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Well-posedness for the Cahn-Hilliard Equation with Neumann Boundary Condition on the Half Space

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Abstract. In this paper, we investigate the Cahn-Hilliard equation defined on the half space and subject to the Neumann boundary and initial condition. For given initial data in some sobolev space, we prove the existence and analytic smoothing effect of the solution.

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1 Introduction

We consider the following Cahn-Hilliard equation on half space

$$u_t = \triangle \psi \quad \text{in} \quad (0,T] \times \mathbb{R}_N^+, \tag{1.1}$$

with

$$\psi = -\triangle u - au + bu^3, \tag{1.2}$$

where $0 < T \leq \infty$, $\mathbb{R}_N^+ = \{x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}_N, x_N > 0\}$, $a > 0$, $b > 0$ are constants, and $u : [0,T] \times \mathbb{R}_N^+ \to \mathbb{R}$ is the unknown. Equations (1.1) and (1.2) are supplemented by the Neumann boundary condition

$$\frac{\partial u}{\partial x_N}\Big|_{x_N=0} = \frac{\partial \triangle u}{\partial x_N}\Big|_{x_N=0} = 0 \tag{1.3}$$

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and the initial value condition

$$u(0,x)=g(x), \quad x\in\mathbb{R}_N^+. \tag{1.4}$$

This paper is concerned with the existence and analytic smoothness of the solutions $u(x,t)$ to the initial boundary value problem (1.1)-(1.4). We show that the regularity of the solution u in x is up to the boundary $\partial\mathbb{R}_N^+=\{x=(x_1,x_2,\cdots,x_N)\in\mathbb{R}_N, x_N=0\}$.

It is well known that the Cahn-Hilliard equation is central to materials science; it describes important qualitative features of two-phase systems, namely, the transport of atoms between unit cells, such as spinodal decomposition of binary mixtures that appears, for example, in cooling processes of alloys, glasses or polymer mixtures. The boundary condition (1.3) has a clear physical meaning: there can not be any exchange of the mixture constituents through the boundary. In the past years, the initial value problem for the Cahn-Hilliard equation on a *bounded* domain with the boundary condition (1.3) was intensively studied, and there were many outstanding results concerning the global existence, uniqueness, regularity and special properties of solutions, etc; see, for example, [1–10]. Established also are the results on global existence and uniqueness of classical solutions to the Cahn-Hilliard equation with decayed mobility, the nonoccurence of finite time blow-up phenomenon for the Cahn-Hilliard equation with non-constant mobility and cubic nonlinearity, and regularity of solution to a Cahn-Hilliard type equation with gradient dependent potential; see the discussions in the papers [11–13]. Recently local Gevrey regularity of solutions to the Cahn-Hilliard equation with *periodic* boundary condition and initial data in a space of distributions is investigated in [14]. Gevrey class is an intermediate space between the spaces of smooth functions and analytic functions, and the Gevrey class function of index 1 is just the real-analytic function. Since the study in [14] concentrates on regularity of periodic solutions, the arguments rely on Fourier expansion and convolution inequalities.

In this work, we study the well-posedness for the Cahn-Hilliard equation, endowed with the Neumann boundary condition (1.3) and initial condition (1.4), on the half space $\mathbb{R}_N^+$. We note that the similar linearized Cahn-Hilliard-Gurtin equations on the half space are discussed in [15] to derive optimal L_p-regularity for the linearized equations in an arbitrary bounded domain. Given an initial data in some Sobolev space, we prove that the problem (1.1)-(1.4) admits a unique solution in the same space, and moreover, similar to the heat equation we have the analytic smoothing effect for the initial-boundary problem, that is, the solution lies in analytic space for $t>0$.

Before state our main result we first introduce the following function space.

Definition 1.1. *Let $0<\rho<1$. We say that a function u belongs to the space X_ρ if*

$$\|u\|_{X_\rho}<+\infty$$