JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **30**, No. 4, pp. 299-316

doi: 10.4208/jpde.v30.n4.2
December 2017

Existence and Nonexistence for Semilinear Equations on Exterior Domains

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Received 5 January 2016; Accepted 20 August 2017

Abstract. In this paper we prove the existence of an infinite number of radial solutions of $\Delta u+K(r)f(u)=0$ on the exterior of the ball of radius $R>0$ centered at the origin in $\mathbb{R}^N$ where f is odd with $f<0$ on $(0,\beta)$, $f>0$ on (β,δ), $f\equiv 0$ for $u>\delta$, and where the function $K(r)$ is assumed to be positive and $K(r)\to 0$ as $r\to\infty$. The primitive $F(u)=\int_0^u f(s)\,\mathrm{d}s$ has a "hilltop" at $u=\delta$ which allows one to use the shooting method to prove the existence of solutions.

AMS Subject Classifications: 34B40, 35B05

Chinese Library Classifications: O175.8, O175.25

Key Words: Semilinear; hilltop.

1 Introduction

In this paper we study radial solutions of:

$$\Delta u+K(r)f(u)=0 \qquad \text{in } \Omega, \tag{1.1}$$

$$u=0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

$$u\to 0 \qquad \text{as } |x|\to\infty, \tag{1.3}$$

where $x\in\Omega=\mathbb{R}^N\backslash B_R(0)$ is the complement of the ball of radius $R>0$ centered at the origin. We assume there exist β,δ with $0<\beta<\delta$ such that $f(0)=f(\beta)=f(\delta)=0$, and $F(u)=\int_0^u f(s)\,\mathrm{d}s$ where:

$$f \text{ is odd and locally Lipschitz, } f<0 \text{ on } (0,\beta), f>0 \text{ on } (\beta,\delta), f\equiv 0 \text{ on } (\delta,\infty), \text{ and } F(\delta)>0. \tag{1.4}$$

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In addition we assume:

$$f'(\beta)>0 \quad \text{if } N>2. \tag{1.5}$$

We note it follows that $F(u)=\int_0^u f(s)\,\mathrm{d}s$ is even (since f is odd) and has a unique positive zero, γ, (since $f<0$ on $(0,\beta)$, $f>0$ on (β,δ), and $F(\delta)>0$) with $\beta<\gamma<\delta$ such that:

$$F<0 \text{ on } (0,\gamma), F>0 \text{ on } (\gamma,\infty), \text{ and } F \text{ is strictly monotone on } (0,\beta) \text{ and on } (\beta,\delta). \tag{1.6}$$

In earlier papers [1,2] we studied (1.1), (1.3) when $\Omega=\mathbb{R}^N$ and $K(r)\equiv 1$. In [3] we studied (1.1) and (1.3) with $K(r)\equiv 1$ and $\Omega=\mathbb{R}^N\backslash B_R(0)$. We proved existence of an infinite number of solutions - one with exactly n zeros for each nonnegative integer n such that $u\to 0$ as $|x|\to\infty$. Interest in the topic for this paper comes from recent papers [4–6] about solutions of differential equations on exterior domains.

When f grows superlinearly at infinity i.e. $\lim\limits_{u\to\infty} f(u)/u=\infty$, and $\Omega=\mathbb{R}^N$ then the problem (1.1), (1.3) has been extensively studied [7–11]. The type of nonlinearity addressed here has not been studied as extensively [1,3].

Since we are interested in radial solutions of (1.1)-(1.3) we assume that $u(x)=u(|x|)=u(r)$ where $x\in\mathbb{R}^N$ and $r=|x|=\sqrt{x_1^2+\cdots+x_N^2}$ so that by the chain rule u solves:

$$u''(r)+\frac{N-1}{r}u'(r)+K(r)f(u(r))=0 \qquad \text{on } (R,\infty), \text{ where } R>0.$$

We now let $b>0$ and we proceed to examine solutions of:

$$u''(r)+\frac{N-1}{r}u'(r)+K(r)f(u(r))=0 \qquad \text{on } (R,\infty), \text{ where } R>0, \tag{1.7}$$

$$u(R)=0,\ u'(R)=b>0. \tag{1.8}$$

We will show that for appropriate values of b we also have $\lim\limits_{r\to\infty} u(r,b)=0$.

We will occasionally denote the solution of the above by $u(r,b)$ in order to emphasize the dependence on the initial parameter b. Also throughout this paper differentiation will always be with respect to the variable r.

We will assume that: there exist constants $c_1>0$, $c_2>0$, and $\alpha>0$ such that:

$$c_1 r^{-\alpha}\le K(r)\le c_2 r^{-\alpha} \quad \text{for } 0<\alpha<2(N-1) \qquad \text{on } [R,\infty). \tag{1.9}$$

In addition, we assume: K is differentiable and $\exists$ constants $d>0, D>0$ s. t.

$$0<d\le \frac{rK'}{K}+2(N-1)\le D \quad \text{on } [R,\infty). \tag{1.10}$$

Note that (1.10) implies $r^{2(N-1)}K(r)$ is non-decreasing since:

$$(r^{2(N-1}K(r))'=r^{2N-3}K\left(2(N-1)+\frac{rK'}{K}\right)>0.$$