JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **31**, No. 4, pp. 333-342

doi: 10.4208/jpde.v31.n4.4
December 2018

Fractional Tikhonov Regularization Method for a Time-Fractional Backward Heat Equation with a Fractional Laplacian

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Received 20 March 2018; Accepted 27 September 2018

Abstract. In this paper, we consider a time-fractional backward problem with a fractional Laplacian. We propose a fractional Tikhonov regularization method for solving this problem under the the a-priori parameter choice rule. Error estimates are proved. Some numerical examples are shown to verify the effectiveness of the proposed method.

AMS Subject Classifications: 65R35

Chinese Library Classifications: O24

Key Words: Inverse problem; Ill-posedness; error estimate; fractional regularization method.

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1 Introduction

The backward heat conduction problem (BHCP) has been studied by many researchers [1,2]. Also there are many mathematical results about fractional backward heat condition problem (FBHCP) in the domain $\mathcal{R}^d$. Deng and Yang [3] considered a discretized Tikhonov regularization methd for solving it, Li and Xiong [4] gave a general regularization method for the FBHCP with application to a deblurring problem. However there are few mathematical results about a time-fractional backward heat equation with fractional Laplacian in a bounded domain. Recently some fractional regularization methods are developed for solving all kinds of ill-posed problems. Please refer to [5-11].

Let Ω be a bound domain in $\mathcal{R}^d$ $(d\in\mathcal{N})$ with sufficient smooth boundary $\partial\Omega$, now we consider the following backward heat equation with fractional Laplacian as follows:

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$$\begin{cases} D_t^\alpha u(x,t) = -A^\beta u(x,t), & (0,T)\times\Omega, \\ u(x,T) = g(x), & \Omega, \\ u(x,t) = 0, & (0,T)\times\partial\Omega, \end{cases} \tag{1.1}$$

where $0\le\alpha\le 1$, $0\le\beta\le 1$ and the Dirichlet-Laplacian A is

$$\begin{cases} Af := -\Delta f = -\Sigma_{j=1}^d \partial_j^2 f, \\ D(A) := \mathcal{L}^2(\Omega)\cap\mathcal{H}_0^1(\Omega)\cap\mathcal{W}^{2,2}, \end{cases} \tag{1.2}$$

here $\mathcal{W}^{m,p}(\Omega)$ is the Sobolev space.

And D_t^α is the Caputo fractional derivative of order $\alpha(0\le\alpha\le 1)$ defined by

$$D_t^\alpha u(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, & 0<\alpha<1, \\ u_t(x,t), & \alpha=1. \end{cases} \tag{1.3}$$

We want to recover the solution $f(x) := u(x,0)$. Due to the ill-posedness, some regularization methods must be applied [12-14]. But in this paper, we use a new fractional Tikhonov method.

The outline of the paper is as follows: in Section 2, we give some preliminary results; In order to overcome the difficulty, a novel a priori bound is introduced in Section 2, a conditional stability result is also given; Meanwhile, we propose a fractional Tikhonov regularization method and give convergence estimate under a priori assumption in section 4.

2 Preliminaries

Throughout this paper, we use the following definitions and Lemmas.

Lemma 2.1. ([15]) *There exist* $\{\lambda_j\}_{j\in\mathcal{N}}\in\mathcal{R}$ *and* $\{\varphi_j\}_{j\in\mathcal{N}}\in D(A)$ *such that*

(i) $A(\varphi_j)=\lambda_j\varphi_j$,

(ii) $0<\lambda_1<\lambda_2<\lambda_3<\cdots$,

(iii) $\lim_{j\to\infty}\lambda_j=\infty$,*furthermore, for each* $f\in\mathcal{L}^2(\Omega)$,

$$f=\sum_{j=1}^\infty\langle f,\varphi_j\rangle\varphi_j \quad \text{in } \mathcal{L}^2 \tag{2.1}$$

and for each $i,j\in\mathcal{N}$,

$$\langle\varphi_i,\varphi_j\rangle = \begin{cases} 1, & \text{if } i=j, \\ 0, & \text{if } i\neq j. \end{cases} \tag{2.2}$$

that is, $\{\varphi_j\}_{j\in\mathcal{N}}$ *is an orthonormal basis of* $\mathcal{L}^2(\Omega)$.