Existence and Nonexistence for Semilinear Equations on Exterior Domains

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Abstract. In this paper we prove the existence of an infinite number of radial solutions of $\Delta u + K(r)f(u) = 0$ on the exterior of the ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$ where $f$ is odd with $f < 0$ on $(0, \beta)$, $f > 0$ on $(\beta, \delta)$, $f \equiv 0$ for $u > \delta$, and where the function $K(r)$ is assumed to be positive and $K(r) \to 0$ as $r \to \infty$. The primitive $F(u) = \int_0^u f(s) \, ds$ has a “hilltop” at $u = \delta$ which allows one to use the shooting method to prove the existence of solutions.

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1 Introduction

In this paper we study radial solutions of:

\begin{align*}
\Delta u + K(r)f(u) &= 0 & \text{in } \Omega, \\
 u &= 0 & \text{on } \partial \Omega, \\
 u &\to 0 & \text{as } |x| \to \infty,
\end{align*}

where $x \in \Omega = \mathbb{R}^N \setminus B_R(0)$ is the complement of the ball of radius $R > 0$ centered at the origin. We assume there exist $\beta, \delta$ with $0 < \beta < \delta$ such that $f(0) = f(\beta) = f(\delta) = 0$, and $F(u) = \int_0^u f(s) \, ds$ where:

\begin{itemize}
  \item $f$ is odd and locally Lipschitz, $f < 0$ on $(0, \beta)$, $f > 0$ on $(\beta, \delta)$, $f \equiv 0$ on $(\delta, \infty)$, and $F(\delta) > 0$.
\end{itemize}

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In addition we assume:

\[ f'(\beta) > 0 \quad \text{if } N > 2. \]

We note it follows that \( F(u) = \int_0^u f(s) \, ds \) is even (since \( f \) is odd) and has a unique positive zero, \( \gamma \), (since \( f < 0 \) on \( (0, \beta) \), \( f > 0 \) on \( (\beta, \delta) \), and \( F(\delta) > 0 \) with \( \beta < \gamma < \delta \) such that:

\[ F < 0 \text{ on } (0, \gamma), F > 0 \text{ on } (\gamma, \infty), \text{ and } F \text{ is strictly monotone on } (0, \beta) \text{ and on } (\beta, \delta). \]  

(1.6)

In earlier papers [1, 2] we studied (1.1), (1.3) when \( \Omega = \mathbb{R}^N \) and \( K(r) \equiv 1 \). In [3] we studied (1.1) and (1.3) with \( K(r) \equiv 1 \) and \( \Omega = \mathbb{R}^N \setminus B_R(0) \). We proved existence of an infinite number of solutions - one with exactly \( n \) zeros for each nonnegative integer \( n \) such that \( u \to 0 \) as \( |x| \to \infty \). Interest in the topic for this paper comes from recent papers [4–6] about solutions of differential equations on exterior domains.

When \( f \) grows superlinearly at infinity i.e. \( \lim_{u \to \infty} f(u)/u = \infty \), and \( \Omega = \mathbb{R}^N \) then the problem \((1.1), (1.3)\) has been extensively studied [7–11]. The type of nonlinearity addressed here has not been studied as extensively [1, 3].

Since we are interested in radial solutions of \((1.1)-(1.3)\) we assume that \( u(x) = u(|x|) = u(r) \) where \( x \in \mathbb{R}^N \) and \( r = |x| = \sqrt{x_1^2 + \cdots + x_N^2} \) so that by the chain rule \( u \) solves:

\[
u''(r) + \frac{N-1}{r} u'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R, \infty), \quad \text{where } R > 0.
\]

We now let \( b > 0 \) and we proceed to examine solutions of:

\[
u''(r) + \frac{N-1}{r} u'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R, \infty), \quad \text{where } R > 0, \quad (1.7)
\]

\[
u(R) = 0, \quad u'(R) = b > 0. \quad (1.8)
\]

We will show that for appropriate values of \( b \) we also have \( \lim_{r \to \infty} u(r, b) = 0 \).

We will occasionally denote the solution of the above by \( u(r, b) \) in order to emphasize the dependence on the initial parameter \( b \). Also throughout this paper differentiation will always be with respect to the variable \( r \).

We will assume that: there exist constants \( c_1 > 0, c_2 > 0, \) and \( \alpha > 0 \) such that:

\[ c_1 r^{-\alpha} \leq K(r) \leq c_2 r^{-\alpha} \quad \text{for } 0 < \alpha < 2(N-1) \quad \text{on } [R, \infty). \]  

(1.9)

In addition, we assume: \( K \) is differentiable and \( \exists \) constants \( d > 0, D > 0 \) s. t.

\[ 0 < d \leq \frac{rK'}{K} + 2(N-1) \leq D \quad \text{on } [R, \infty). \]  

(1.10)

Note that \((1.10)\) implies \( r^{2(N-1)} K(r) \) is non-decreasing since:

\[ (r^{2(N-1)} K(r))' = r^{2N-3} K \left( 2(N-1) + \frac{rK'}{K} \right) > 0. \]