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Optimal Regularity and Control of the Support for the Pullback Equation

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Abstract. Given f,g two $C^{r,\alpha}$ either symplectic forms or volume forms on a bounded open set $\Omega\subset\mathbb{R}^n$ with $0<\alpha<1$ and $r\geq 0$, we give natural conditions for the existence of a map $\varphi\in\mathrm{Diff}^{r+1,\alpha}(\Omega;\Omega)$ satisfying

$$\varphi^*(g)=f \text{ in } \Omega \quad \text{and} \quad \mathrm{supp}(\varphi-\mathrm{id})\subset\Omega.$$

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1 Introduction

The pullback equation $\varphi^*(g)=f$ where g and f are both symplectic forms or both volume forms has been studied a lot. One could consult [1] for an extensive survey for the pullback equation in general. We start by giving a very brief summary for the symplectic case: Darboux [2] proved that any two symplectic forms can be pulled back locally one to another. This result has been reproved by Moser [3] using an elegant flow method. These two proofs do not produce any gain in regularity: the map φ is at most as regular as the data g and f. Later Bandyopadhyay-Dacorogna [4] established in particular a local existence result with optimal regularity in the Hölder spaces $C^{r,\alpha}$, $0<\alpha<1$. Since the pullback equation is a system of first order PDE's, optimality means here that for $g,f\in C^{r,\alpha}$ there exists a solution $\varphi\in C^{r+1,\alpha}$.

For the global case coupled with a Dirichlet condition,

$$\varphi^*(g)=f \text{ in } \Omega \quad \text{and} \quad \varphi=\mathrm{id} \text{ on } \partial\Omega, \tag{1.1}$$

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the following (quasi) optimal result has been proved in [5] (see also [4] for a slightly weaker result): given $\omega \in C^{r,\alpha}([0,1]\times\overline{\Omega};\Lambda^2)$ an homotopy of symplectic forms between g and f such that, for every $t\in[0,1]$,

$$\omega_t-\omega_0 \text{ is exact in } \Omega \quad \text{and} \quad \omega_t\wedge\nu=\omega_0\wedge\nu\in C^{r+1,\alpha}(\partial\Omega;\Lambda^3),$$

then there exists $\varphi\in\mathrm{Diff}^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ solving (1.1), where ν denotes the outward unit normal of some smooth bounded open set Ω and is identified with a $1-$form. Note that, for a solution to (1.1) to exist, we necessarily have

$$g\wedge\nu=f\wedge\nu \quad \text{on } \partial\Omega.$$

Note also that the only non natural condition (whose necessity is still an open problem) is the extra regularity of $\omega_t\wedge\nu$ on the boundary.

Concerning the case of volume forms (in which case, identifying volume forms with functions, $\varphi^*(g)=f$ reads as the single equation $g(\varphi)\det\nabla\varphi=f$) the first existence result (with no gain in regularity) for (1.1) is due to Moser [3]. Afterwards Dacorogna and Moser proved in [6] that given any $g,f\in C^{r,\alpha}(\overline{\Omega})$ strictly positive where Ω is a smooth connected bounded open set with

$$\int_\Omega g=\int_\Omega f, \tag{1.2}$$

there exists $\varphi\in\mathrm{Diff}^{r+1,\alpha}(\overline{\Omega};\overline{\Omega})$ satisfying (1.1). Note that (1.2) is obviously necessary to solve (1.1). Other proofs of this optimal regularity result have been established in [7–9].

In this paper we give conditions to solve the pullback equation $\varphi^*(g)=f$ in Ω with optimal regularity in Hölder space and imposing that $\varphi=\mathrm{id}$ near the boundary (and not only on $\partial\Omega$ as in (1.1)). An obvious necessary condition for this problem is then

$$\mathrm{supp}(g-f)\subset\Omega. \tag{1.3}$$

We prove (cf. Theorems 3.1 and 3.2) that the above condition is to some extent also sufficient:

Theorem 1.1. *(i) given Ω a bounded open set in $\mathbb{R}^n$ star-shaped with respect with some open ball and ω a continuous homotopy of $C^{r,\alpha}(\Omega;\Lambda^2)$ symplectic forms between g and f such that*

$$\mathrm{supp}(\omega_t-f)\subset\Omega \quad \text{for every } t\in[0,1],$$

there exists $\varphi\in\mathrm{Diff}^{r+1,\alpha}(\Omega;\Omega)$ verifying

$$\varphi^*(g)=f \text{ in } \Omega \quad \text{and} \quad \mathrm{supp}(\varphi-\mathrm{id})\subset\Omega. \tag{1.4}$$

(ii) given g,f two non vanishing $C^{r,\alpha}(\Omega)$ functions in some bounded connected open set Ω verifying (1.2) *and* (1.3), *there exists $\varphi\in\mathrm{Diff}^{r+1,\alpha}(\Omega;\Omega)$ verifying* (1.4).