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Extremal Functions of the Singular Moser-Trudinger Inequality Involving the Eigenvalue

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Abstract. In this paper, we derive the singular Moser-Trudinger inequality which involves the first eigenvalue and several singular points, and further prove the existence of the extremal functions for the relative Moser-Trudinger functional. Since the problems involve more complicated norm and multiple singular points, not only we can't use the symmetrization to deal with a one-dimensional inequality, but also the processes of the blow-up analysis become more delicate. In particular, the new inequality is more general than that of [1,2].

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1 Introduction

Let $\Omega\subset\mathbb{R}^2$ be a smooth bounded domain. The famous Moser-Trudinger inequality [3–5] says that

$$\sup_{u\in W_0^{1,2}(\Omega),||\nabla u||_{L^2(\Omega)\leq 1}}\int_\Omega e^{\sigma|u|^2}\mathrm{d}x<+\infty \tag{1.1}$$

for any $\sigma\leq 4\pi$. Moreover, for any fixed $u\in W_0^{1,2}(\Omega)$, it also holds that

$$\int_\Omega e^{\sigma|u|^2}\mathrm{d}x<+\infty$$

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for any $\sigma > 0$. In particular, the constant $\sigma = 4\pi$ is optimal in (1.1), which implies that, for any $\sigma > 4\pi$, the inequality (1.1) is invalid and there exists a sequence of $\{u_\epsilon\}$ in $W_0^{1,2}(\Omega)$ and $||\nabla u_\epsilon||_{L^2(\Omega)} = 1$ such that

$$\int_\Omega e^{\sigma|u_\epsilon|^2} \mathrm{d}x \to \infty \qquad \text{as } \epsilon \to 0.$$

Moser-Trudinger inequality (1.1), as a limit case of the Sobolev embedding, plays an important role in two-dimensional analytic and geometric problems. The further interesting subject is the existence of extremal functions to (1.1). By using the blow-up method Carleson and Chang [6] showed that the supremum is actually attained if Ω is a ball. Flucher [7] generalized this result to arbitrary bounded domains in $\mathbb{R}^2$. See also Adimurthi-Tintarev [8], Malchiodi-Martinazzi [9] and Mancini-Sandeep [10] and the references in these papers for recent developments on this subject.

This inequality was generalized in many ways. One kind of generalization of (1.1) is the so-called singular Moser-Trudinger inequality, which was originally established by Adimurthi-Sandeep [11]. They proved that

$$\sup_{u \in W_0^{1,2}(\Omega), ||\nabla u||_{L^2(\Omega)} \le 1} \int_\Omega \frac{e^{su^2} - 1}{|x|^{2t}} \mathrm{d}x < +\infty,$$

for $s \in (0, 4\pi(1-t))$ and $t \in [0,1)$. Further, Csató-Roy [12] proved that the supremum is attained for this singular Moser-Trudinger embedding.

For the case of several singular points, Iula-Mancini [1] proved that the supremum

$$\sup_{u \in W_0^{1,2}(\Omega), \int_\Omega |\nabla u|^2 \mathrm{d}x \le 1} \int_\Omega V(x) e^{4\pi(1+\alpha)(1+\lambda||u||^2_{L^q(\Omega)})} \mathrm{d}x \tag{1.2}$$

is finite and is attained for $\lambda \in [0, \lambda_q(\Omega))$. Here

$$\lambda_q(\Omega) = \inf_{u \in W_0^{1,2}(\Omega), \int_\Omega |\nabla u|^2 \mathrm{d}x \le 1} \frac{\int_\Omega |\nabla u|^2 \mathrm{d}x}{||u||^2_{L^q(\Omega)}}$$

for $q > 1$, and

$$V(x) = K(x) \prod_{i=1}^m |x - p_i|^{2\alpha_i}, \tag{1.3}$$

where $K(x) > 0$, $K(x) \in C^0(\overline{\Omega})$; $p_1, p_2, \cdots, p_m$ are the different points in Ω; and $\alpha_i \in (-1, +\infty)$, $\alpha_i \notin \mathbb{Z}$ such that

$$\alpha = \min_{1 \le i \le m} \{\alpha_i\} \quad \text{and } \alpha \in (-1, 0).$$