JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **32**, No. 1, pp. 1-19

doi: 10.4208/jpde.v32.n1.1
March 2019

Existence and Regularity of a Weak Solution to a Class of Systems in a Multi-Connected Domain

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Received 28 March 2017; Accepted 13 March 2019

Abstract. We consider the existence and regularity of a weak solution to a class of systems containing a p-curl system in a multi-connected domain. This paper extends the result of the regularity theory for a class containing a p-curl system that is given in the author's previous paper. The optimal $C^{1+\alpha}$-regularity of a weak solution is shown in a multi-connected domain.

AMS Subject Classifications: 35A05, 35A15, 35D05, 35D10, 35J20

Chinese Library Classifications: O175.27

Key Words: Regularity of a weak solution; variational problem; p-curl system.

1 Introduction

In this paper, we consider the existence and regularity of a weak solution to a class of systems containing a p-curl system in a bounded multi-connected domain Ω in $\mathbb{R}^3$.

In a bounded simply connected domain Ω in $\mathbb{R}^3$ without holes, Yin [1] considered the existence of a unique solution for the so-called p-curl system

$$\begin{cases} \operatorname{curl}[|\operatorname{curl}\boldsymbol{v}|^{p-2}\operatorname{curl}\boldsymbol{v}]=\boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v}=0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v}=\boldsymbol{0} & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where Γ denotes the $C^{2+\alpha}$ $(\alpha\in(0,1))$ boundary of Ω, $p>1$, $\boldsymbol{n}$ the outer normal unit vector field to Γ, and $\boldsymbol{f}$ is a given vector field satisfying $\operatorname{div}\boldsymbol{f}=0$ in Ω. If $\boldsymbol{f}$ is a C^α-vector function, then he showed the optimal $C^{1+\beta}$-regularity for some $\beta\in(0,1)$ of a weak solution in Yin [2], see also Yin et al. [3].

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Eq. (1.1) is a steady-state approximation of Bean's critical state model for type II superconductors. For further physical background, see [3], Chapman [4] and Prigozhin [5].

Aramaki [6] extended the result of [2] on the $C^{1+\beta}$ regularity of a weak solution to a more general equation, in a simply connected domain without holes to the following system.

$$\begin{cases} \mathrm{curl}[S_t(x,|\mathrm{curl}\boldsymbol{v}|^2|)\mathrm{curl}\boldsymbol{v}]=\boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\boldsymbol{v}=0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v}=\boldsymbol{0} & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where the function $S(x,t)\in C^2(\Omega\times(0,\infty))\cap C^0(\Omega\times[0,\infty))$ satisfies some structure conditions. Now and from now on, we denote $\frac{\partial}{\partial t}S(x,t)$ and $\frac{\partial^2}{\partial t^2}S(x,t)$ by $S_t(x,t)$ and $S_{tt}(x,t)$, respectively.

However, in a multi-connected domain, the systems (1.1) and (1.2) are not well posed. In fact, if the second Betti number is positive, for a weak solution $\boldsymbol{v}$ of (1.1) or (1.2), $\boldsymbol{v}+\boldsymbol{z}$, where $\boldsymbol{z}$ satisfies $\mathrm{curl}\boldsymbol{z}=\boldsymbol{0},\mathrm{div}\boldsymbol{z}=0$ in Ω and $\boldsymbol{z}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, is also a weak solution. Thus it is necessary to add some conditions to (1.1) and (1.2).

In this paper, we show the unique existence and optimal $C^{1+\beta}$-regularity of a weak solution to the system (1.2) with additive conditions.

The paper is organized as follows. In Section 2, we give some preliminaries and the main theorem. In Section 3, we give the existence of a weak solution of (2.10) below. Section 4 is devoted to the regularity of the weak solution obtained in Section 3.

2 Preliminaries and the main theorem

Since we allow that Ω is a multi-connected domain, we assume that Ω has the following conditions as in Amrouche and Seloula [7] (cf. Amrouche and Seloula [8], Dautray and Lions [9] and Girault and Raviart [10]). Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{2+\alpha}$ with the boundary Γ and Ω be locally situated on one side of Γ.

(1) Γ has a finite number of connected components $\Gamma_0,\Gamma_1,\ldots,\Gamma_m$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3\setminus\overline{\Omega}$.

(2) There exist n connected open surfaces $\Sigma_j,(j=1,\ldots,n)$, called cuts, contained in Ω such that

   (a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.

   (b) $\partial\Sigma_j\subset\Gamma$ $(j=1,\ldots,n)$, where $\partial\Sigma_j$ denotes the boundary of Σ_j, and Σ_j is non-tangential to Γ.

   (c) $\overline{\Sigma_i}\cap\overline{\Sigma_j}=\emptyset(i\neq j)$.

   (d) The open set $\dot{\Omega}=\Omega\setminus(\cup_{i=1}^n\Sigma_i)$ is simply connected and pseudo $C^{1,1}$ class.

The number n is called the first Betti number which is equal to the number of handles of Ω, and m is called the second Betti number which is equal to the number of holes. We say that if $n=0$, Ω is simply connected, and if $m=0$, Ω has no holes.

From now on we use the notations $L^p(\Omega), W^{m,p}(\Omega)$ ($m\geq 0$,integer), $W^{s,p}(\Gamma)$ ($s\in\mathbb{R}$), and so on, for the standard Sobolev spaces of functions. For any Banach space B, we denote $B\times B\times B$ by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$ by $\boldsymbol{u}\cdot\boldsymbol{v}$.

Define two spaces by

$$\begin{aligned}
&\mathbb{K}_N^p(\Omega)=\{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\mathrm{div}\boldsymbol{v}=0,\mathrm{curl}\boldsymbol{v}=\boldsymbol{0}\text{ in }\Omega,\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\ \text{ on }\ \Gamma\},\\
&\mathbb{K}_T^p(\Omega)=\{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\mathrm{div}\boldsymbol{v}=0,\mathrm{curl}\boldsymbol{v}=\boldsymbol{0}\text{ in }\Omega,\boldsymbol{v}\cdot\boldsymbol{n}=0\ \text{ on }\ \Gamma\}.
\end{aligned}$$

For any function $q\in W^{1,p}(\dot\Omega)$, we write an extension of $\nabla q\in \boldsymbol{L}^p(\dot\Omega)$ to $\boldsymbol{L}^p(\Omega)$ by $\widetilde{\nabla}q$. Let q_j^T ($j=1,\dots n$) be the unique solution in $W^{2,p}(\dot\Omega)$ of the system

$$\begin{cases}
-\Delta q_j^T=0 & \text{in } \dot\Omega,\\
\boldsymbol{n}\cdot\nabla q_j^T=0 & \text{on } \Gamma,\\
\big[q_j^T\big]_{\Sigma_k}=\text{const.},\quad \big[\boldsymbol{n}\cdot\nabla q_j^T\big]_{\Sigma_k}=0 & k=1,\dots,n\\
\langle \boldsymbol{n}\cdot\nabla q_j^T,1\rangle_{\Sigma_k}=\delta_{jk} & k=1,\dots n,
\end{cases}\tag{2.1}$$

where $\big[q_j^T\big]_{\Sigma_k}$ is the jump of the function of q_j^T across Σ_k. Then according to [7, Corollary 4.1], $\{\widetilde{\nabla}q_j^T;j=1,\dots,n\}$ is a basis of $\mathbb{K}_T^p(\Omega)$. In fact, $\widetilde{\nabla}q_j^T\in C^{1+\alpha}(\overline{\Omega})$.

On the other hand, let q_i^N ($i=1,\dots m$) be the unique solution in $W^{2,p}(\Omega)$ of the system

$$\begin{cases}
-\Delta q_i^N=0 & \text{in } \Omega,\\
q_i^N\big|_{\Gamma_0}=0,\quad q_i^N\big|_{\Gamma_k}=\text{const.} & k=1,\dots m,\\
\langle \boldsymbol{n}\cdot\nabla q_i^N,1\rangle_{\Gamma_k}=\delta_{ik}\,(k=1,\dots n) & \langle \boldsymbol{n}\cdot\nabla q_i^N,1\rangle_{\Gamma_0}=-1.
\end{cases}\tag{2.2}$$

Then $\{\nabla q_i^N;i=1,\dots m\}$ is a basis of $\mathbb{K}_N^p(\Omega)$ (cf. [7, Corollary 4.2]). In fact, $\nabla q_i^N\in \boldsymbol{C}^{1+\alpha}(\overline{\Omega})$. Thus we can see that $\dim\mathbb{K}_T^p(\Omega)=n$ and $\dim\mathbb{K}_N^p(\Omega)=m$.

