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Semi-linear Elliptic Equations on Graph

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Abstract. Let $G=(V,E)$ be a locally finite graph, $\Omega\subset V$ be a finite connected set, Δ be the graph Laplacian, and suppose that $h:V\to\mathbb{R}$ is a function satisfying the coercive condition on Ω, namely there exists some constant $\delta>0$ such that

$$\int_\Omega u(-\Delta+h)u\,\mathrm{d}\mu\geq\delta\int_\Omega|\nabla u|^2\mathrm{d}\mu,\quad\forall u:V\to\mathbb{R}.$$

By the mountain-pass theorem of Ambrosette-Rabinowitz, we prove that for any $p>2$, there exists a positive solution to

$$-\Delta u+hu=|u|^{p-2}u\qquad\text{in }\Omega.$$

Using the same method, we prove similar results for the p-Laplacian equations. This partly improves recent results of Grigor'yan-Lin-Yang.

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1 Introduction and main results

Let $G=(V,E)$ be a locally finite graph, where V denotes the vertex set and E denotes the edge set. The weight of xy is supposed that $w_{xy}>0$ and $w_{xy}=w_{yx}$, where $xy\in E$. Here and throughout this paper we write $y\sim x$ if $xy\in E$. Let $\deg(x)=\sum_{y\sim x}w_{xy}$ be the degree of $x\in V$. We can define the μ-Laplacian on G and the associated gradient form as

$$\Delta u(x)=\frac{1}{\mu(x)}\sum_{y\sim x}w_{xy}(u(y)-u(x)),\tag{1.1}$$

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$$\Gamma(u,v)(x)=\frac{1}{2\mu(x)}\sum_{y\sim x}w_{xy}(u(y)-u(x))(v(y)-v(x)), \tag{1.2}$$

where $\mu:V\to\mathbb{R}$ is a finite measure, and $\Gamma(u,u)$ is written as $\Gamma(u)$. We can also define the length of $\nabla u(x)$ as

$$|\nabla u|(x)=\sqrt{\Gamma(u)(x)}=\left(\frac{1}{2\mu(x)}\sum_{y\sim x}w_{xy}(u(y)-u(x))^2\right)^{1/2}. \tag{1.3}$$

For any function $u:V\to\mathbb{R}$, we denote,

$$\int_\Omega u\mathrm{d}\mu=\sum_{x\in\Omega}\mu(x)u(x). \tag{1.4}$$

In this note, we consider existence results for the semi-linear elliptic equation

$$-\Delta u+hu=|u|^{p-2}u \qquad \text{in } \Omega, \tag{1.5}$$

where h satisfies the coercive condition on Ω, namely, there exists some constant $\delta>0$ such that

$$\int_\Omega u(-\Delta+h)u\mathrm{d}\mu\geq\delta\int_\Omega|\nabla u|^2\mathrm{d}\mu. \tag{1.6}$$

for all functions $u:V\to\mathbb{R}$ with zero boundary condition.

Recently the equation (1.5) has been studied by Grigor'yan-Lin-Yang [1] in the case that $h=-\alpha$ is a constant. They proved that if $\alpha<\lambda_1(\Omega)$, then for any $p>2$, there exists a positive solution to the equation

$$\begin{cases}-\Delta u-\alpha u=|u|^{p-2}u & \text{in } \Omega^0,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.7}$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian with respect to the Dirichlet boundary condition, and it reads

$$\lambda_1(\Omega)=\inf_{u\not\equiv 0,u|_{\partial\Omega=0}}\frac{\int_\Omega|\nabla u|^2\mathrm{d}\mu}{\int_\Omega u^2\mathrm{d}\mu} \tag{1.8}$$

where $\partial\Omega$ is the boundary of Ω, namely $\partial\Omega=\{x\in\Omega:\exists y\notin\Omega \text{ such that } xy\in E\}$. Moreover the interior of Ω is denoted by $\Omega^0=\Omega\setminus\partial\Omega$.

Our first result is the following:

Theorem 1.1. *Let $G=(V,E)$ be a locally finite graph, $\Omega\subset V$ be a finite connected set with $\Omega^0\neq\varnothing$. Suppose that $h:V\to\mathbb{R}$ satisfies the coercive condition, namely there exists some constant $\delta>0$ such that for all $u\in W_0^{1,2}(\Omega)$*

$$\int_\Omega u(-\Delta+h)u\mathrm{d}\mu\geq\delta\int_\Omega|\nabla u|^2\mathrm{d}\mu.$$