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The Generalization of the Classical Weighted Pseudo-Almost Periodic Solutions Under the Measure Theory to some Classes of Nonautonomous Partial Evolution Equations

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Abstract. In this paper, we use a recent works [5], where the authors provide a new approach for pseudo almost periodic solution under the measure theory, under Acquistpace-Terreni conditions, we make extensive use of interpolation spaces and exponential dichotomy techniques to obtain the existence of μ-pseudo almost periodic solutions to some classes of nonautonomous partial evolution equations.

AMS Subject Classifications: 34G20; 34B05; 42A75; 47D06; 35L90.

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1 Introduction

In this work, we propose to study the existence of μ-pseudo almost periodic solutions under the measure theory to the class of abstract nonautonomous differential equations

$$\frac{d}{dt}\left[u(t)+f(t,B(t)u(t))\right]=A(t)u(t)+g(t,C(t)u(t)),\ t\in\mathbb{R}, \tag{1.1}$$

where $A(t)$ for $t\in\mathbb{R}$ is a family of closed linear operators on $D(A(t))$ satisfying the well-known Acquistapace-Terreni conditions, $B(t)$, $C(t)$ $(t\in\mathbb{R})$ are families of (possibly unbounded) linear operators, and $f:\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}^t_\beta$, $g:\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}$ are μ-pseudo almost periodic

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in $t\in\mathbb{R}$ uniformly in the second variable. Recall that the concept of μ-pseudo almost periodicity introduced by [5] is a natural generalization of the classical concept of weighted pseudo almost periodicity in the sense of Diagana [12, 13]. In recent paper [11], results on the existence and uniqueness of weighted pseudo almost periodic solutions for equation (1.1) are developed. Classical definition and properties of μ-pseudo almost periodic function solutions introduced in [5] are used.

The organization of this works is as follows. In section 2, we introduce the basic notations and recall the definitions and lemmas of μ-pseudo almost periodic functions introduced in [5], and we introduce the basic notations of evolution family and exponential dichotomy. Some preliminary results on intermediate spaces are also stated there. In Section 3, we study the existence and uniqueness of μ-pseudo almost periodic mild solution of (1.1).

2 Preliminaries

2.1 μ-pseudo almost periodic functions

Let $(\mathbb{X},||.||)$, $(\mathbb{Y},||.||)$ be two Banach spaces, and $BC(\mathbb{R},\mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y},\mathbb{X})$) be the space of bounded continuous functions $f:\mathbb{R}\longrightarrow\mathbb{X}$ (respectively, $f:\mathbb{R}\times\mathbb{Y}\longrightarrow\mathbb{X}$). $BC(\mathbb{R},\mathbb{X})$ equipped with the norm $||f||=\sup_{t\in\mathbb{R}}||f(t)||$ is a Banach space. $B(\mathbb{X},\mathbb{Y})$ denotes the Banach spaces of all bounded linear operator from $\mathbb{X}$ into $\mathbb{Y}$ equipped with natural topology. If $\mathbb{Y}=\mathbb{X}$, $B(\mathbb{X},\mathbb{Y})$ is simply denoted by $B(\mathbb{X})$.

Definition 2.1. ([6, 7]) *A continuous function $f:\mathbb{R}\mapsto\mathbb{X}$ is said to be almost periodic if for every $\epsilon>0$ there exists a positive number l such that every interval of length l contains a number τ such that*

$$\|f(t+\tau)-f(t)\|<\epsilon \ \ for \ \ t\in\mathbb{R}.$$

The set of all almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$ will be denoted by a continuous function $f:\mathbb{R}\times\mathbb{Y}\mapsto\mathbb{X}$ is said to be almost periodic in t uniformly for $y\in\mathbb{Y}$, if for every $\epsilon>0$, and any compact subset K of $\mathbb{Y}$, there exists a positive number l such that every interval of length l contains a number τ such that

$$\|f(t+\tau,y)-f(t,y)\|<\epsilon \ \ for \ \ (t,y)\in\mathbb{R}\times K.$$

We denote the set of such functions $APU(\mathbb{R}\times\mathbb{Y};\mathbb{X})$.

Notice that $(AP(\mathbb{R};\mathbb{X}),\|.\|_\infty)$, is a Banach space with supremum norm given by

$$\|u\|_\infty=\sup_{t\in\mathbb{R}}\|u(t)\|.$$

Next, we give the new concept of the ergodic functions developed in [5], and generalizing the ergodicity given before [12, 13].

We denote by $\mathcal{B}$ the Lebesque σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R})=+\infty$ and $\mu([a,b])<\infty$, for all $a,b\in\mathbb{R}$ $(a\leq b)$.

Definition 2.2. ([1, 5]) *Let $\mu\in\mathcal{M}$. A bounded continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be μ-ergodic if*

$$\lim_{r\longrightarrow+\infty}\frac{1}{\mu([-r,r])}\int_{[-r,r]}\|f(s)\|d\mu(s)=0.$$

We denote the space of all such functions by $\mathcal{E}(\mathbb{R};\mathbb{X},\mu)$.
A continuous function $f:\mathbb{R}\to\mathbb{X}$ is said to be μ-pseudo almost periodic if it is written in the form

$$f=g+h,$$

where $g\in AP(\mathbb{R};\mathbb{X})$ and $h\in\mathcal{E}(\mathbb{R};\mathbb{X},\mu)$. The collection of such functions will be denoted by $PAP(\mathbb{R};\mathbb{X},\mu)$.