We assume that a function $S(x,t)\in C^2(\Omega\times(0,\infty))\cap C(\Omega\times[0,\infty))$ satisfies the following structural conditions: There exist a constant $1<p<\infty$ and positive constants $0<\lambda<\Lambda$ such that for all $x\in\Omega$

$$S(x,0)=0 \text{ and } \lambda t^{(p-2)/2}\leq S_t(x,t)\leq \Lambda t^{(p-2)/2}\quad \text{for }\ t>0.\tag{2.3}$$

$$\lambda t^{(p-2)/2}\leq S_t(x,t)+2tS_{tt}(x,t)\leq \Lambda t^{(p-2)/2}\quad \text{for }\ t>0.\tag{2.4}$$

$$\text{If } 1<p<2, S_{tt}(x,t)<0, \text{ and if } p\geq 2, S_{tt}(x,t)\geq 0\quad \text{for }\ t>0.\tag{2.5}$$

$$\text{There exists a constant } C>0 \text{ such that } |S_{tx}(x,t)|\leq Ct^{(p-2)/2}\quad \text{for }\ t>0.\tag{2.6}$$

We note that from (2.3), we have

$$\frac{2}{p}\lambda t^{p/2}\le S(x,t)\le \frac{2}{p}\Lambda t^{p/2} \quad \text{for } t\ge 0.$$

When $S(x,t)=t^{p/2}$, system (1.2) becomes (1.1), and by elementary calculations, we see that $S(x,t)=\nu(x)t^{p/2}$, where $\nu\in C^2(\Omega)$ and $0<\nu_*\le\nu(x)\le\nu^*<\infty$ and $|\nabla\nu(x)|\le C$ for all $x\in\Omega$, satisfies (2.3)-(2.6).

We note from (2.3) that $S(x,t)$ is a strictly increasing function with respect to $t\in[0,\infty)$, and from (2.3) and (2.4), $|S_{tt}(x,t)|\le\Lambda t^{(p-4)/2}$ for $x\in\Omega, t>0$. Define

$$\Phi(x,t)=\begin{cases} t(S_t(x,t))^2 & \text{for } x\in\Omega, t>0,\\ 0 & \text{for } x\in\Omega, t=0.\end{cases} \tag{2.7}$$

Then $\Phi\in C^1(\Omega\times(0,\infty))\cap C^0(\Omega\times[0,\infty))$, and from (2.4), we see that Φ satisfies

$$\Phi_t(x,t)=S_t(x,t)(S_t(x,t)+2tS_{tt}(x,t))>0 \quad \text{for } x\in\Omega, t>0. \tag{2.8}$$

Thus let $t=\Psi(x,\rho)\in C^1(\Omega\times(0,\infty))$ be the implicit function of $\rho=\Phi(x,t)$ and define

$$f(x,\rho)=\frac{1}{S_t(x,\Psi(x,\rho))}. \tag{2.9}$$

Then we obtain the following lemma whose proof is given in [6, Lemma 2.1].

Lemma 2.1. *Assume that the hypotheses* (2.3)-(2.6) *hold and let $q=p/(p-1)$ be the conjugate exponent to p. Then we have the following.*

(i) $\Lambda^{-(q-1)}\rho^{(q-2)/2}\le f(x,\rho)\le\lambda^{-(q-1)}\rho^{(q-2)/2}$ *and* $|f_x(x,\rho)|\le C\rho^{(q-2)/2}$.

(ii) *There exist positive constants c and C depending only on λ and Λ such that*

$$c\rho^{(q-2)/2}\le f(x,\rho)+2\rho f_\rho(x,\rho)\le C\rho^{(q-2)/2}.$$

Since we allow that Ω is multi-connected, thus (1.2) is not well posed, we consider the following system instead of (1.2).

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}]=\boldsymbol{f} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v}=0 & \text{in } \Omega,\\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0} & \text{on } \Gamma,\\ \langle \boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & \text{for } i=1,\dots m,\end{cases} \tag{2.10}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_i}$ denotes the duality bracket of $W^{-1/p.p}(\Gamma_i)$ and $W^{1-1/q,q}(\Gamma_i)$.

We are in a position to state the main theorem.

Theorem 2.1. *Assume that Ω is a bounded domain in $\mathbb{R}^3$ with $C^{2+\alpha}$ boundary satisfying* (1) *and* (2) for some $\alpha\in(0,1)$, *and that a function $S(x,t)$ satisfies the conditions* (2.3)-(2.6). *Moreover, assume that $\boldsymbol{f}\in\boldsymbol{L}^q(\Omega)$ satisfies*

$$\operatorname{div}\boldsymbol{f}=0 \text{ in } \Omega \text{ and } \langle \boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,\ i=1,\dots,m. \tag{2.11}$$

Then the system (2.10) *has a unique weak solution $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ in the sense of* (3.3) *below. Furthermore, if $\boldsymbol{f}\in\boldsymbol{C}^\alpha(\overline{\Omega})$ satisfies* (2.11), *then the weak solution $\boldsymbol{v}$ belongs to $\boldsymbol{C}^{1+\beta}(\overline{\Omega})$ for some $\beta\in(0,1)$, and there exists a constant $C>0$ depending only on α,Ω and $\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{C}^{1+\beta}(\overline{\Omega})}\le C.$$

This theorem contains [2, Theorem] and [6, Theorem 1.1] as a corollary.

Corollary 2.2. *Assume that Ω is a bounded, simply connected domain in $\mathbb{R}^3$ without holes, and with $C^{2+\alpha}$ boundary Γ for some $\alpha\in(0,1)$. Assume moreover that $\boldsymbol{f}$ satisfies that $\boldsymbol{f}\in\boldsymbol{C}^\alpha(\overline{\Omega})$ and* $\operatorname{div}\boldsymbol{f}=0$ *in Ω. Then the weak solution $\boldsymbol{v}$ of* (1.2) *belongs to $\boldsymbol{C}^{1+\beta}(\overline{\Omega})$ for some $\beta\in(0,1)$. In addition, there exists a constant C depending only on p,Ω and $\|\boldsymbol{f}\|_{C^\alpha(\overline{\Omega})}$ such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{C}^{1+\beta}(\overline{\Omega})}\le C.$$

3 Existence of a weak solution

In order to obtain the existence of a weak solution of (2.10), we consider the space

$$\boldsymbol{V}^p(\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\operatorname{curl}\boldsymbol{v}\in\boldsymbol{L}^p(\Omega),\operatorname{div}\boldsymbol{v}=0 \text{ in } \Omega,\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0} \text{ on } \Gamma,\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,i=1,\dots m\}.$$

Then we have

Lemma 3.1. *Let $1<p<\infty$. Then $\boldsymbol{V}^p(\Omega)$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$, and we can regard $\boldsymbol{V}^p(\Omega)$ as a separable, reflexive Banach space with the norm*

$$\|\boldsymbol{v}\|_{\boldsymbol{V}^p(\Omega)}:=\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$$

which is equivalent to $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$.

Proof. By [7, Corollary 3.2], we can easily see that $\boldsymbol{V}^p(\Omega)$ is a closed subspace of $\boldsymbol{W}^{1,p}(\Omega)$ and that the norm $\|\operatorname{curl}\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$ is equivalent to the norm $\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p}(\Omega)}$. Since $\boldsymbol{W}^{1,p}(\Omega)$ is separable and reflexive and $\boldsymbol{V}^p(\Omega)$ is a closed subspace, $\boldsymbol{V}^p(\Omega)$ is separable and reflexive (see, e.g., Brezis [11]). □

We consider the following minimization problem: to find $\boldsymbol{v}\in\boldsymbol{V}^p(\Omega)$ such that $\boldsymbol{v}$ is a minimizer of

$$\inf_{\boldsymbol{u}\in\boldsymbol{V}^p(\Omega)} I[\boldsymbol{u}],$$

where

$$I[\boldsymbol{u}]=\int_\Omega S(x,|\mathrm{curl}\boldsymbol{u}|^2)\mathrm{d}x-2\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\mathrm{d}x$$

for some given $\boldsymbol{f}$. Then we have the following.

