$u_{tt}-u_{xx}-(\mu+1)u_{xxxx}+u_{xxxxxx}=(u^2)_{xx}. \tag{1.10}$$