It is well known [5] that $(\mathcal{E}(\mathbb{R};\mathbb{X},\mu),\|.\|_\infty)$ is a Banach space. In the sequel as in [5], we need the following assumptions.
(M1) For all a,b and $c\in\mathbb{R}$, such that $0\leq a<b\leq c$, there exist $\tau_0\geq 0$ and $\alpha_0\geq 0$ such that

$$|\tau|\geq\tau_0\Longrightarrow\mu((a+\tau,b+\tau))\geq\alpha_0([\tau,c+\tau]).$$

(M2) For all $\tau\in\mathbb{R}$, there exist $\beta>0$ and a bounded interval I such that

$$\mu(\{a+\tau:\ a\in A\})\leq\beta\mu(A)\quad\text{when } A\in\mathcal{B}\text{ satisfies } A\cap I=\emptyset.$$

It is proved in [5] that whenever $\mu\in\mathcal{M}$ satisfies the assumption (M1), the decomposition of a μ-pseudo almost periodic function in the form $f=g+h$, where $g\in AP(\mathbb{R};\mathbb{X})$ and $h\in\mathcal{E}(\mathbb{R};\mathbb{X},\mu)$, is unique. Furthermore, the space $(PAP(\mathbb{R};\mathbb{X},\mu),\|.\|_\infty)$, is a Banach space. Whenever $\mu\in\mathcal{M}$ satisfies the assumption (M2), $PAP(\mathbb{R};\mathbb{X},\mu)$ is translation invariant, that is $f\in PAP(\mathbb{R};\mathbb{X},\mu)$ implies $f_\tau=f(.+\tau)\in PAP(\mathbb{R};\mathbb{X},\mu)$ for all $\tau\in\mathbb{R}$.

Definition 2.3. ([5]) *Let $\mu\in\mathcal{M}$. A continuous function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ is said to be μ-ergodic in t uniformly with respect to $y\in\mathbb{Y}$ if the following conditions are true.*
(i) For all $y\in\mathbb{Y}$, $f(.,y)\in\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$.
(ii) f is uniformly continuous on each compact set K in $\mathbb{Y}$ with respect to the second variable y. The collection of such functions will be denoted by $\mathcal{E}U(\mathbb{R}\times\mathbb{Y};\mathbb{X},\mu)$.
A continuous function $f:\mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ is said to be uniformly μ-pseudo almost periodic if is written in the form

$$f=g+h,$$

where $g\in APU(\mathbb{R}\times\mathbb{Y};\mathbb{X})$ and $h\in\mathcal{E}U(\mathbb{R}\times\mathbb{Y};\mathbb{X},\mu)$. The collection of such functions denoted by $PAPU(\mathbb{R}\times\mathbb{Y};\mathbb{X},\mu)$.

Theorem 2.1. ([5]) *Let $\mu\in\mathcal{M}$, $F\in PAPU(\mathbb{R}\times\mathbb{Y};\mathbb{X},\mu)$ and $h\in PAP(\mathbb{R};\mathbb{Y},\mu)$. Assume that, for all bounded subset B of $\mathbb{Y}$, F is bounded on $\mathbb{R}\times B$. Then $t\longmapsto F(t,h(t))\in PAP(\mathbb{R};\mathbb{X},\mu)$.*

2.2 Evolution family and exponential dichotomy

Definition 2.4. ([8–10]) *A family of bounded linear operators* $(U(t,s))_{t\geq s}$, *on a Banach space* $\mathbb{X}$ *is called a strongly continuous evolution family if*
(1) $U(t,r)U(r,s)=U(t,s)$ *and* $U(s,s)=I$ *for all* $t\geq r\geq s$ *and* $t,r,s\in\mathbb{R}$,
(2) The map $(t,s)\to U(t,s)x$ *is continuous for all* $x\in\mathbb{X}$, $t\geq s$ *and* $t,s\in\mathbb{R}$.
(3) $U(\cdot,s)\in C^1((s,\infty),B(\mathbb{X}))$, $\dfrac{\partial U}{\partial t}(t,s)=A(t)U(t,s)$ *and*

$$\left\|A(t)^k U(t,s)\right\|\leq K(t-s)^{-k}$$

for $0<t-s\leq 1$, $k=0,1$.
(4) $\partial_s^+ U(t,s)x=-U(t,s)A(s)x$ *for* $t>s$ *and* $x\in D(A(s))$ *with* $A(s)x\in\overline{D(A(s))}$.
$A(t)$ *is as in (1.1).*

Definition 2.5. *An evolution family* $(U(t,s))_{t\geq s}$ *on a Banach space* $\mathbb{X}$ *is called hyperbolic (or has exponential dichotomy) if there exist projections* $P(t),t\in\mathbb{R}$, *uniformly bounded and strongly continuous in* t, *and constants* $M>0$, $\delta>0$ *such that*
(1) $U(t,s)P(s)=P(t)U(t,s)$ *for* $t\geq s$ *and* $t,s\in\mathbb{R}$,
(2) The restriction $U_Q(t,s):Q(s)\mathbb{X}\to Q(t)\mathbb{X}$ *of* $U(t,s)$ *is invertible for* $t\geq s$ *and* $t,s\in\mathbb{R}$ *(and we set* $U_Q(t,s)=U(s,t)^{-1}$*).*
(3)

$$\|U(t,s)P(s)\|\leq Ne^{-\delta(t-s)} \tag{2.1}$$

$$\|U_Q(s,t)Q(t)\|\leq Ne^{-\delta(t-s)} \tag{2.2}$$

for $t\geq s$ *and* $t,s\in\mathbb{R}$.
Here and below we set $Q:=I-P$.

To introduce the inter and extrapolation spaces for $A(t)$, we need the following assumptions.