Proposition 3.2. *Let $\boldsymbol{f}\in\boldsymbol{L}^q(\Omega)$, where q is the conjugate exponent of p. Then the functional I on $\boldsymbol{V}^p(\Omega)$ has a unique minimizer $\boldsymbol{v}\in\boldsymbol{V}^p(\Omega)$.*

Proof. From the Hölder inequality, for any $\boldsymbol{u}\in\boldsymbol{V}^p(\Omega)$, we have

$$I[\boldsymbol{u}]\geq\frac{2}{p}\lambda\int_\Omega|\mathrm{curl}\boldsymbol{u}|^p\mathrm{d}x-2\|\boldsymbol{f}\|_{\boldsymbol{L}^q(\Omega)}\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}.$$

By Lemma 3.1 and the Young inequality, there exists a constant $c>0$ such that for any $\varepsilon>0$, there exists $C(\varepsilon)>0$ such that

$$I[\boldsymbol{u}]\geq 2c\|\boldsymbol{u}\|^p_{\boldsymbol{V}^p(\Omega)}-C(\varepsilon)\|\boldsymbol{f}\|^q_{\boldsymbol{L}^q(\Omega)}-\varepsilon\|\boldsymbol{u}\|^p_{\boldsymbol{V}^p(\Omega)}.$$

Choosing $\varepsilon=c>0$,

$$I[\boldsymbol{u}]\geq c\|\boldsymbol{u}\|^p_{\boldsymbol{V}^p(\Omega)}-C(c)\|\boldsymbol{f}\|^q_{\boldsymbol{L}^q(\Omega)}. \tag{3.1}$$

Thus the functional I is coercive, that is,

$$\lim_{\boldsymbol{u}\in\boldsymbol{V}^p(\Omega),\|\boldsymbol{u}\|_{\boldsymbol{V}^p(\Omega)}\to\infty}I[\boldsymbol{u}]=\infty.$$

Now if we put $F(x,t)=S(x,t^2)$, it follows from (2.3) and (2.4) that

$$\begin{aligned}&F_t(x,t)=2tS_t(x,t^2)\geq 2\lambda t^{p-1}>0 \quad \text{for } x\in\Omega,\ t>0,\\ &F_{tt}(x,t)=2(S_t(x,t^2)+2t^2S_{tt}(x,t^2))\geq 2\lambda t^{p-2}>0 \quad \text{for } x\in\Omega,\ t>0.\end{aligned}$$

Thus I is a strictly convex functional on $\boldsymbol{V}^p(\Omega)$. We claim that I is lower semicontinuous. In fact, let $\boldsymbol{w}_j\to\boldsymbol{w}$ in $\boldsymbol{V}^p(\Omega)$. Then $\mathrm{curl}\boldsymbol{w}_j\to\mathrm{curl}\boldsymbol{w}$ in $\boldsymbol{L}^p(\Omega)$. Thus there exists a subsequence $\{\boldsymbol{w}_{j_k}\}$ of $\{\boldsymbol{w}_j\}$ such that

$$\lim_{k\to\infty}\int_\Omega S(x,|\mathrm{curl}\boldsymbol{w}_{j_k}|^2)\mathrm{d}x=\liminf_{j\to\infty}\int_\Omega S(x,|\mathrm{curl}\boldsymbol{w}_j|^2)\mathrm{d}x,$$

and $\mathrm{curl}\boldsymbol{w}_{j_k}\to\mathrm{curl}\boldsymbol{w}$ a.e. in Ω. Since S is continuous, $S(x,|\mathrm{curl}\boldsymbol{w}_{j_k}|^2)\to S(x,|\mathrm{curl}\boldsymbol{w}|^2)$ a.e. in Ω. Since $S(x,t)\geq 0$, it follows from the Fatou lemma that

$$\begin{aligned}\int_\Omega S(x,|\mathrm{curl}\boldsymbol{w}|^2)\mathrm{d}x&\leq\liminf_{k\to\infty}\int_\Omega S(x,|\mathrm{curl}\boldsymbol{w}_{j_k}|^2)\mathrm{d}x=\lim_{k\to\infty}\int_\Omega S(x,|\mathrm{curl}\boldsymbol{w}_{j_k}|^2)\mathrm{d}x\\ &=\liminf_{j\to\infty}\int_\Omega S(x,|\mathrm{curl}\boldsymbol{w}_j|^2)\mathrm{d}x.\end{aligned}$$

Therefore

$$I[\boldsymbol{w}] \le \liminf_{j\to\infty} I[\boldsymbol{w}_j],$$

so I is lower semicontinuous. We note that a convex, lower semi-continous functional on non-empty Banach space is weakly lower semi-continous, so I is weakly lower semi-continuous. From (3.1), I is lower semi-continuous, coercive and strictly convex on the reflexive Banach space $\boldsymbol{V}^p(\Omega)$. Hence $\inf_{\boldsymbol{u}\in \boldsymbol{V}^p(\Omega)} I[\boldsymbol{u}]$ is attained by a unique $\boldsymbol{u}_0 \in \boldsymbol{V}^p(\Omega)$, that is,

$$I[\boldsymbol{u}_0] = \inf_{\boldsymbol{u}\in \boldsymbol{V}^p(\Omega)} I[\boldsymbol{u}].$$

For example, see Ekeland and Témam [12, Chapter 2, Proposition 1.2]. □

Define a space

$$\mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div}) = \{\boldsymbol{v}\in \boldsymbol{L}^p(\Omega);\ \mathrm{curl}\,\boldsymbol{v}\in \boldsymbol{L}^p(\Omega),\ \mathrm{div}\,\boldsymbol{v}\in L^p(\Omega),\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\ \text{ on }\ \Gamma\}.$$

Then we claim the following lemma.

Lemma 3.3. *Assume that $\boldsymbol{f}\in \boldsymbol{L}^q(\Omega)$, $\mathrm{div}\,\boldsymbol{f}=0$ in Ω and $\langle \boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$ for $i=1,\dots,m$. Then we have*

$$\inf_{\boldsymbol{u}\in \boldsymbol{V}^p(\Omega)} I[\boldsymbol{u}] = \inf_{\boldsymbol{w}\in \mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div})} I[\boldsymbol{w}].$$

Proof. Since $\boldsymbol{V}^p(\Omega)\subset \mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div})$, it is trivial that

$$\inf_{\boldsymbol{u}\in \boldsymbol{V}^p(\Omega)} I[\boldsymbol{u}] \ge \inf_{\boldsymbol{w}\in \mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div})} I[\boldsymbol{w}].$$

For any $\boldsymbol{u}\in \mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div})$, we consider the following div-curl system.

$$\begin{cases} \mathrm{curl}\,\boldsymbol{v}=\mathrm{curl}\,\boldsymbol{u} & \text{in }\ \Omega,\\ \mathrm{div}\,\boldsymbol{v}=0 & \text{in }\ \Omega,\\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0} & \text{on }\ \Gamma. \end{cases} \tag{3.2}$$

Since $\mathrm{div}(\mathrm{curl}\,\boldsymbol{u})=0$ in Ω, $\boldsymbol{n}\cdot\mathrm{curl}\,\boldsymbol{u}=\boldsymbol{n}\cdot\mathrm{curl}\,\boldsymbol{u}_T=0$ on Γ, where $\boldsymbol{u}_T$ is the tangent component of $\boldsymbol{u}$ (cf. Monneau [13]), it follows from Aramaki [14, Theorem 3.5] that (3.2) has a solution $\boldsymbol{v}\in \boldsymbol{W}^{1,p}(\Omega)$. Define $\boldsymbol{w}=\boldsymbol{v}-\sum_{k=1}^m \langle \boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_k}\boldsymbol{e}_k$, where $\boldsymbol{e}_k=\nabla q_k^N$. Since $\langle \boldsymbol{e}_k\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=\delta_{ki}$ from (2.2), we have, for $i=1,\dots,m$,

$$\langle \boldsymbol{w}\cdot\boldsymbol{n},1\rangle_{\Gamma_i} = \langle \boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i} - \sum_{k=1}^m \langle \boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_k}\langle \boldsymbol{e}_k\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0.$$

Since $\mathrm{div}\,\boldsymbol{w}=0$, $\mathrm{curl}\,\boldsymbol{w}=\mathrm{curl}\,\boldsymbol{v}=\mathrm{curl}\,\boldsymbol{u}$ in Ω and $\boldsymbol{w}\times\boldsymbol{n}=\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, we see that $\boldsymbol{w}\in \boldsymbol{V}^p(\Omega)$ and $\mathrm{curl}\,\boldsymbol{w}=\mathrm{curl}\,\boldsymbol{u}$. Since $\boldsymbol{f}\in \boldsymbol{L}^q(\Omega)$ satisfies $\mathrm{div}\,\boldsymbol{f}=0$ in Ω and $\langle \boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$ for $i=1,\dots,m$, it follows from the divergence theorem that

$$0=\int_\Omega \mathrm{div}\,\boldsymbol{f}\,dx = \int_\Gamma \boldsymbol{f}\cdot\boldsymbol{n}\,dS = \sum_{i=0}^m \langle \boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Gamma_i} = \langle \boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Gamma_0}=0.$$

Therefore it follows from [7, Lemma 4.1] that there exists $\boldsymbol{g}\in \boldsymbol{W}^{1,q}(\Omega)$ such that $\boldsymbol{f}=\mathrm{curl}\boldsymbol{g}$ in Ω. By integration by parts,

