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Existence and Uniqueness Results for Caputo Fractional Differential Equations with Integral Boundary Value Conditions

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Abstract. In the paper, we consider the existence and uniqueness results for Caputo fractional differential equations with integral boundary value condition. The sufficient conditions of existence and uniqueness are obtained by applying the contraction mapping principle, Krasnoselskii's fixed point theorem and Leray-Schauder degree theory, which partly improves and extends the associated results of fractional differential equations. Four examples illustrating our main results are included.

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1 Introduction

Fractional differential equations can be used to describe many phenomena in a number of fields. For examples in physics, polymer rheology, chemistry, electrodynamics of complex medium, regular variation in thermodynamics, control theory, signal and image processing, biophysics, and so forth. There are many papers dealing with the existence and uniqueness results of boundary value problems for nonlinear fractional differential equations [1–5]. Meanwhile, boundary value problems with integral boundary value conditions of nonlinear fractional differential equations have aroused considerable attention. Boundary value problems with integral boundary value conditions have various

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applications in population dynamics,chemical engineering, etc. For some recent development on the integral boundary conditions, see the texts [6–9] and the references cited therein.

Recently, in [10] Bai and Lü used classical fixed point theorems to prove the multiple positive solutions for the following nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, & 0<t<1, \\ u(0)=u(1)=0, \end{cases}$$

where $1<\alpha\le 2$, D^{α} is the Riemann-Liouville fractional derivative of order α and

$$f\in C([0,1]\times[0,\infty),[0,\infty)).$$

In [11], Xu et al. investigated the existence of positive solutions for the following fractional boundary value problem

$$\begin{cases} D^{\alpha}u(t)+f(t,u(t))=0, & 0<t<1, \\ u(0)=u'(0)=u(1)=u'(1)=0, \end{cases}$$

where $2<\alpha\le 3$, D^{α} is a fractional derivative in the sense of Riemann-Liouville and $f\in C([0,1]\times[0,\infty),[0,\infty))$. The existence, multiplicity, uniqueness of positive solutions are established by using some fixed point theorems.

In [12], Cabada and Wang used Guo–Krasnoselskii fixed point theorem to show the existence of positive solutions for a class of nonlinear boundary value problem with integral boundary conditions as

$$\begin{cases} {}^{c}D^{\alpha}u(t)+f(t,u(t))=0, & 0<t<1, \\ u(0)=u''(0)=0, \quad u(1)=\lambda\displaystyle\int_0^1 u(s)\mathrm{d}s, \end{cases}$$

where $2<\alpha<3, 0<\lambda<2$, ${}^{c}D^{\alpha}$ is the Caputo fractional derivative of order α, and $f\in C([0,1]\times[0,\infty),[0,\infty))$.

Motivated by the results of [10–12], we consider the existence and uniqueness results for the following Caputo fractional differential equations with integral boundary value condition

$$\begin{cases} {}^{c}D^{\alpha}u(t)+f(t,u(t))=0, & 0<t<1, \ n<\alpha<n+1, \\ u(0)=u'(0)=\cdots=u^{(n-2)}(0)=u^{(n)}(0)=0, \\ u(1)=\lambda\displaystyle\int_0^1 u(s)\mathrm{d}s, & 0<\lambda<n, \end{cases} \tag{1.1}$$