(H0) The family of closed linear operators $A(t)$ for $t\in\mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace-Terreni conditions, that is, there exist constants $\omega\in\mathbb{R}$, $\theta\in(\frac{\pi}{2},\pi)$, $K,L\geq 0$ and $\mu,\nu\in(0,1]$ with $\mu+\nu>1$ such that

$$\Sigma_\theta\cup\{0\}\subset\rho\Big(A(t)-\omega\Big)\ni\lambda,\quad \Big\|R(\lambda,A(t)-\omega)\Big\|\leq\frac{K}{1+|\lambda|} \tag{2.3}$$

and

$$\Big\|\Big(A(t)-\omega\Big)R\Big(\lambda,A(t)-\omega\Big)\Big[R\Big(\omega,A(t)\Big)-R\Big(\omega,A(s)\Big)\Big]\Big\|\leq L\frac{|t-s|^\mu}{|\lambda|^\nu} \tag{2.4}$$

for $t,s\in\mathbb{R}$, $\lambda\in\Sigma_\theta:=\{\lambda\in\mathbb{C}\setminus\{0\}:|\arg\lambda|\leq\theta\}$.

Note that in the particular case when $A(t)$ has a constant domain $D = D(A(t))$, it is well-known [2] that equation (2.4) can be replaced with the following: There exist constants L and $0 < \gamma \leq 1$ such that

$$\|(A(t) - A(s))R(\omega, A(r)))\| \leq L|t-s|^{\gamma}, s,t,r \in \mathbb{R}. \tag{2.5}$$

2.3 Interpolation Spaces

This setting requires some estimates related to $U(t,s)$. For that, we make extensive use of the real interpolation spaces of order (α,∞) between $\mathbb{X}$ and $D(A(t))$, where $\alpha \in (0,1)$. We refer the reader to [2–4] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{X}$ (assumption (H0) holds when $A(t)$ is replaced with A) and let $\alpha \in (0,1)$. Define the new norm on $D(A)$ (the real interpolation space) by

$$\mathbb{X}_{\alpha}^{A} = \{x \in \mathbb{X},\ \|x\|_{\alpha}^{A} := \sup_{r>0} \|r^{\alpha}(A-\omega)R(r, A-\omega)x\| < \infty\},$$

and which consider the continuous interpolations spaces $\mathbb{X}_{\alpha}^{A}$ by the way, is a Banach space when endowed with the norm $\|\cdot\|_{\alpha}^{A}$. For convenience we further write

$$\mathbb{X}_{0}^{A} := \mathbb{X},\ \|x\|_{0}^{A} := \|x\|,\ \mathbb{X}_{1}^{A} := D(A)$$

and $\|x\|_{1}^{A} := \|(\omega - A)x\|$. Moreover, let $\hat{\mathbb{X}}^{A} := \overline{D(A)}$ of $\mathbb{X}$. In particular, we will frequently use the following continuous embedding.

$$D(A) \hookrightarrow \mathbb{X}_{\beta}^{A} \hookrightarrow D((\omega - A)^{\alpha}) \hookrightarrow \mathbb{X}_{\alpha}^{A} \hookrightarrow \hat{\mathbb{X}}^{A} \subset \mathbb{X}, \tag{2.6}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_{\alpha}^{A}$ and $\mathbb{X}$. However, we have the following continuous injection

$$\mathbb{X}_{\beta}^{A} \hookrightarrow \overline{D(A)}^{\|\cdot\|_{\alpha}^{A}} \tag{2.7}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (H1), we set

$$\mathbb{X}_{\alpha}^{t} := \mathbb{X}_{\alpha}^{A(t)}, \quad \hat{\mathbb{X}}^{t} := \hat{\mathbb{X}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in (2.7) hold with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_{\alpha}$ [[4], Definition 1.1.1] and hence there is a constant $c(\alpha)$ such that

$$\|y\|_{\alpha}^{t} \leq c(\alpha)\|y\|^{1-\alpha}\|A(t)y\|^{\alpha},\ y \in D(A(t)). \tag{2.8}$$

We have the following fundamental estimates for the evolution family $U(t,s)$.

Proposition 2.1. ([11]) *For $x\in\mathbb{X}$, $0\le\alpha\le 1$ and $t>s$, the following assertions hold.*

(i) There is a constant $c(\alpha)$, such that

$$\|U(t,s)P(s)x\|_\alpha^t \le c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \tag{2.9}$$

(ii) There is a constant $m(\alpha)$, such that

$$\|U_Q(s,t)Q(t)x\|_\alpha^s \le m(\alpha)e^{-\delta(t-s)}\|x\|. \tag{2.10}$$

3 Main results

To study the existence and uniqueness of μ-pseudo almost periodic solutions of equation (1.1) we need the following additional assumptions.

(H1) The evolution family $(U(t,s))_{t\ge s}$ generated by $A(t)$ has an exponential dichotomy with constants $N>0$, $\delta>0$, dichotomy projections $P(t)$, $t\in\mathbb{R}$.

(H2) $R(\omega,A(\cdot))\in AP(B(\mathbb{X}_\alpha))$. Moreover, there exists a function $H:[0,\infty)\mapsto[0,\infty)$ with $H\in L^1[0,\infty)$ such that for every $\varepsilon>0$ there exists $l(\varepsilon)$ such that every interval of length $l(\varepsilon)$ contains a τ with the property

$$\|A(t+\tau)U(t+\tau,s+\tau)-A(t)U(t,s)\|_{B(\mathbb{X},\mathbb{X}_\alpha)}\le\varepsilon H(t-s)$$

for all $t,s\in\mathbb{R}$ with $t>s$.

(H3) There exists $0\le\alpha<\beta<1$ such that

$$\mathbb{X}_\alpha^t=\mathbb{X}_\alpha,\quad \mathbb{X}_\beta^t=\mathbb{X}_\beta$$

for all $t\in\mathbb{R}$, with uniform equivalent norm
If $0\le\alpha<\beta<1$, then we let $k(\alpha)$ denote the bound of the embedding $\mathbb{X}_\beta\hookrightarrow\mathbb{X}_\alpha$, that is

$$\|u\|_\alpha\le k(\alpha)\|u\|_\beta$$

for each $u\in\mathbb{X}_\beta$

(H4) Let $\mu\in\mathcal{M}$ and let $0<\alpha<\beta<1$. We suppose $f:\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}_\beta$ belongs to $PAP(\mathbb{X},\mathbb{X}_\beta,\mu)$ and satisfy
i) For all bounded subset B of $\mathbb{X}$, f is bounded on $\mathbb{R}\times B$.
ii) There exists $K_f>0$ such that

$$\|f(t,u)-f(t,v)\|_\beta\le K_f\|u-v\|,$$

for all $u,v\in\mathbb{X}$ and $t\in\mathbb{R}$.