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}\mathrm{d}x&=\int_\Omega \mathrm{curl}\boldsymbol{g}\cdot\boldsymbol{w}\mathrm{d}x=\int_\Gamma(\boldsymbol{g}\times\boldsymbol{n})\cdot\boldsymbol{w})\mathrm{d}S+\int_\Omega \boldsymbol{g}\cdot\mathrm{curl}\boldsymbol{w}\mathrm{d}x\\&=\int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{n}\times\boldsymbol{w})\mathrm{d}S+\int_\Omega \boldsymbol{g}\cdot\mathrm{curl}\boldsymbol{u}\mathrm{d}x=\int_\Omega \boldsymbol{g}\cdot\mathrm{curl}\boldsymbol{u}\mathrm{d}x\\&=\int_\Omega \mathrm{curl}\boldsymbol{g}\cdot\boldsymbol{u}\mathrm{d}x=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\mathrm{d}x.\end{aligned}$$

Hence $I[\boldsymbol{w}]=I[\boldsymbol{u}]$. So

$$\inf_{\boldsymbol{w}\in \boldsymbol{V}^p(\Omega)} I[\boldsymbol{w}]\le I[\boldsymbol{u}]\ \text{for all}\ \boldsymbol{u}\in\mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div}).$$

Thus we have

$$\inf_{\boldsymbol{w}\in \boldsymbol{V}^p(\Omega)} I[\boldsymbol{w}]\le \inf_{\boldsymbol{u}\in\mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div})} I[\boldsymbol{u}].$$

Therefore we get the conclusion. □

Let $\boldsymbol{v}\in\boldsymbol{V}^p(\Omega)$ be the minimizer of

$$\inf_{\boldsymbol{u}\in \boldsymbol{V}^p(\Omega)} I[\boldsymbol{u}]$$

and $\boldsymbol{w}\in\mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div})$. Then by the Euler-Lagrange equation, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}I[\boldsymbol{v}+\varepsilon\boldsymbol{w}]\big|_{\varepsilon=0}=2\int_\Omega\big(S_t(x,|\mathrm{curl}\boldsymbol{v}|^2)\mathrm{curl}\boldsymbol{v}\cdot\mathrm{curl}\boldsymbol{w}-\boldsymbol{f}\cdot\boldsymbol{w}\big)\mathrm{d}x=0.$$

Hence $\boldsymbol{v}$ is a weak solution in the sense of

$$\begin{cases}\int_\Omega S_t(x,|\mathrm{curl}\boldsymbol{v}|^2)\mathrm{curl}\boldsymbol{v}\cdot\mathrm{curl}\boldsymbol{w}\mathrm{d}x=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{w}\mathrm{d}x & \text{for all}\ \ \boldsymbol{w}\in\mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div}),\\ \mathrm{div}\boldsymbol{v}=0 & \text{in}\ \ \Omega,\\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0} & \text{on}\ \ \Gamma,\\ \langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & \text{for}\ \ i=1,\dots,m.\end{cases}\tag{3.3}$$

Since $\boldsymbol{C}_0^\infty(\Omega)\subset\mathcal{H}_0^p(\Omega,\mathrm{curl},\mathrm{div})$, $\boldsymbol{v}$ satisfies (in the distribution sense),

$$\begin{cases}\mathrm{curl}[S_t(x,|\mathrm{curl}\boldsymbol{v}|^2)\mathrm{curl}\boldsymbol{v}]=\boldsymbol{f} & \text{in}\ \ \Omega,\\ \mathrm{div}\boldsymbol{v}=0 & \text{in}\ \ \Omega,\\ \boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0} & \text{on}\ \ \Gamma,\\ \langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & \text{for}\ \ i=1,\dots,m.\end{cases}\tag{3.4}$$

If we choose a test function $\boldsymbol{w}=\boldsymbol{v}$ in (3.3), it follows from the Hölder inequality that for any $\varepsilon>9$,

$$\lambda\|\mathrm{curl}\boldsymbol{v}\|^p_{L^p(\Omega)}\le\|\boldsymbol{f}\|_{L^q(\Omega)}\|\boldsymbol{v}\|_{L^p(\Omega)}\le C(\varepsilon)\|\boldsymbol{f}\|^q_{L^q(\Omega)}+\varepsilon\|\boldsymbol{v}\|^p_{L^p(\Omega)}.$$

Here if we note that $\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)} \le C\|\mathrm{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}$, then choosing $\varepsilon > 0$ small enough, we have

$$\|\boldsymbol{v}\|_{\boldsymbol{V}^p(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^q(\Omega)}^{q-1},$$

where $C > 0$ is a constant depending only on λ and p.

Remark 3.4. If Ω has no holes, the last conditions of (3.3) and (3.4) are unnecessary.

We get the following proposition.

Proposition 3.5. *Assume that $\boldsymbol{f} \in \boldsymbol{L}^q(\Omega)$ satisfies $\mathrm{div}\,\boldsymbol{f} = 0$ in Ω and $\langle \boldsymbol{f} \cdot \boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for $i = 1, \ldots, m$. Then the system* (3.4) *has a unique weak solution $\boldsymbol{v} \in \boldsymbol{V}^p(\Omega)$, and there exists a constant $C > 0$ depending only on λ and p such that*

$$\|\boldsymbol{v}\|_{\boldsymbol{V}^p(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{L}^q(\Omega)}^{q-1}.$$

To complete the proof, it suffices to prove the uniqueness of the weak solution of (3.4). In order to do so, we use the following lemma with respect to monotonicity.

Lemma 3.6. *There exists a constant $c > 0$ such that for all vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$\big(S_t(x, |\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x, |\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a} - \boldsymbol{b}|^p & \text{if } p > 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p-2}|\boldsymbol{a} - \boldsymbol{b}|^2 & \text{if } 1 < p \le 2. \end{cases}$$

Proof. Put $J(x,p) = \big(S_t(x, |\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x, |\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b})$. Then

$$\begin{aligned} J(x,p) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} S_t(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)(s\boldsymbol{a} + (1-s)\boldsymbol{b})\mathrm{d}s \cdot (\boldsymbol{a} - \boldsymbol{b}) \\ &= \int_0^1 S_t(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)\mathrm{d}s|\boldsymbol{a} - \boldsymbol{b}|^2 \\ &\qquad + \int_0^1 2S_{tt}(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)\big((|s\boldsymbol{a} + (1-s)\boldsymbol{b}) \cdot (\boldsymbol{a} - \boldsymbol{b})\big)^2 \mathrm{d}s. \end{aligned}$$

When $p > 2$, since $S_{tt}(x,t) \ge 0$, we have

$$J(x,p) \ge \int_0^1 S_t(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)\mathrm{d}s|\boldsymbol{a} - \boldsymbol{b}|^2 \ge \lambda \int_0^1 |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^{p-2}|\boldsymbol{a} - \boldsymbol{b}|^2 \mathrm{d}s.$$

From DiBenedetto [15, p. 14], we can show that $J(x,p) \ge c|\boldsymbol{a} - \boldsymbol{b}|^p$ for some $c > 0$.

When $1 < p < 2$, since $S_{tt}(x,t) < 0$, we have

$$\begin{aligned} J(x,p) \ge &\int_0^1 S_t(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)|\boldsymbol{a} - \boldsymbol{b}|^2 \mathrm{d}s \\ &+ \int_0^1 2S_{tt}(x, |s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2)|s\boldsymbol{a} + (1-s)\boldsymbol{b}|^2|\boldsymbol{a} - \boldsymbol{b}|^2. \end{aligned}$$

From (2.4), we have

$$J(x,p) \geq \lambda \int_0^1 |s\boldsymbol{a}+(1-s)\boldsymbol{b}|^{p-2}\mathrm{d}s|\boldsymbol{a}-\boldsymbol{b}|^2 \geq c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2$$

for some $c>0$. □

End of the proof of Proposition 3.5

Assume that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are weak solutions in $\boldsymbol{V}^p(\Omega)$ of (3.4). Choosing $\boldsymbol{w}=\boldsymbol{v}_1-\boldsymbol{v}_2$ as a test function of the first equation of (3.3), we have

$$\int_\Omega S_t(x,|\mathrm{curl}\boldsymbol{v}_1|^2)\mathrm{curl}\boldsymbol{v}_1\cdot\mathrm{curl}(\boldsymbol{v}_1-\boldsymbol{v}_2)\mathrm{d}x=\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}_1-\boldsymbol{v}_2)\mathrm{d}x,$$
$$\int_\Omega S_t(x,|\mathrm{curl}\boldsymbol{v}_2|^2)\mathrm{curl}\boldsymbol{v}_2\cdot\mathrm{curl}(\boldsymbol{v}_1-\boldsymbol{v}_2)\mathrm{d}x=\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}_1-\boldsymbol{v}_2)\mathrm{d}x.$$

Thus

$$\int_\Omega \big(S_t(x,|\mathrm{curl}\boldsymbol{v}_1|^2)\mathrm{curl}\boldsymbol{v}_1-S_t(x,|\mathrm{curl}\boldsymbol{v}_2|^2)\mathrm{curl}\boldsymbol{v}_2\big)\cdot\mathrm{curl}(\boldsymbol{v}_1-\boldsymbol{v}_2)\mathrm{d}x=0.$$

