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Level Sets of Certain Subclasses of α-Analytic Functions

DAGHIGHI Abtin[1,*] and WIKSTRÖM Frank[2]

[1] *Mid Sweden University, Holmgatan 10, SE-851 70 Sundsvall, Sweden.*
[2] *Center for Mathematical Sciences, Lund University, Box 118, SE-221 00 Lund, Sweden.*

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Abstract. For an open set $V\subset\mathbb{C}^n$, denote by $\mathscr{M}_\alpha(V)$ the family of α-analytic functions that obey a boundary maximum modulus principle. We prove that, on a bounded "harmonically fat" domain $\Omega\subset\mathbb{C}^n$, a function $f\in\mathscr{M}_\alpha(\Omega\setminus f^{-1}(0))$ automatically satisfies $f\in\mathscr{M}_\alpha(\Omega)$, if it is C^{α_j-1}-smooth in the z_j variable, $\alpha\in\mathbb{Z}_+^n$ up to the boundary. For a submanifold $U\subset\mathbb{C}^n$, denote by $\mathfrak{M}_\alpha(U)$, the set of functions locally approximable by α-analytic functions where each approximating member and its reciprocal (off the singularities) obey the boundary maximum modulus principle. We prove, that for a C^3-smooth hypersurface, Ω, a member of $\mathfrak{M}_\alpha(\Omega)$, cannot have constant modulus near a point where the Levi form has a positive eigenvalue, unless it is there the trace of a polyanalytic function of a simple form. The result can be partially generalized to C^4-smooth submanifolds of higher codimension, at least near points with a Levi cone condition.

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1 Introduction

A higher order generalization of the holomorphic functions are the solutions of the equation $\partial^q f/\partial\bar{z}^q=0$, for a positive integer q. These functions are called *polyanalytic functions of order q* or simply *q-analytic functions*. An excellent introduction to polyanalytic functions can be found in the survey article by Balk [1]. For several recent results on

*Corresponding author. *Email addresses:* abtindaghighi@gmail.com (A. Daghighi), Frank.Wikstrom@math.lth.se (F. Wikström)

polyanalytic functions, see, e.g.,[2, 3]. A higher dimensional generalization of q-analytic functions is the set of *α-analytic functions*, $\alpha \in \mathbb{Z}_+^n$. In this paper, we consider the set of functions which can be locally uniformly approximated by α-analytic functions satisfying a boundary maximum modulus principle. We prove an extension of Radó's theorem for such functions on complex manifolds. Secondly, we prove, for a special subclass on (not necessarily complex) submanifolds, a generalization of the fact that q-analytic functions cannot have constant modulus on open sets unless they are of a particularly simple form.

Our main results are Theorems 3.4, 5.1 and 5.2. Theorem 5.1 is a generalization of the fact that q-analytic functions cannot have constant modulus on open sets unless they are of the form $\lambda\overline{Q(z)}/Q(z)$ for some polynomial Q of degree $<q$ and some constant $\lambda \in \mathbb{C}$. In particular we consider families, $\mathfrak{M}_\alpha$, (defined on submanifolds) whose members and their reciprocals (except at singularities) satisfy a boundary maximum modulus principle, and can be locally uniformly approximated by α-analytic functions. We show that such families cannot have members with constant modulus near points with at least one positive Levi eigenvalue, unless these members are the there the trace of an α-analytic function of the form $\lambda\overline{Q(z)}/Q(z)$ for some holomorphic polynomial $Q(z)$ of degree $<\alpha_j$ in $z_j, 1 \le j \le n$ and constant λ. For hypersurfaces, this is done for the C^3-smooth case but we also give a partial generalization for C^4-smooth real submanifolds of higher codimension, see Theorem 5.2. Theorem 3.4, is that functions in n complex variables which are C^{α_j-1}-smooth in the z_j variable, and which, off their zero set, are α-analytic functions that obey the boundary maximum modulus principle, extend across their zero set to functions of the same class.

2 Preliminaries

For general background on polyanalytic functions, see Balk [1] and references therein (in particular we want to mention the earlier work by Balk & Zuev [4]).

Definition 2.1. *Let $U \subset \mathbb{C}$ be a domain. A function $f: U \to \mathbb{C}$ is called* q-analytic (*or* polyanalytic of order q) *if it can be written as*

$$f(z) = \sum_{j=0}^{q-1} a_j(z)\bar{z}^j, \ a_j \in \mathscr{O}(U). \tag{2.1}$$

If $a_{q-1} \not\equiv 0$, then q is called the exact *order.*

It is well-known, and almost self-evident, that $f(z) = u(x,y) + iv(x,y)$ of class $C^q(U)$ is q-analytic on U if and only if

$$\frac{\partial^q f}{\partial \bar{z}^q} = 0 \quad on\ U. \tag{2.2}$$

(*see Balk [1, p. 198]*).