(H5) Let $\mu\in\mathcal{M}$ and let $0<\alpha<\beta<1$. We suppose $g:\mathbb{R}\times\mathbb{X}\mapsto\mathbb{X}$ belongs to $PAP(\mathbb{X},\mathbb{X},\mu)$ and satisfy
i) For all bounded subset B of $\mathbb{X}$, g is bounded on $\mathbb{R}\times B$.
ii) There exists $K_g>0$ such that

$$\|g(t,u)-g(t,v)\|_\beta\leq K_g\|u-v\|,$$

for all $u,v\in\mathbb{X}$ and $t\in\mathbb{R}$.

(H6) We suppose that the linear operators $B(t),C(t):\mathbb{X}_\alpha\mapsto\mathbb{X}$ for all $t\in\mathbb{R}$, are bounded and set

$$\varpi:=\max\left(\sup_{t\in\mathbb{R}}\|B(t)\|_{B(\mathbb{X}_\alpha,\mathbb{X})},\sup_{t\in\mathbb{R}}\|C(t)\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\right).$$

Furthermore, $t\mapsto B(t)u$ and $t\mapsto C(t)u$ are almost periodic for each $u\in\mathbb{X}_\alpha$.

To study the existence and uniqueness of pseudo almost periodic solutions to equation (1.1), we first introduce the notion of mild solution.

Definition 3.1. *A function $u:\mathbb{R}\mapsto\mathbb{X}_\alpha$ is said to be a mild solution to equation* (1.1) *provided that the function $s\to A(s)U(t,s)P(s)f(s,B(s)u(s))$ is integrable on (s,t), $s\to A(s)U(t,s)Q(s)f(s,B(s)u(s))$ is integrable on (t,s) and*

$$\begin{aligned}
u(t)&=-f(t,B(t)u(t))+U(t,s)\Big(u(s)+f(s,B(s)u(s))\Big)\\
&-\int_s^t A(s)U(t,s)P(s)f(s,B(s)u(s))ds+\int_t^s A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds\\
&+\int_s^t U(t,s)P(s)g(s,C(s)u(s))ds-\int_t^s U_Q(t,s)Q(s)g(s,C(s)u(s))ds,
\end{aligned}\tag{3.1}$$

for $t\geq s$ and for all $t,s\in\mathbb{R}$.

In a first step, we proved the following result.

Theorem 3.1. *Assume that assumptions* (H0)-(H1) *hold and let u be a mild solution of* (1.1) *on* $\mathbb{R}$. *Then, for all $t\in\mathbb{R}$*

$$\begin{aligned}
u(t)=&-f(t,B(t)u(t))-\int_{-\infty}^t A(s)U(t,s)P(s)f(s,B(s)u(s))ds\\
&+\int_t^\infty A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds\\
&+\int_{-\infty}^t U(t,s)P(s)g(s,C(s)u(s))ds\\
&-\int_t^\infty U_Q(t,s)Q(s)g(s,C(s)u(s))ds.
\end{aligned}$$

Proof. Let u be the mild solution of (1.1) on $\mathbb{R}$. For all $t\geq s$ and all $s\in\mathbb{R}$, we have

$$\begin{aligned}u(s)=&-f(s,B(s)u(s))-\int_{-\infty}^{s}A(\sigma)U(s,\sigma)P(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma\\&+\int_{s}^{\infty}A(\sigma)U_Q(s,\sigma)Q(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma\\&+\int_{-\infty}^{s}U(s,\sigma)P(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma\\&-\int_{s}^{\infty}U_Q(s,\sigma)Q(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma.\end{aligned}$$

Multiply both sides of the equality by $U(t,s)$, we get

$$\begin{aligned}&U(t,s)u(s)=-U(t,s)f(s,B(s)u(s))\\&-\int_{-\infty}^{s}A(\sigma)U(t,\sigma)P(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma+\int_{s}^{\infty}A(\sigma)U_Q(t,\sigma)Q(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma\\&+\int_{-\infty}^{s}U(t,\sigma)P(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma-\int_{s}^{\infty}U_Q(t,\sigma)Q(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma\\&=-U(t,s)f(s,B(s)u(s))\\&-\int_{-\infty}^{t}A(\sigma)U(t,\sigma)P(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma+\int_{s}^{t}A(\sigma)U(t,\sigma)P(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma\\&+\int_{t}^{\infty}A(\sigma)U_Q(t,\sigma)Q(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma+\int_{s}^{t}A(\sigma)U_Q(t,\sigma)Q(\sigma)f(\sigma,B(\sigma)u(\sigma))d\sigma\\&+\int_{-\infty}^{t}U(t,\sigma)P(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma-\int_{s}^{t}U(t,\sigma)P(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma\\&-\int_{t}^{\infty}U_Q(t,\sigma)Q(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma-\int_{s}^{t}U_Q(t,\sigma)Q(\sigma)g(\sigma,C(\sigma)u(\sigma))d\sigma\\&=-U(t,s)f(s,B(s)u(s))+u(t)+f(t,B(t)u(t))\\&+\int_{s}^{t}A(s)U(t,s)P(s)f(s,B(s)u(s))ds-\int_{t}^{s}A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds\\&-\int_{s}^{t}U(t,s)P(s)g(s,C(s)u(s))ds+\int_{t}^{s}U_Q(t,s)Q(s)g(s,C(s)u(s))ds,\end{aligned}$$

Hence u is a mild solution of equation (1.1). □

Throughout the rest of the paper we denote by $\Gamma_1,\Gamma_2,\Gamma_3$ and Γ_4, the nonlinear integral operators defined by

$$(\Gamma_1 u)(t)=\int_{-\infty}^{t}A(s)U(t,s)P(s)f(s,B(s)u(s))ds,$$

$$(\Gamma_2 u)(t)=\int_{t}^{\infty}A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds,$$

$$(\Gamma_3 u)(t) = \int_{-\infty}^{t} U(t,s)P(s)g(s,C(s)u(s))ds$$

and

$$(\Gamma_4 u)(t) := \int_{t}^{\infty} U_Q(t,s)Q(s)g(s,C(s)u(s))ds.$$

We next ned the following preliminary technical results.