By Lemma 3.6,

$$\int_\Omega |\mathrm{curl}(\boldsymbol{v}_1-\boldsymbol{v}_2)|^p\mathrm{d}x=0, \qquad \text{if } \ p>2,$$
$$\int_\Omega (|\mathrm{curl}\boldsymbol{v}_1|+|\mathrm{curl}\boldsymbol{v}_2|)^{p-2}|\mathrm{curl}(\boldsymbol{v}_1-\boldsymbol{v}_2)|^2\mathrm{d}x=0, \qquad \text{if } \ 1<p\leq 2.$$

This implies $\mathrm{curl}(\boldsymbol{v}_1-\boldsymbol{v}_2)=\boldsymbol{0}$. Since $\boldsymbol{v}_1-\boldsymbol{v}_2\in\boldsymbol{V}^p(\Omega)$, we get $\boldsymbol{v}_1=\boldsymbol{v}_2$ in Ω. This completes the proof of Proposition 3.5. □

4 Regularity of the weak solution of (3.4)

In this section, we consider the regularity of the solution of (3.4). Throughout this section, we assume that

$$\boldsymbol{f}\in\boldsymbol{C}^\alpha(\overline{\Omega}),\mathrm{div}\boldsymbol{f}=0 \text{ in } \Omega \text{ and } \langle \boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 \quad \text{for } \ i=1,\dots,m \tag{4.1}$$

and q denotes the conjugate exponent of p. We consider the following div-curl system.

$$\begin{cases} \mathrm{curl}\boldsymbol{G}=\boldsymbol{f} & \text{in } \ \Omega, \\ \mathrm{div}\boldsymbol{G}=0 & \text{in } \ \Omega, \\ \boldsymbol{\nu}\cdot\boldsymbol{G}=0 & \text{on } \ \partial\Omega. \end{cases} \tag{4.2}$$

By hypotheses (4.1) and the divergence theorem, it holds that $\langle \boldsymbol{f}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0$ for $i=0,1,\dots,m$ as in the preceding arguments. Therefore from Pan [16, Lemma 5.7 (ii)], the system (4.2) has a solution $\boldsymbol{G}\in\boldsymbol{C}^{1+\alpha}(\overline{\Omega})$ and there exists a constant $C=C(\Omega,\alpha)$ such that

$$\|\boldsymbol{G}\|_{\boldsymbol{C}^{1+\alpha}(\overline{\Omega})}\leq C\Big(\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}+\|\boldsymbol{G}\|_{\boldsymbol{C}^0(\overline{\Omega})}\Big).$$

Here $\boldsymbol{G}$ is uniquely determined up to an additive element of $\mathbb{K}_T^p(\Omega)$. Thus the weak solution $\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega)$ of (3.4) satisfies

$$\begin{cases} \operatorname{curl}[S_t(x,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}-\boldsymbol{G}]=\boldsymbol{0} & \text{in } \Omega,\\ \operatorname{div}\boldsymbol{v}=0 & \text{in } \Omega,\\ \boldsymbol{\nu}\times\boldsymbol{v}=\boldsymbol{0} & \text{on } \Gamma,\\ \langle \boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & i=1,\dots m. \end{cases} \tag{4.3}$$

By hypothesis (2.3), we have

$$S_t(x,|\operatorname{curl}\boldsymbol{v}|^2)|\operatorname{curl}\boldsymbol{v}|\le\Lambda|\operatorname{curl}\boldsymbol{v}|^{p-1}\in L^q(\Omega).$$

Put $\boldsymbol{h}=S_t(x,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}-\boldsymbol{G}\in\boldsymbol{L}^q(\Omega)$. Then $\boldsymbol{h}$ satisfies $\operatorname{curl}\boldsymbol{h}=\boldsymbol{0}$ in Ω. By the Helmholtz decomposition of $\boldsymbol{h}$ (cf. [7, Theorem 6.1]), we can write

$$\boldsymbol{h}=\boldsymbol{z}+\nabla\psi+\operatorname{curl}\boldsymbol{w},$$

where $\boldsymbol{z}\in\mathbb{K}_T^q(\Omega)$ is unique, $\psi\in W^{1,q}(\Omega)$ is unique up to an additive constant and $\boldsymbol{w}\in\boldsymbol{W}^{1,q}(\Omega)$ satisfies $\operatorname{div}\boldsymbol{w}=0$ in Ω and $\boldsymbol{w}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ, and is unique up to an additive element of $\mathbb{K}_N^q(\Omega)$. However, since $\operatorname{curl}\boldsymbol{h}=\boldsymbol{0}$ in Ω, we have $\operatorname{curl}^2\boldsymbol{w}=\boldsymbol{0}$ in the distribution sense in Ω. Therefore

$$\begin{aligned} 0&=\int_\Omega \operatorname{curl}^2\boldsymbol{w}\cdot\boldsymbol{w}\,\mathrm{d}x=\int_\Gamma(\boldsymbol{n}\times\operatorname{curl}\boldsymbol{w})\cdot\boldsymbol{w}\,\mathrm{d}S+\int_\Omega|\operatorname{curl}\boldsymbol{w}|^2\mathrm{d}x\\ &=\int_\Gamma(\boldsymbol{w}\times\boldsymbol{n})\cdot\operatorname{curl}\boldsymbol{w}\,\mathrm{d}S+\int_\Omega|\operatorname{curl}\boldsymbol{w}|^2\mathrm{d}x=\int_\Omega|\operatorname{curl}\boldsymbol{w}|^2\mathrm{d}x, \end{aligned}$$

where dS denotes the surface element of Γ. So we have $\operatorname{curl}\boldsymbol{w}=\boldsymbol{0}$ in Ω. Thus we can write

$$\boldsymbol{h}=\boldsymbol{z}+\nabla\psi, \tag{4.4}$$

and the following estimate holds.

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^q(\Omega)}+\|\psi\|_{W^{1,q}(\Omega)/\mathbb{R}}\le C\Big(\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}+\|\boldsymbol{G}\|_{\boldsymbol{C}^0(\overline{\Omega})}\Big).$$

Since Γ is $C^{2+\alpha}$ class, we note that $\mathbb{K}_T^q(\Omega)\subset\boldsymbol{C}^{1+\alpha}(\overline{\Omega})$. Hence we can write

$$S_t(x,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}=\boldsymbol{G}+\boldsymbol{z}+\nabla\psi,$$

where $\boldsymbol{G},\boldsymbol{z}\in\boldsymbol{C}^{1+\alpha}(\overline{\Omega})$. By (2.7), we have

$$\Phi(x,|\operatorname{curl}\boldsymbol{v}|^2)=S_t(x,|\operatorname{curl}\boldsymbol{v}|^2)^2|\operatorname{curl}\boldsymbol{v}|^2=|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2.$$

Therefore we can write

$$|\operatorname{curl}\boldsymbol{v}|^2=\Psi\big(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2\big).$$

Moreover, we can write

$$\mathrm{curl}\boldsymbol{v}=\frac{\boldsymbol{G}+\boldsymbol{z}+\nabla\psi}{S_t(x,|\mathrm{curl}\boldsymbol{v}|^2)}=f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\psi).$$

Since $\boldsymbol{n}\cdot\mathrm{curl}\boldsymbol{v}=\boldsymbol{n}\cdot\mathrm{curl}\boldsymbol{v}_T=0$ on Γ, $\psi\in W^{1,q}(\Omega)$ is a weak solution of the following system.

$$\begin{cases} \mathrm{div}[f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\psi)]=0 & \text{in } \Omega, \\ \boldsymbol{n}\cdot f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\psi)=0 & \text{on } \Gamma. \end{cases} \tag{4.5}$$

If we show that $\psi\in C^{1+\beta}(\overline{\Omega})$ for some $\beta\in(0,1)$, then $\boldsymbol{v}$ is a solution of the following div-curl system

$$\begin{cases} \mathrm{curl}\boldsymbol{v}=f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\psi) & \text{in } \Omega, \\ \mathrm{div}\boldsymbol{v}=0 & \text{in } \Omega, \\ \boldsymbol{n}\times\boldsymbol{v}=\boldsymbol{0} & \text{on } \Gamma, \\ \langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0 & i=1,\dots,m. \end{cases} \tag{4.6}$$

Since $f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\psi)\in\boldsymbol{C}^\beta(\overline{\Omega})$, it follows from the regularity of the div-curl system (cf. Bolik and Wahl [20]) that $\boldsymbol{v}\in C^{1+\beta}(\overline{\Omega})$, and

$$\|\boldsymbol{v}\|_{C^{1+\beta}(\overline{\Omega})}\le C\Big(\|f(x,|\boldsymbol{G}+\boldsymbol{z}+\nabla\psi|^2)(\boldsymbol{G}+\boldsymbol{z}+\nabla\psi)\|_{\boldsymbol{C}^\beta(\overline{\Omega})}+\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}\Big).$$

