JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **31**, No. 3, pp. 214-223

doi: 10.4208/jpde.v31.n3.2
September 2018

A Boundary Value Problem for the Degenerated Elliptic Equation with Singular Coefficient and Spectral Parameter

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Received 1 May 2017; Accepted 10 February 2018

Abstract. In this paper, the Dirichlet problem in the vertical half-band for the degenerated elliptic equation with singular coefficient and spectral parameter is investigated. Using the Hankel transformation and the method of separation of variables, the solution of the Dirichlet problem is presented in the explicit form.

AMS Subject Classifications: 35J25, 35J70, 35J75

Chinese Library Classifications: O175.27

Key Words: Half-band; Hankel transformation; series; the Fourier method; uniqueness; existence.

1 Introduction and formulation of the problem

Boundary value problems in various unbounded domains for the Gellerstedt equation were studied in [1,2]. The Dirichlet problem for the degenerated elliptic equation

$$u_{xx}+u_{yy}+\frac{k}{y}u_y=0,$$

where $k\in R$ was considered in [3]. Boundary value problems for the elliptic equations were studied in works [4,5]. Singular partial differential equations appear at studying various problems of aerodynamics and gas dynamics [6] and irrigation problems [7]. A series of interesting results,devoted to studying boundary-value problems for partial differential equations were obtained in works [8–11].

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We consider the equation

$$y^m u_{xx}+u_{yy}+\frac{\beta_0}{y}u_y-\lambda^2 y^m u=0, \tag{1.1}$$

where $m>0$, $-\frac{m}{2}<\beta_0<1$, $\lambda\in R$, in the vertical half-band $D=\{(x,y):0<x<1,y>0\}$. Let $O(0,0),B(1,0)$, $J_0=\{(x,y):x=0,y>0\}$, $J_1=\{(x,y):x=1,y>0\}$. Let $\overline{D}=D\cup\bar{J}_0\cup\overline{OB}\cup\bar{J}_1$.

Problem D. Find a function $U(x,y)$, with the following properties:
1. $U(x,y)\in C(\overline{D})\cap C^2(D)$ and satisfies the equation (1.1) in the domain D;
2. it satisfies the relation

$$\lim_{y\to\infty}U(x,y)=0 \text{ uniformly with respect to } x\in[0,1]; \tag{1.2}$$

3. it satisfies the boundary conditions

$$U(0,y)=\varphi_1(y), \qquad 0\le y<\infty; \tag{1.3}$$
$$U(1,y)=\varphi_2(y), \qquad 0\le y<\infty; \tag{1.4}$$
$$U(x,0)=\tau(x), \qquad 0\le x\le 1, \tag{1.5}$$

where $\varphi_1(y)$, $\varphi_2(y)$, $\tau(x)$ are giving functions, moreover, $\varphi_1(0)=\varphi_2(0)=\tau(0)=\tau(1)=0$.

2 Uniqueness of the solution for the problem

Theorem 2.1. *The solution of the Dirichlet problem for equation* (1.1) *is unique.*

Proof. Let D_h be a finite domain, intercepted from the domain D by the straight line $y=h$, $h>0$, $O_h(0,h)$, $B_h(1,h)$.

Let $U(x,y)$ be the solution of the homogeneous Dirichlet problem. Suppose that $U(x,y)\neq 0$ in $\overline{D}$. Then there exists a domain D_h, such that $U(x,y)\neq 0$ in D_h. Then the function $U(x,y)$ can attain its positive maximum and negative minimum in the domain $\overline{D}_h$ only on $\overline{O_hB_h}$. By virtue of the maximum principle for elliptic equations [12], the function $U(x,y)$ doesn't attain its positive maximum in inner points of the domain D_h. According to the homogeneous boundary condition it cannot also be attained on $\overline{OB}\cup OO_h\cup BB_h$.

Thus, $\max_{\overline{D}_h}|u(x,y)|$ should be attained on $\overline{O_hB_h}$. We take an arbitrary small number $\xi>0$ and taking into account (1.2), we choose h so large that $|U(x,y)|<\xi$ on $y=h$. Let (x_0,y_0) be an arbitrary point of the domain D. For sufficiently large h, this point falls into the domain D_h and therefore $|U(x_0,y_0)|<\xi$. By virtue of arbitrariness of ξ, we have $U(x_0,y_0)=0$. Then $U(x,y)\equiv 0$ in the domain D. Theorem 2.1 is proved. □