Lemma 3.1. *Let* $\mu \in \mathcal{M}$ *satisfying* (M1)-(M2) *and* $u \in PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$, *if the linear operators* $C(.)$ *satisfy* (H6) *then* $C(.)u(.) \in PAP(\mathbb{R},\mathbb{X},\mu)$.

Proof. Let $u \in PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$ then $u = u_1 + u_2$ where $u_1 \in AP(\mathbb{R},\mathbb{X}_\alpha)$ and $u_2 \in \mathcal{E}(\mathbb{R},\mathbb{X}_\alpha,\mu)$. We have, $C(t)u(t) = C(t)u_1(t) + C(t)u_2(t)$ for all $t \in \mathbb{R}$. Since $u_1 \in AP(\mathbb{R},\mathbb{X}_\alpha)$, for every $\epsilon > 0$ there exists l_ϵ such that every interval of length l_ϵ contains a τ such that

$$\Big\| u_1(t+\tau) - u_1(t) \Big\|_\alpha < \frac{\epsilon}{\Big(\sup_{t\in\mathbb{R}} \|u_1(t)\|_\alpha + \varpi\Big)}, \quad t \in \mathbb{R}.$$

Similarly, since $C(t) \in AP(B(\mathbb{X}_\alpha,\mathbb{X}))$, we have

$$\Big\| C(t+\tau) - C(t) \Big\|_{B(\mathbb{X}_\alpha,\mathbb{X})} < \frac{\epsilon}{\Big(\sup_{t\in\mathbb{R}} \|u_1(t)\|_\alpha + \varpi\Big)}, \quad t \in \mathbb{R}.$$

Now

$$\begin{aligned}
&\|C(t+\tau)u_1(t+\tau) - C(t)u_1(t)\| \\
&\leq \| \Big[C(t+\tau) - C(t)\Big] u_1(t+\tau)\| + \|C(t)\Big[u_1(t+\tau) - u_1(t)\Big]\| \\
&\leq \|C(t+\tau) - C(t)\|_{B(\mathbb{X}_\alpha,\mathbb{X})} \|u_1(t+\tau)\|_\alpha + \varpi \|u_1(t+\tau) - u_1(t)\|_\alpha \\
&\leq \epsilon,
\end{aligned}$$

and hence $t \mapsto C(t)u_1(t)$ belongs to $AP(\mathbb{R},\mathbb{X})$.
To complete the proof, it suffices to prove that $t \mapsto C(t)u_2(t)$ belongs to $\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$. Indeed, we have

$$\frac{1}{\mu([-r,r])} \int_{-r}^{r} \Big\| C(t)u_2(t) \Big\| d\mu(t) \leq \frac{\varpi}{\mu([-r,r])} \int_{-r}^{r} \|u_2(t)\|_\alpha d\mu(t)$$

and hence

$$\lim_{r\to\infty} \frac{1}{\mu([-r,r])} \int_{-r}^{r} \Big\| C(t)u_2(t) \Big\| d\mu(t) = 0.$$

□

Lemma 3.2. ([11]) *Assume that assumptions* (H0)-(H1)and (H3 *hold and let* $0 \leq \theta < \alpha < \beta < 1$ *with* $2\alpha > \theta + 1$. *Then, there exist two constants* $m(\alpha,\beta), n(\alpha,\theta) > 0$ *such that*

$$\|A(s)U_Q(t,s)Q(s)x\|_\alpha \leq m(\alpha,\beta) e^{\delta(s-t)} \|x\|_\beta \qquad \text{for } t \leq s, \tag{3.2}$$

and

$$\|A(s)U(t,s)P(s)x\|_\alpha \leq n(\alpha,\theta)(t-s)^{-\alpha} e^{-\frac{\delta}{4}(t-s)} \|x\|_\beta, \qquad \text{for } t > s. \tag{3.3}$$

Lemma 3.3. *Let assumptions* (H0)-(H4) and (H6) *hold, then the integral operators* Γ_1 *and* Γ_2 *defined above map* $PAP(\mathbb{X}_\alpha,\mu)$ *into itself.*

Proof. Let $u\in PAP(\mathbb{X}_\alpha,\mu)$. From Lemma 3.1 it follows that the function $t\mapsto B(t)u(t)$ belongs to $PAP(\mathbb{X})$. Using assumption (H4) and Theorem 2.1 it follows that $\psi(\cdot)=f(\cdot,Bu(\cdot))$ is in $PAP(\mathbb{X}_\beta,\mu)$ whenever $u\in PAP(\mathbb{X}_\alpha,\mu)$. In particular,

$$\|\psi\|_{\infty,\beta}=\sup_{t\in\mathbb{R}}\|f(t,Bu(t))\|_\beta<\infty.$$

Since $\psi(\cdot)=f(\cdot,Bu(\cdot))$ is in $PAP(\mathbb{X}_\beta,\mu)$ then $\psi=\phi_1+\phi_2$, where $\phi_1\in AP(\mathbb{R},\mathbb{X}_\beta)$ and $\phi_2\in\mathcal{E}(\mathbb{R},\mathbb{X}_\beta,\mu)$, that is, $\Gamma_1\psi=\Xi(\phi_1)+\Xi(\phi_2)$ where