If we put $\boldsymbol{F}=\boldsymbol{G}+\boldsymbol{z}\in\boldsymbol{C}^{1+\alpha}(\overline{\Omega})$, then we note that

$$\|\boldsymbol{G}\|_{\boldsymbol{C}^{1+\alpha}(\overline{\Omega})}+\|\boldsymbol{z}\|_{\boldsymbol{C}^{1+\alpha}(\overline{\Omega})}\le C\Big(\|\boldsymbol{f}\|_{\boldsymbol{C}^\alpha(\overline{\Omega})}+\|\boldsymbol{G}\|_{\boldsymbol{C}^0(\overline{\Omega})}\Big).$$

Now we consider the regularity of weak solution $\psi\in W^{1,q}(\Omega)$ of the equation (4.5). In order to do so, we define

$$\boldsymbol{A}(x,\boldsymbol{p})=f(x,|\boldsymbol{F}(x)+\boldsymbol{p}|^2)(\boldsymbol{F}(x)+\boldsymbol{p}),\quad (\boldsymbol{p}=(p_1,p_2,p_3)\in\mathbb{R}^3).$$

Then (4.5) is written by

$$\begin{cases} \mathrm{div}\boldsymbol{A}(x,\nabla\psi)=0 & \text{in } \Omega, \\ \boldsymbol{n}\cdot\boldsymbol{A}(x,\nabla\psi)=0 & \text{on } \Gamma. \end{cases} \tag{4.7}$$

We show the following structure conditions in [15]: There exist positive constants $c,C>0$ such that

$$\begin{cases} \boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p}\ge c|\boldsymbol{p}|^q-Cg_1(x), \\ |\boldsymbol{A}(x,\boldsymbol{p})|\le C(|\boldsymbol{p}|^{q-1}+g_2(x)), \end{cases} \tag{4.8}$$

where $g_i\ge 0$ and $g_i\in L^\infty(\Omega)$ for $i=1,2$. In fact,

$$\boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p}=f(x,|\boldsymbol{F}(x)+\boldsymbol{p}|^2)(\boldsymbol{F}(x)+\boldsymbol{p})\cdot\boldsymbol{p}$$

$$
\begin{aligned}
&= f(x,|\boldsymbol{F}(x)+\boldsymbol{p}|^2)(\boldsymbol{F}(x)+\boldsymbol{p})\cdot(\boldsymbol{F}(x)+\boldsymbol{p}) - f(x,|\boldsymbol{F}(x)+\boldsymbol{p}|^2)(\boldsymbol{F}(x)+\boldsymbol{p})\cdot\boldsymbol{F}(x)\\
&\geq f(x,|\boldsymbol{F}(x)+\boldsymbol{p}|^2)|\boldsymbol{F}(x)+\boldsymbol{p}|^2 - f(x,|\boldsymbol{F}(x)+\boldsymbol{p}|^2)|\boldsymbol{F}(x)+\boldsymbol{p}||\boldsymbol{F}(x)|.
\end{aligned}
$$

It follows from Lemma 2.1 and the Young inequality that

$$\boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p} \geq \Lambda^{-(q-1)}|\boldsymbol{F}(x)+\boldsymbol{p}|^q - \varepsilon|\boldsymbol{F}(x)+\boldsymbol{p}|^q - C(\varepsilon)|\boldsymbol{F}(x)|^q$$

for any $\varepsilon>0$. If we choose $\varepsilon>0$ small enough, then there exist constants $c,C>0$ such that

$$\boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p} \geq c|\boldsymbol{F}(x)+\boldsymbol{p}|^q - C|\boldsymbol{F}(x)|^q.$$

Since $|\boldsymbol{p}|^q = |\boldsymbol{F}(x)+\boldsymbol{p}-\boldsymbol{F}(x)|^q \leq 2^{q-1}(|\boldsymbol{F}(x)+\boldsymbol{p}|^q + |\boldsymbol{F}(x)|^q)$, we have

$$\boldsymbol{A}(x,\boldsymbol{p})\cdot\boldsymbol{p} \geq \frac{c}{2^{q-1}}|\boldsymbol{p}|^q - (C+1)|\boldsymbol{F}(x)|^q.$$

Since $\boldsymbol{F}\in C^{1+\alpha}(\overline{\Omega})$, we can see $|\boldsymbol{F}|^q \in L^\infty(\Omega)$. Thus the first inequality of (4.8) holds. For the second inequality of (4.8),

$$|\boldsymbol{A}(x,\boldsymbol{p})| \leq \lambda^{-(q-1)}|\boldsymbol{F}(x)+\boldsymbol{p}|^{q-1} \leq \lambda^{-(q-1)}\max\{1,2^{q-2}\}\Big(|\boldsymbol{p}|^{q-1} + |\boldsymbol{F}(x)|^{q-1}\Big).$$

Here we used the inequalities: for $q\geq 2$,

$$|\boldsymbol{F}(x)+\boldsymbol{p}|^{q-1} \leq 2^{q-2}(|\boldsymbol{F}(x)|^{q-1} + |\boldsymbol{p}|^{q-1}),$$

and for $1<q<2$,

$$|\boldsymbol{F}(x)+\boldsymbol{p}|^{q-1} \leq |\boldsymbol{F}(x)|^{q-1} + |\boldsymbol{p}|^{q-1}.$$

In order to apply [15, Theorem 1.3, p. 43], we have to show the global boundedness of a weak solution of (4.7) under the structural conditions.

4.1 Global boundedness of weak solution

Proposition 4.1. *Consider the equation* (4.7) *under structure conditions* (4.8) *and let* $\psi \in W^{1,q}(\Omega)$ *be a weak solution of* (4.7). *Then* $\psi \in L^\infty(\Omega)$, *and there exists a constant* $C>0$ *such that*

$$\sup_\Omega |\psi| \leq C\Big(\|\psi\|_{L^q(\Omega)} + \|g_1\|^{1/q}_{L^\infty(\Omega)}\Big).$$

Proof. For $s>1$ and $Z\geq 1$, define a function

$$\gamma(t) = \begin{cases} \min\{t,Z\}^{qs-q} & \text{if } t\geq 0,\\ 0 & \text{if } t<0. \end{cases} \tag{4.9}$$

Then $\gamma(t)$ is a bounded function in t and there exists a constant $C_\gamma>0$ such that

$$0 \leq \gamma'(t)t \leq C_\gamma \gamma(t) \quad \text{a.e. } t. \tag{4.10}$$

We note that we can take $C_\gamma = qs$. We take $\varphi = \gamma(\psi)\psi_+$ as a test function of (4.7), where $\psi_+ = \max\{0,\psi\}$. Since

$$\nabla\varphi = \begin{cases} (\gamma'(\psi)\psi + \gamma(\psi))\nabla\psi & \text{if } \psi \geq 0, \\ 0 & \text{if } \psi < 0, \end{cases}$$

it follows from (4.8) and (4.10) that

$$\begin{aligned} \boldsymbol{A}(x,\nabla\psi)\cdot\nabla\varphi &\geq (\gamma'(\psi)\psi + \gamma(\psi))(c|\nabla\psi|^q - Cg_1(x)) \\ &\geq c\gamma(\psi)|\nabla\psi|^q - C(1+C_\gamma)\gamma(\psi)g_1(x). \end{aligned}$$

Since

$$\int_\Omega \boldsymbol{A}(x,\nabla\psi)\cdot\nabla\varphi \mathrm{d}x = 0,$$

we have

$$c\int_\Omega \gamma(\psi)|\nabla\psi|^q \mathrm{d}x \leq C(1+C_\gamma)\int_\Omega \gamma(\psi)g_1(x)\mathrm{d}x. \tag{4.11}$$

Now define $h = \chi_{\{\psi>0\}}\min\{\psi,Z\}^{s-1}\psi = \min\{\psi,Z\}^{s-1}\psi_+$, where $\chi_{\{\psi>0\}}$ is the characteristic function of the set $\{x\in\Omega;\psi(x)>0\}$. Then we see that

$$\nabla h = \begin{cases} s\psi^{s-1}\nabla\psi & \text{if } 0<\psi\leq Z, \\ Z^{s-1}\nabla\psi & \text{if } \psi > Z, \\ 0 & \text{if } \psi \leq 0. \end{cases}$$

