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A Remark on Hardy-Trudinger-Moser Inequality

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Abstract. Let $\mathbb{B}$ be the unit disc in $\mathbb{R}^2$, $\mathscr{H}$ be the completion of $C_0^\infty(\mathbb{B})$ under the norm

$$||u||_{\mathscr{H}}=\Big(\int_{\mathbb{B}}|\nabla u|^2\mathrm{d}x-\int_{\mathbb{B}}\frac{u^2}{(1-|x|^2)^2}\mathrm{d}x\Big)^{\frac{1}{2}},\quad \forall u\in C_0^\infty(\mathbb{B}).$$

Using blow-up analysis, we prove that for any $\gamma\leqslant 4\pi$, the supremum

$$\sup_{u\in\mathscr{H},||u||_{1,h}\leqslant 1}\int_{\mathbb{B}}\mathrm{e}^{\gamma u^2}\mathrm{d}x$$

can be attained by some function $u_0\in\mathscr{H}$ with $||u_0||_{1,h}=1$, where h is a decreasingly nonnegative, radially symmetric function, and satisfies a coercive condition. Namely there exists a constant $\delta>0$ satisfying

$$||u||_{1,h}^2=\|u\|_{\mathscr{H}}^2-\int_{\mathbb{B}}hu^2\mathrm{d}x\geq\delta\|u\|_{\mathscr{H}}^2,\quad \forall u\in\mathscr{H}.$$

This extends earlier results of Wang-Ye [1] and Yang-Zhu [2].

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1 Introduction

Let $\mathbb{B}$ be the unit disc in $\mathbb{R}^2$. The Trudinger-Moser inequality [3–7] is known as

$$\sup_{u\in W_0^{1,2}(\mathbb{B}),||\nabla u||_2\leqslant 1}\int_{\mathbb{B}}\mathrm{e}^{\gamma u^2}\mathrm{d}x<\infty,\quad \forall\gamma\leq 4\pi.$$

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When $\gamma > 4\pi$, the above integrals are still finite, but the supremum is infinite. Here and in the sequel, for any real number $p \geqslant 1$, $||\cdot||_p$ denotes the L^p-norm with respect to the Lebesgue measure. Another well-known inequality in analysis is the Hardy inequality

$$\int_{\mathbb{B}} |\nabla u|^2 \mathrm{d}x \geqslant \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} \mathrm{d}x, \quad \forall u \in W_0^{1,2}(\mathbb{B}),$$

which was improved by Brezis and Marcus [8] to the following form: there exists some constant C satisfying

$$\int_{\mathbb{B}} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} \mathrm{d}x \geqslant C \int_{\mathbb{B}} u^2 \mathrm{d}x, \quad \forall u \in W_0^{1,2}(\mathbb{B}). \tag{1.1}$$

In view of (1.1), one can define the function space $\mathscr{H}$ as a completion of $C_0^\infty(\mathbb{B})$ under the norm

$$||u||_{\mathscr{H}} = \Big(\int_{\mathbb{B}} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} \mathrm{d}x \Big)^{\frac{1}{2}}.$$

Clearly, $\mathscr{H}$ is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ induced by the norm $\|\cdot\|_{\mathscr{H}}$. Several important properties of the space $\mathscr{H}$ we will use in our article could be found in Wang-Ye [1, Lemma 1] and Yang-Zhu [2, Lemma 4, Lemma 5].

According to Wang-Ye [1] and Mancini-Sandeep [9, the inequality (1.2)], we have that for any $p > 1$, there exists a constant $C_p > 0$ satisfying

$$\|u\|_p \leqslant C_p \|u\|_{\mathscr{H}}, \quad \forall u \in \mathscr{H}.$$

This together with the inequality $\|u\|_{\mathscr{H}} \leqslant \|\nabla u\|_2$ leads to

$$W_0^{1,2}(\mathbb{B}) \subset \mathscr{H} \subset \cap_{p \geqslant 1} L^p(\mathbb{B}).$$

Obviously $\mathscr{H} \nsubseteq L^\infty(\mathbb{B})$. Using the blow-up analysis, Wang-Ye [1] obtained the following Hardy-Trudinger-Moser inequality

$$\sup_{u \in \mathscr{H}, ||u||_{\mathscr{H}} \leqslant 1} \int_{\mathbb{B}} \mathrm{e}^{4\pi u^2} \mathrm{d}x < +\infty. \tag{1.2}$$

Moreover, the above supremum can be attained.

Let

$$\lambda_1(\mathbb{B}) = \inf_{u \in \mathscr{H}, u \not\equiv 0} ||u||_{\mathscr{H}}^2 / ||u||_2^2 \tag{1.3}$$

be the first eigenvalue of the Hardy-Laplace operator

$$\mathscr{L}_{\mathscr{H}} = -\Delta - \frac{\mathscr{I}}{(1-|x|^2)^2},$$