$$\Xi\phi_1(t):=\int_{-\infty}^t A(s)U(t,s)P(s)\phi_1(s)ds$$

and

$$\Xi\phi_2(t):=\int_{-\infty}^t A(s)U(t,s)P(s)\phi_2(s)ds.$$

Firstly, we show that $\Xi\phi_1\in BC(\mathbb{R},\mathbb{X}_\beta)$. Indeed, using estimate (3.3), we obtain

$$\begin{aligned}\|\,\Xi\phi_1(t)\,\|_\beta&\le\int_{-\infty}^t\|A(s)U(t,s)P(s)\phi_1(s)\|ds\\&\le n(\alpha,\theta)\int_{-\infty}^t(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|\phi_1\|_\beta ds\\&\le n(\alpha,\theta)(\frac{\delta}{4})^{1-\alpha}\Gamma(1-\alpha)\|\phi_1\|_\beta\end{aligned}$$

Then $\Xi\phi_1\in BC(\mathbb{R},\mathbb{X}_\beta)$. Next, we prove that $\Xi(\phi_1)\in AP(\mathbb{R},\mathbb{X}_\alpha)$. Since $\phi_1\in AP(\mathbb{R},\mathbb{X}_\beta)$, then for every $\epsilon>0$ there exists $l(\epsilon)>0$ such that every interval of length $l(\epsilon)$ contains a τ with the property

$$\|\phi_1(t+\tau)-\phi_1(t)\|_\beta<\epsilon\nu\ \text{ for each } t\in\mathbb{R}$$

where $\nu=\dfrac{4^{\alpha-1}\delta^{1-\alpha}}{n(\alpha,\theta)\Gamma(1-\alpha)}$. Hence,

$$\begin{aligned}&\Xi\phi_1(t+\tau)-\Xi\phi_1(t)\\&=\int_{-\infty}^{t+\tau}A(s)U(t+\tau,s)P(s)\phi_1(s)ds-\int_{-\infty}^{t}A(s)U(t,s)P(s)\phi_1(s)ds\\&=\int_{-\infty}^{t}A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)\Big(\phi_1(s+\tau)-\phi_1(s)\Big)ds\\&+\int_{-\infty}^{t}\Big(A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)-A(s)U(t,s)P(s)\Big)\phi_1(s)ds.\end{aligned}$$

Using equation (3.3) it follows that

$$\|\int_{-\infty}^{t} A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)\Big(\phi_1(s+\tau)-\phi_1(s)\Big)ds\|_\alpha \le \epsilon.$$

Similarly, using assumption (H2), it follows that

$$\begin{aligned}\|\int_{-\infty}^{t} &(A(s+\tau)U(t+\tau,s+\tau)P(s+\tau)\\ &-A(s)U(t,s)P(s))\phi_1(s)ds\|_\alpha \le \epsilon N\|H\|_{L^1}\|\phi_1\|_\infty\end{aligned}$$

where $\|H\|_{L^1}=\int_0^\infty H(s)ds<\infty$. Therefore,

$$\|\Xi(\phi_1)(t+\tau)-\Xi(\phi_1)(t)\|_\alpha \le \Big(1+N\|H\|_{L^1}\|\phi_1\|_\infty\Big)\epsilon$$

for each $t\in\mathbb{R}$, and hence $\Xi(\phi_1)\in AP(\mathbb{R},\mathbb{X}_\alpha)$.
Now, we show that $\Xi(\phi_2)\in BC(\mathbb{R},\mathbb{X}_\beta)$. Using estimate (3.3) and replacing $\Xi(\phi_1)$ by $\Xi(\phi_2)$ in the previous case we get the result. To complete the proof, we will prove that $\Xi(\phi_2)\in \mathcal{E}(\mathbb{R},\mathbb{X}_\beta,\mu)$. Now, let $r>0$. Again from equation (3.3), we have

$$\begin{aligned}&\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|(\Xi\phi_2)(t)\|_\alpha d\mu(t)\\ &\le \frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_0^{+\infty}\|A(t-s)U(t,t-s)P(t-s)\phi_2(t-s)\|_\alpha ds d\mu(t)\\ &\le \frac{n(\alpha,\theta)}{\mu([-r,r])}\int_{-r}^{r}\int_0^{+\infty} s^{-\alpha}e^{-\frac{\delta}{4}s}\|\phi_2(t-s)\|_\beta ds d\mu(t)\\ &\le n(\alpha,\theta).\int_0^{+\infty} s^{-\alpha}e^{-\frac{\delta}{4}s}\left(\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|\phi_2(t-s)\|_\beta d\mu(t)\right)ds.\end{aligned}$$

Now

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|\phi_2(t-s)\|_\beta d\mu(t)=0,$$

Since μ satisfy (M2) then $t\mapsto\phi_2(t-s)\in\mathcal{E}(\mathbb{R},\mathbb{X}_\beta,\mu)$ for every $s\in\mathbb{R}$. To complete the proof, we use the well known Lebesgue's dominated convergence theorem.