Thus $|\nabla h|^q = s^q\gamma(\psi)|\nabla\psi|^q$ and $h^q = \gamma(\psi)\psi^q$. From the Gagliardo-Nirenberg inequality (cf. Lieberman [17, Theorem 5.8]), for $N > q \geq 1, N \geq 3$, there exists a constant $C = C(\Omega,N,q)$ such that

$$\left(\int_\Omega |v|^{Nq/(N-q)}\mathrm{d}x\right)^{(N-q)/N} \leq C\left(\int_\Omega |v|^q \mathrm{d}x\right)^{(N-3)/N}\left(\int_\Omega (|\nabla v|^q + |v|^q)\mathrm{d}x\right)^{3/N}$$

for any $v\in W^{1,q}(\Omega)$. Put $\kappa = N/(N-q)(>1)$. Since $h\in W^{1,q}(\Omega)$, by applying this inequality, and using inequality $(a+b)^p \leq a^p + b^p$ for $0<p\leq 1, a,b\geq 0$, (4.11) and the Young inequality, we have

$$\begin{aligned} &\left(\int_\Omega |h|^{\kappa q}\mathrm{d}x\right)^{1/\kappa} \leq C\left(\int_\Omega |h|^q\mathrm{d}x\right)^{(N-3)/N}\left(\int_\Omega (|\nabla h|^q + |h|^q)\mathrm{d}x\right)^{3/N} \\ &\leq C\left[\int_\Omega |h|^q \mathrm{d}x + \left(\int_\Omega |h|^q\mathrm{d}x\right)^{(N-3)/N}\left(\int_\Omega |\nabla h|^q\mathrm{d}x\right)^{3/N}\right] \\ &\leq C\left[\int_\Omega \gamma(\psi)\psi^q\mathrm{d}x + \left(\int_\Omega \gamma(\psi)\psi^q\mathrm{d}x\right)^{(N-3)/N}\left(qs^{q+1}\int_\Omega \gamma(\psi)g_1(x)\mathrm{d}x\right)^{3/N}\right] \\ &\leq C\left[\int_\Omega \gamma(\psi)\psi^q\mathrm{d}x + q^{3/N}\left(s^{3(q+1)/(N-3)}\int_\Omega \gamma(\psi)\psi^q\mathrm{d}x\right)^{(N-3)/N}\left(\int_\Omega \gamma(\psi)g_1(x)\mathrm{d}x\right)^{3/N}\right] \end{aligned}$$

$$\leq Cs^{3(q+1)/(N-3)}\int_\Omega \gamma(\psi)\psi^q \mathrm{d}x + C\int_\Omega \gamma(\psi) g_1 \mathrm{d}x.$$

Now since $\min\{\psi, Z\} \leq \psi$ for $\psi > 0$, so $\frac{\psi}{\min\{\psi,Z\}} \geq 1$ and $\kappa > 1$, we have

$$\begin{aligned}
h^{q\kappa} &= \gamma(\psi)^\kappa \psi^{q\kappa} \\
&= \chi_{\{\psi>0\}} \min\{\psi,Z\}^{q\kappa(s-1)}\psi^{q\kappa} = \chi_{\{\psi>0\}}\min\{\psi,Z\}^{q\kappa s}\left(\frac{\psi^q}{\min\{\psi,Z\}^q}\right)^\kappa \\
&\geq \chi_{\{\psi>0\}}\min\{\psi,Z\}^{q\kappa s}\frac{\psi^q}{\min\{\psi,Z\}^q}.
\end{aligned}$$

Therefore we have

$$\left(\int_\Omega \min\{\psi,Z\}^{q\kappa s}\chi_{\{\psi>0\}}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x\right)^{1/\kappa} \leq \left(\int_\Omega h^{q\kappa}\mathrm{d}x\right)^{1/\kappa}. \tag{4.12}$$

On the other hand, since $\frac{\psi}{\min\{\psi,Z\}} \geq 1$ for $\psi > 0$, we can write

$$\begin{aligned}
\int_\Omega \gamma(\psi)\psi^q \mathrm{d}x &= \int_\Omega \min\{\psi,Z\}^{qs}\chi_{\{\psi>0\}}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x, \qquad (4.13)\\
\int_\Omega \gamma(\psi) g_1 \mathrm{d}x &= \int_\Omega \min\{\psi,Z\}^{qs-q} g_1 \chi_{\{\psi>0\}}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x.
\end{aligned}$$

Here we define a measure $\mathrm{d}\mu = \chi_{\{\psi>0\}}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x$. Then using the Hölder inequality and the Young inequality, we can write

$$\begin{aligned}
\int_\Omega \gamma(\psi) g_1 \mathrm{d}x &\leq \left(\int_\Omega \min\{\psi,Z\}^{qs}\mathrm{d}\mu\right)^{(qs-q)/qs}\left(\int_\Omega g_1^s \mathrm{d}\mu\right)^{1/s} \\
&\leq \int_\Omega g_1^s \mathrm{d}\mu + \int_\Omega \min\{\psi,Z\}^{qs}\mathrm{d}\mu. \qquad (4.14)
\end{aligned}$$

If we put $w = \min\{\psi_+, Z\}^q$, it follows from (4.12) that

$$\left(\int_\Omega w^{\kappa s}\mathrm{d}\mu\right)^{1/\kappa} \leq \left(\int_\Omega h^{q\kappa}\mathrm{d}x\right)^{1/\kappa} \leq C_1 s^{3(q+1)/(N-3)}\int_\Omega w^s \mathrm{d}\mu + C_2\int_\Omega g_1^s \mathrm{d}\mu.$$

We choose N so that $m := 3(q+1)/(N-3) \geq 1$. Then we see that

$$\left(\int_\Omega w^{\kappa s}\mathrm{d}\mu\right)^{1/\kappa} \leq Cs^m\left\{\int_\Omega w^s \mathrm{d}\mu + \int_\Omega g_1^s \mathrm{d}\mu\right\}. \tag{4.15}$$

Since $s > 1$, it follow from (4.15) that

$$\left(\int_\Omega w^{\kappa s}\mathrm{d}\mu + \|g_1\|^{\kappa s}_{L^\infty(\Omega,d\mu)}\right)^{1/\kappa} \leq Cs^m\left\{\int_\Omega w^s \mathrm{d}\mu + \|g_1\|^s_{L^\infty(\Omega,d\mu)}\right\}.$$

We apply the following proposition due to the Moser iteration method, whose proof appears in [6, Appendix A] (cf. [17, Lemma 5.30]).

Proposition 4.2. *Let (X,dv) be a measure space and suppose that $0\le w\in L^p(X,dv)$ for all $p\ge 1$. Assume that there exist constants $C>0,\kappa>1,K\ge 0,m\ge 1,s_0\ge 1$ such that for all $s\ge s_0$,*

$$\left(\int_X w^{\kappa s}\mathrm{d}\nu+K^{\kappa s}\right)^{1/\kappa}\le Cs^m\left(\int_X w^s\mathrm{d}\nu+K^s\right). \tag{4.16}$$

Then we can see that $w\in L^\infty(\Omega,dv)$, and there exists a constant $C_1>0$ such that

$$\|w\|_{L^\infty(\Omega,dv)}\le C_1\big(\|w\|_{L^1(\Omega,dv)}+K\big).$$

We apply this proposition with $X=\Omega,dv=d\mu$ as above, $w=\min\{\psi_+,Z\}^q,\kappa=N/(N-q)(>1),s_0=m=3(q+1)/(N-3)(\ge 1)$ and $K=\|g_1\|_{L^\infty(\Omega,\mu)}$. Hence we can see that $w\in L^\infty(\Omega,d\mu)$ and

$$\|w\|_{L^\infty(\Omega,d\mu)}\le C\Big(\|w\|_{L^1(\Omega,d\mu)}+\|g_1\|_{L^\infty(\Omega,d\mu)}\Big).$$

Here we note

$$\begin{aligned}\int_\Omega g_1^s\mathrm{d}\mu&=\int_\Omega g_1^s\chi_{\{\psi>0\}}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x=\|g_1\|^s_{L^\infty(\Omega)}\int_\Omega\chi_{\{\psi>0\}}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x\\&=\|g_1\|^s_{L^\infty(\Omega)}\left\{\int_{0<\psi\le Z}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x+\int_{\psi\ge Z}\frac{\psi^q}{\min\{\psi,Z\}^q}\mathrm{d}x\right\}\\&\le\|g_1\|^s_{L^\infty(\Omega)}(|\Omega|+Z^{-q}\|\psi\|^q_{L^q(\Omega)}).\end{aligned}$$

Thus we have

$$\left(\int_\Omega g_1^s\mathrm{d}\mu\right)^{1/s}\le\|g_1\|_{L^\infty(\Omega)}(|\Omega|+\|\psi\|^q_{L^q(\Omega)})^{1/s}.$$

