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Global Existence and Blow Up of Solutions in a Cauchy Viscoelastic Problem

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Abstract. The Cauchy problem for a viscoelastic equation with nonlinear damping and source terms is considered. We establish the nonexistence result of global solutions with initial energy controlled above by a critical value by modifying the method introduced in a work by Autuori *et al.* in 2010. Then we establish global existence for arbitrary initial data or the energy in potential well. These improve earlier results in the literatures.

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Key Words: Cauchy problem; viscoelastic equation; global existence; blow up.

1 Introduction

In [1], Messaoudi considered the following initial-boundary value problem

$$\begin{cases} u_{tt}-\triangle u+\int_0^t g(t-s)\triangle u(s)\mathrm{d}s+|u_t|^{m-2}u_t=|u|^{p-2}u, & x\in\Omega\ t>0,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\Omega,\\ u(t,x)=0, \quad (t,x)\in\mathbb{R}_0^+\times\partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N(N\geq 1)$ with a smooth boundary $\partial\Omega$, $p>2,m\geq 2$, and g is a nonincreasing positive function. He showed, under suitable assumptions on g, that the solutions with negative initial energy blow up in finite time if $p>m$ and continue to exist if $m\geq p$ provided that $\max\{m,p\}\leq\frac{2(N-1)}{N-2}$ if $N\geq 3$.

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In the absence of the viscoelastic term, the problem (1.1) has been extensively studied and results concerning the global existence and nonexistence have been established by many authors. Levine [2,3] showed that solutions with negative initial energy blow up in finite time in the case of linear damping. Georgiev and Todorova [4] extended Levine's results to the nonlinear damping via a method different from the one known as the concavity method used by Levine. Vitillaro [5] extended the result in the situation where the solution has positive energy. In the presence of the viscoelastic term (i.e. $g \neq 0$), the blow up result [1] was improved by the same author [6] and Wu [7]. Recently, Wu *et al.* [8] considered the problem (1.1) with general damping and source terms, they established the nonexistence result of global solutions with the initial energy controlled by a critical value by modifying the method in a work Autuori *et al.* [9] in 2010, see [10,11] for more general results.

For the problem in $\mathbb{R}^N$, Todorova [12] studied the following problem

$$u_{tt} - \triangle u + q^2(x)u + |u_t|^{m-2}u_t = |u|^{p-2}u, \qquad \text{in } (0,\infty)\times\mathbb{R}^N. \tag{1.2}$$

He showed the results similar to [4] when $q(x)$ is a decaying function. For more related results, we refer the readers to [13–15]. In [15], the author considered the linear damping case and obtained that the solution blows up in finite time when the initial energy is nonpositive. We also mention the work [16], Todorova considered the Cauchy problem (1.2) with $q(x)=1$ and obtained the solution is global and blows up in finite time under suitable conditions by the ideas of the potential well theory.

In the presence of the viscoelastic term, Kafini and Messaoudi [17] considered problem (1.1) when $m=2$ and $\Omega=\mathbb{R}^N$, and obtained the results similar to [15]. Later, Lu and Li [18] considered the following Cauchy problem with nonlinear damping term

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)\mathrm{d}s + u + |u_t|^{m-2}u_t = |u|^{p-2}u, \quad \text{in } (0,\infty)\times\mathbb{R}^N$$

with compactly supported initial data $u(0,x)=u_0(x), u_t(0,x)=u_1(x)$. They also obtained that the solution blows up in finite time when the initial energy is negative and a global existence result under suitable assumptions.

Motivated by these papers, in this work, we intend to study the following Cauchy problem

$$\begin{cases} u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)\mathrm{d}s + u + |u_t|^{m-2}u_t = f(x,u), & (t,x)\in(0,\infty)\times\mathbb{R}^N, \\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\mathbb{R}^N, \end{cases} \tag{1.3}$$

where g, f, u_0 and u_1 are functions to be specified later. Such problems arise in viscoelasticity and in systems governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann's model. Our aim is to extend the results of [8,12,17,18] to our problem.