The proof for $\Gamma_2 u(\cdot)$ is similar to that of $\Gamma_1 u(\cdot)$ except that one makes use of equation (3.2) instead of equation (3.3). □

Lemma 3.4. *Let $\mu\in\mathcal{M}$ satisfying* (M1) and (M2). *Assume further that* (H0)-(H3), (H5) and (H6) *hold, then the integral operators Γ_3 and Γ_4 defined above map $PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$ into itself.*

Proof. Let $u\in PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$. From Lemma 3.1 we get $C(\cdot)u(\cdot)\in PAP(\mathbb{R},\mathbb{X},\mu)$. Let $h(t)=g(t,Cu(t))$. Using assumption (H5 and Theorem 2.1 it follows that $h\in PAP(\mathbb{R},\mathbb{X},\mu)$. Now write $h=\psi_1+\psi_2$ where $\psi_1\in AP(\mathbb{R},\mathbb{X})$ and $\psi_2\in\mathcal{E}(\mathbb{R},\mathbb{X},\mu)$, that is, $\Gamma_3 h=\Xi(\psi_1)+\Xi(\psi_2)$ where

$$\Xi\psi_1(t):=\int_{-\infty}^t U(t,s)P(s)\psi_1(s)ds$$

and

$$\Xi\psi_2(t):=\int_{-\infty}^t U(t,s)P(s)\psi_2(s)ds.$$

Firstly, we show that $\Xi\psi_1\in BC(\mathbb{R},\mathbb{X}_\beta)$. Indeed, using estimate (2.9), we obtain

$$\begin{aligned}\|\Xi\psi_1(t)\|_\beta&\le\int_{-\infty}^t\|U(t,s)P(s)\psi_1(s)\|ds\\&\le c(\alpha)\int_{-\infty}^t(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|\psi_1\|_\beta ds\\&\le c(\alpha)(\frac{\delta}{2})^{1-\alpha}\Gamma(1-\alpha)\|\psi_1\|_\beta.\end{aligned}$$

Then $\Xi\psi_1\in BC(\mathbb{R},\mathbb{X}_\beta)$. Next, we prove that $\Xi(\psi_1)\in AP(\mathbb{R},\mathbb{X}_\alpha)$. Since $\psi_1\in AP(\mathbb{R},\mathbb{X}_\beta)$, then for every $\epsilon>0$ there exists $l(\epsilon)>0$ such that every interval of length $l(\epsilon)$ contains a τ with the property

$$\|\psi_1(t+\tau)-\psi_1(t)\|_\beta<\epsilon\eta\quad\text{for each } t\in\mathbb{R},$$

where $\eta=\dfrac{1}{2^\alpha\delta^{1-\alpha}c(\alpha)\Gamma(1-\alpha)}$. Hence,

$$\begin{aligned}&\Xi\psi_1(t+\tau)-\Xi\psi_1(t)\\&=\int_{-\infty}^{t+\tau}U(t+\tau,s)P(s)\psi_1(s)ds-\int_{-\infty}^t U(t,s)P(s)\psi_1(s)ds\\&=\int_{-\infty}^t U(t+\tau,s+\tau)P(s+\tau)\Big(\psi_1(s+\tau)-\psi_1(s)\Big)ds\\&\quad+\int_{-\infty}^t\Big(U(t+\tau,s+\tau)P(s+\tau)-U(t,s)P(s)\Big)\psi_1(s)ds.\end{aligned}$$

Using equation (2.9) it follows that

$$\|\int_{-\infty}^t U(t+\tau,s+\tau)P(s+\tau)\Big(\phi_1(s+\tau)-\phi_1(s)\Big)ds\|_\alpha\le\frac{\epsilon}{2}.$$

Similarly, using assumption (H2). Let $\epsilon>0$, from [10] we know that $r\to\Gamma(t+r,s+r)\in AP(\mathbb{B}(\mathbb{X}))$ for $t,s\in\mathbb{R}$, where we may take the same almost periods for t,s with $\|t-s\|\le h>0$. Hence, there exists $l(\epsilon)>0$ such that every interval of length $l(\epsilon)$ contains a number $\tau>0$ with the properties that, for $t\in\mathbb{R}$, $\sigma>0$:

$$\|U(t+\tau,s+\tau)P(s+\tau)-U(t,s)P(s)\|\le\frac{\epsilon}{2\|\psi_1\|_\beta}$$

and

$$\|U_Q(t+\tau,s+\tau)P(s+\tau)-U_Q(t,s)P(s)\|\leq \frac{\epsilon}{2\|\psi_1\|_\beta}.$$

Therefore,

$$\|\Xi(\psi_1)(t+\tau)-\Xi(\psi_1)(t)\|_\alpha\leq\epsilon,$$

for each $t\in\mathbb{R}$, and hence $\Xi(\psi_1)\in AP(\mathbb{R},\mathbb{X}_\alpha)$. Next, using similar techniques as previously, we get $\Xi(\psi_2)\in BC(\mathbb{R},\mathbb{X}_\beta)$. In fact, using estimate (2.10) and replacing $\Xi(\psi_1)$ by $\Xi(\psi_2)$ we get the result. Now, to complete the proof, we will prove that $\Xi(\psi_2)\in\mathcal{E}(\mathbb{R},\mathbb{X}_\beta,\mu)$. Let $r>0$. Again from equation (2.10), we have

$$\begin{aligned}
&\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|(\Xi\psi_2)(t)\|_\alpha d\mu(t)\\
&\leq\frac{1}{\mu([-r,r])}\int_{-r}^{r}\int_0^{+\infty}\|U(t,t-s)P(t-s)\psi_2(t-s)\|_\alpha ds d\mu(t)\\
&\leq\frac{c(\alpha)}{\mu([-r,r])}\int_{-r}^{r}\int_0^{+\infty}s^{-\alpha}e^{-\frac{\delta}{2}s}\|\psi_2(t-s)\|_\beta ds d\mu(t)\\
&\leq c(\alpha).\int_0^{+\infty}s^{-\alpha}e^{-\frac{\delta}{2}s}\left(\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|\psi_2(t-s)\|_\beta d\mu(t)\right)ds.
\end{aligned}$$

Now observe that

$$\lim_{r\to\infty}\frac{1}{\mu([-r,r])}\int_{-r}^{r}\|\psi_2(t-s)\|_\beta d\mu(t)=0.$$

Since μ satisfy (M2) then $t\mapsto\psi_2(t-s)\in\mathcal{E}(\mathbb{R},\mathbb{X}_\beta,\mu)$ for every $s\in\mathbb{R}$. Finally the proof is acheived using the as well the Lebesgue's dominated convergence theorem.