Letting $s\to\infty$, we get $\|g_1\|_{L^\infty(\Omega,d\mu)}\le\|g_1\|_{L^\infty(\Omega)}$. Also, we have $\|w\|_{L^1(\Omega,d\mu)}=\|\psi_+\|^q_{L^q(\Omega)}$. From Proposition 4.2, we can see that

$$\min\{\psi_+,Z\}\le C\Big(\|\psi_+\|_{L^q(\Omega)}+\|g_1\|^{1/q}_{L^\infty(\Omega)}\Big).$$

Since $Z\ge 1$ is arbitrary, letting $Z\to\infty$, we get

$$\sup_\Omega\psi\le\sup_\Omega\psi_+\le C\Big(\|\psi_+\|_{L^q(\Omega)}+\|g_1\|^{1/q}_{L^\infty(\Omega)}\Big).$$

Since $g_1=|\boldsymbol{F}(x)|^q$, we have

$$\sup_\Omega\psi\le\sup_\Omega\psi_+\le C\big(\|\psi_+\|_{L^q(\Omega)}+\|\boldsymbol{F}\|_{L^\infty(\Omega)}\big).$$

Since $\boldsymbol{A}(x,\boldsymbol{p})=f(x,|\boldsymbol{F}(x)+\boldsymbol{p}|^2)(\boldsymbol{F}(x)+\boldsymbol{p})$, if we replace $\boldsymbol{F}$ with $-\boldsymbol{F}$ in (4.7), then $-\psi$ is a weak solution of (4.7). Therefore from the above arguments, we get

$$\sup_\Omega(-\psi)\le\sup_\Omega\psi_-\le C\big(\|\psi_-\|_{L^q(\Omega)}+\|\boldsymbol{F}\|_{L^\infty(\Omega)}\big).$$

As a result, we get

$$\sup_{\Omega}|\psi|\le C(\|\psi\|_{L^q(\Omega)}+\|\boldsymbol{F}\|_{L^\infty(\Omega)}).$$

Thus we have proved the global boundedness of the weak solution ψ of the equation (4.7). □

4.2 Hölder continuity of weak solution of (4.7)

According to (4.8), the structural conditions (A_1)-(A_5) in [15] hold. Since the weak solution ψ of (4.7) is globally bounded, it follows from [15, Theorem 1.3] that ψ is Hölder continuous in $\overline{\Omega}$, and there exists constants $C>0$ and $\gamma\in(0,1)$ such that

$$|\psi(x)-\psi(y)|\le C\|\psi\|_{L^\infty(\Omega)}|x-y|^\gamma,\quad x,y\in\overline{\Omega}.$$

4.3 Hölder continuity of the gradient of weak solution of (4.7)

We follows the idea of Lieberman [18]. When $\boldsymbol{F}(x)=\boldsymbol{k}=$constant, we put $\phi(x)=\boldsymbol{k}\cdot x$, and $\varphi=\phi+\psi$. Then we can reduce equation (4.7) into the form

$$\begin{cases} \operatorname{div}\boldsymbol{B}(x,\nabla\varphi)=0 & \text{in } \Omega,\\ \boldsymbol{n}\cdot\boldsymbol{B}(x,\nabla\varphi)=0 & \text{on } \Gamma \end{cases} \tag{4.17}$$

where $\boldsymbol{B}(x,\boldsymbol{p})=(B_1(x,\boldsymbol{p}),B_2(x,\boldsymbol{p}),B_3(x,\boldsymbol{p}))=f(x,|\boldsymbol{p}|^2)\boldsymbol{p}$. We see that

$$\sum_{i,j=1}^{3}\frac{\partial B_i}{\partial p_j}\xi_i\xi_j=f(x,|\boldsymbol{p}|^2)|\xi|^2+2f_\rho(x,|\boldsymbol{p}|^2)(\boldsymbol{p}\cdot\xi)^2.$$

Here $f_\rho(x,|\boldsymbol{p}|^2)$ may change the sign. When $f_\rho(x,|\boldsymbol{p}|^2)\ge 0$, from Lemma 2.1,

$$\sum_{i,j=1}^{3}\frac{\partial B_i}{\partial p_j}\xi_i\xi_j\ge f(x,|\boldsymbol{p}|^2)|\xi|^2\ge\Lambda^{-(q-1)}|\boldsymbol{p}|^{q-2}|\xi|^2.$$

When $f_\rho(x,|\boldsymbol{p}|^2)<0$,

$$\begin{aligned}\sum_{i,j=1}^{3}\frac{\partial B_i}{\partial p_j}\xi_i\xi_j&\ge f(x,|\boldsymbol{p}|^2)|\xi|^2+2f_\rho(x,|\boldsymbol{p}|^2)|\boldsymbol{p}|^2|\xi|^2\\&=\{f(x,|\boldsymbol{p}|^2)+2f_\rho(x,|\boldsymbol{p}|^2)|\boldsymbol{p}|^2\}|\xi|^2.\end{aligned}$$

Since $f(x,\rho)+2\rho f_\rho(x,\rho)\ge c\rho^{(q-2)/2}$ for some $c>0$, there exists a constant $c>0$ such that

$$\sum_{i,j=1}^{3}\frac{\partial B_i}{\partial p_j}\xi_i\xi_j\ge c|\boldsymbol{p}|^{q-2}|\xi|^2.$$

On the other hand,

$$\left|\frac{\partial B_i}{\partial p_j}\right| \le f(x,|\boldsymbol{p}|^2)+2|f_\rho(x,|\boldsymbol{p}|^2)||\boldsymbol{p}|^2.$$

If we use the relations $f(x,\rho)\le\lambda^{-(q-1)}\rho^{(q-2)/2}$ by Lemma 2.1 and

$$\begin{aligned}\rho|f_\rho(x,\rho)|&=\frac{\Phi(x,t)|S_{tt}(x,t)|}{(S_t(x,t))^2\Phi_t(x,t)}=\frac{\Phi(x,t)|S_{tt}(x,t)|}{(S_t(x,t))^2S_t(x,t)(S_t(x,t)+2tS_{tt}(x,t))}\\&\le C\frac{t^{p-1}t^{(p-4)/2}}{t^{3(p-2)/2}t^{(p-2)/2}}\le C_1t^{-(p-2)/2}\le C_2\rho^{-(p-2)/(2(p-1))}\le C_3\rho^{(q-2)/2},\end{aligned}$$

we have

$$\left|\frac{\partial B_i}{\partial p_j}\right|\le C|\boldsymbol{p}|^{q-2}.$$

Finally, from (2.6), we have

$$|\boldsymbol{B}(x,\boldsymbol{p})-\boldsymbol{B}(y,\boldsymbol{p})|\le C|x-y||\boldsymbol{p}|^{q-1}.$$

Thus we see that the structural conditions of [18] hold. Therefore, when $\boldsymbol{F}(x)=\boldsymbol{k}=$ constant, if we apply [18, Theorem 2], there exist $\beta\in(0,1)$ and a constant C dependent on $\lambda,\Lambda,p,\sup_{\overline{\Omega}}|\psi|$ and Ω such that $\varphi\in C^{1+\beta}(\overline{\Omega})$, and

$$\|\varphi\|_{C^{1+\beta}(\overline{\Omega})}\le C. \tag{4.18}$$

Next we use the perturbation method to verify the regularity of weak solution of (4.7). Fix $x_0\in\Omega$ and choose a ball $B_{R_0}(x_0)$ with center x_0 and radius $R_0>0$ such that $B_{R_0}(x_0)\subset\Omega$. For any $0<R\le R_0$, we consider the following equation

$$\begin{cases}\operatorname{div}[f(x_0,|\boldsymbol{F}(x_0)+\nabla\overline{\psi}|^2)(\boldsymbol{F}(x_0)+\nabla\overline{\psi})]=0 & \text{in } B_R(x_0),\\ \overline{\psi}=\varphi & \text{on } \partial B_R(x_0),\end{cases} \tag{4.19}$$

where $\partial B_R(x_0)$ denotes the boundary of $B_R(x_0)$. By [18, Lemma 5], equation (4.19) has a unique solution $\overline{\psi}$ in $W^{1,q}(B_R(x_0))$. Moreover, $\overline{\psi}\in C^{1+\beta}(\overline{B_R(x_0)})$ and satisfies

$$\|\overline{\psi}\|_{C^{1+\beta}(\overline{B_R(x_0)})}\le C.$$

Using this fact and the perturbation method, the weak solution ψ of (4.7) is in $C^{1+\beta}_{\rm loc}(\Omega)$ for some $\beta\in(0,1)$ and for any $\Omega'\Subset\Omega$, there exists a constant C depending only on the known data and $\operatorname{dist}(\Omega',\partial\Omega)$ such that

$$\|\psi\|_{C^{1+\beta}(\overline{\Omega'})}\le C.$$

Here we use a variant of the perturbation method developed by Choe [19, pp. 36-38] (cf. [6, Appendix B]). Finally $C^{1+\beta}$ regularity near the boundary Γ follows from [18, Lemma 6] and the perturbation method.

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