The proof for $\Gamma_4 u(\cdot)$ is similar to that of $\Gamma_3 u(\cdot)$ except that one makes use of equation (2.9) instead of equation (2.10). □

Now, we are able to state our second main result.

Theorem 3.2. *Let $\mu\in\mathcal{M}$ satisfying* (M1) and (M2). *Assume further that assumptions* (H0)-(H6) *hold and that $\kappa<1$. Then, the equation* (1.1) *has a unique μ-pseudo almost periodic mild solution, where*

$$\kappa=K_g\varpi\Big[\delta^{-1}m(\alpha)+c(\alpha)2^{1-\alpha}\delta^{\alpha-1}\Gamma(1-\alpha)\Big]
+K_f\varpi\Big[1+\delta^{-1}m(\alpha,\beta)+4^{1-\alpha}\delta^{\alpha-1}n(\alpha,\theta)\Gamma(1-\alpha)\Big].$$

Proof. Consider the nonlinear operator $\mathbb{M}$ defined on $PAP(\mathbb{X}_\alpha,\mu)$ by

$$\begin{aligned}\mathbb{M}u(t)= &-f(t,B(t)u(t))-\int_{-\infty}^{t} A(s)U(t,s)P(s)f(s,B(s)u(s))ds\\ &+\int_{t}^{\infty} A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds\\ &+\int_{-\infty}^{t} U(t,s)P(s)g(s,C(s)u(s))ds\\ &-\int_{t}^{\infty} U_Q(t,s)Q(s)g(s,C(s)u(s))ds\end{aligned}$$

for all $t\in\mathbb{R}$. Next, in view of Lemma 3.4 and Lemma 3.3, it follows that $\mathbb{M}$ maps $PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$ into itself. To complete the proof one has to show that $\mathbb{M}$ is a contractive on $PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$. Let $u,v\in PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$. Firstly, we have

$$\begin{aligned}&\|\Gamma_1(v)(t)-\Gamma_1(u)(t)\|_\alpha\\ \leq\ & n(\alpha,\theta)K_f\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|B(s)v(s)-B(s)u(s)\|ds\\ \leq\ & n(\alpha,\theta)K_f\varpi\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|v(s)-u(s)\|_\alpha ds\\ \leq\ & n(\alpha,\theta)K_f\varpi\|v-u\|_{\infty,\alpha}\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}ds\\ =\ & 4^{1-\alpha}\delta^{\alpha-1}n(\alpha,\theta)\Gamma(1-\alpha)K_f\varpi\|v-u\|_{\infty,\alpha}.\end{aligned}$$

Next, we have

$$\begin{aligned}&\|\Gamma_2(v)(t)-\Gamma_2(u)(t)\|_\alpha\\ \leq\ & m(\alpha,\beta)\int_{t}^{\infty}\|f(s,B(s)v(s))-f(s,B(s)u(s))\|_\beta ds\\ \leq\ & m(\alpha,\beta)K_f\varpi\int_{t}^{+\infty}e^{\delta(t-s)}\|B(s)v(s)-B(s)u(s)\|ds\\ \leq\ & m(\alpha,\beta)K_f\varpi\int_{t}^{+\infty}e^{\delta(t-s)}\|v(s)-u(s)\|_\alpha ds\\ \leq\ & m(\alpha,\beta)K_f\varpi\|v-u\|_{\infty,\alpha}\int_{t}^{+\infty}e^{\delta(t-s)}ds\\ =\ & \delta^{-1}m(\alpha,\beta)K_f\varpi\|v-u\|_{\infty,\alpha}.\end{aligned}$$

Now, we have

$$\begin{aligned}\|\Gamma_3(v)(t)-\Gamma_3(u)(t)\|_\alpha &\leq \int_{-\infty}^{t}\|U(t,s)P(s)[g(s,C(s)v(s))-g(s,C(s)u(s))]\|_\alpha ds\\ &\leq Kc(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|C(s)v(s)-C(s)u(s)\|ds\\ &\leq \varpi K_g c(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|v(s)-u(s)\|_\alpha ds\\ &\leq K_g\varpi c(\alpha)2^{1-\alpha}\delta^{\alpha-1}\Gamma(1-\alpha)\|v-u\|_{\infty,\alpha},\end{aligned}$$

Finally, we have

$$\begin{aligned}\|\Gamma_4(v)(t)-\Gamma_4(u)(t)\|_\alpha &\leq \int_{t}^{\infty}m(\alpha)e^{\delta(t-s)}\|g(s,C(s)v(s))-g(s,C(s)u(s))\|ds\\ &\leq \int_{t}^{\infty}m(\alpha)K_g e^{\delta(t-s)}\|C(s)v(s)-C(s)u(s)\|ds\\ &\leq \varpi m(\alpha)K_g\int_{t}^{\infty}e^{\delta(t-s)}\|v(s)-u(s)\|_\alpha ds\\ &\leq K_g m(\alpha)\varpi\|v-u\|_{\infty,\alpha}\int_{t}^{+\infty}e^{\delta(t-s)}ds\\ &\leq K_g\delta^{-1}\varpi m(\alpha)\|v-u\|_{\infty,\alpha}.\end{aligned}$$

Combining previous approximations it follows that

$$\|\mathbb{M}v-\mathbb{M}u\|_{\infty,\alpha}\leq\kappa\|v-u\|_{\infty,\alpha}.$$

Then $\mathbb{M}$ is a contraction map on $PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$. Therefore, $\mathbb{M}$ has unique fixed point in $PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$, that is, there exist unique $u\in PAP(\mathbb{R},\mathbb{X}_\alpha,\mu)$ such that $\mathbb{M}u=u$. Therefore, (1.1), has a unique μ-pseudo almost periodic mild solution. □

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