On a Nonlinear Heat Equation with Degeneracy on the Boundary

ZHAN Huashui∗

School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China.

Received 29 May 2017; Accepted 11 March 2019

Abstract. The paper studies the stability of weak solutions of a nonlinear heat equation with degenerate on the boundary. A new kind of weak solutions are introduced. By the new weak solution, the stability of weak solutions is proved only dependent on the initial value.

AMS Subject Classifications: 35L65 35R35
Chinese Library Classifications: O175.26
Key Words: Nonlinear heat equation; weak solution; stability, initial value.

1 Introduction

Consider the nonlinear heat equation

\[ u_t = \text{div}(k(u,x,t)\nabla u) + f(u,x,t), \quad (x,t) \in Q_T = \Omega \times (0,T), \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with appropriately smooth boundary, the function \( k(u,x,t) \) has the meaning of nonlinear thermal conductivity, which depends on the temperature \( u = u(x,t) \). It is generally assume that the matrix \( k(u,x,t) \) is semidefinite. If \( k(u,x,t) = k(x) \), Eq. (1.1) becomes a linear parabolic equation, we would like to suggest that, for linear equations, any boundedness estimate is equivalent to a stability result (i.e., control of differences of solutions in terms of differences of data), but this is not the truth for nonlinear equations generally. One can see the well-known monographs or textbooks [1–7] and the references therein. However, in some special case, if we add some restrictions to \( k(u,x,t) \), the character may be still true. For simplicity, the paper limits to consider

\[ k(u,x,t) = ma(x)u^{m-1}, \quad m > 0, \]

∗Corresponding author. Email address: 2012111007@xmut.edu.cn (H.S. Zhan)
On a Nonlinear Heat Equation with Degeneracy on the Boundary

with \( a(x) > 0 \) when \( x \in \Omega \), but \( a(x) = 0 \) when \( x \in \partial \Omega \). In other words, we will consider the following nonlinear equation

\[
    u_t = \text{div}(a(x) \nabla u^m) + f(u, x, t), \quad (x, t) \in Q_T. \tag{1.2}
\]

From my own perspective, the initial value

\[
    u(x, 0) = u_0(x), \quad x \in \Omega, \tag{1.3}
\]

is indispensable. While, the usual boundary value

\[
    u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \tag{1.4}
\]

may be superfluous. To see that, if \( f \equiv 0 \) in (1.2), we suppose that \( u \) and \( v \) are two classical solutions of equation (1.2) with the initial values \( u(x, 0) \) and \( v(x, 0) \) respectively. Then we have

\[
    \int_{\Omega} S_\eta(u^m - v^m)(u - v) dx + \int_{\Omega} a(x) S'_\eta(u^m - v^m) |\nabla u^m - \nabla v^m|^2 dx \\
    = \int_{\partial \Omega} a(x) S_\eta(u^m - v^m)(\nabla u - \nabla v) \cdot \vec{n} d\Sigma = 0,
\]

where \( \vec{n} \) is the outer unit normal vector of \( \Omega \), \( S_\eta(s) \) is the approximate function of the sign function (the details are given (3.1)-(3.2) later). Then

\[
    \int_{\Omega} S_\eta(u^m - v^m)(u - v) dx \leq 0,
\]

\[
    \lim_{\eta \to 0} \int_{\Omega} S_\eta(u^m - v^m)(u - v) dx = \int_{\Omega} \text{sign}(u^m - v^m)(u - v) dx
\]

\[
    = \int_{\Omega} \text{sign}(u - v)(u - v) dx = \frac{d}{dt} \int_{\Omega} |u - v| dx.
\]

Then, even without any boundary value condition (1.4), the classical solutions have the stability

\[
    \int_{\Omega} |u(x, t) - v(x, t)| dx \leq \int_{\Omega} |u_0(x) - v_0(x)| dx. \tag{1.5}
\]

Certainly, since \( |\nabla u^m| \) may be singular or degenerate on \( \overline{\Omega} \), equation (1.2) only has a weak solution generally.

Thus, to study the well-posedness of weak solutions to equation (1.2), or a more general reaction-diffusion equation with the type

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x_i} \left( a^{ij}(u, x, t) \frac{\partial u}{\partial x_j} \right) + \text{div}(b(u, x, t)) + f(u, x, t), \quad (x, t) \in \Omega \times (0, T), \tag{1.6}
\]

the whole boundary value condition (1.4) is overdetermined. For the linear case, the problem had been completely solved by Fichera [8], Oleinik [9] et al., for nonlinear case,
the situation is far from success. Many important papers, for examples, references [8,10–17] have been devoting to the corresponding problem. In our paper, we introduce a new kind of weak solutions for Eq. (1.6). By the new definition, we can deal with the stability, or the uniqueness of the weak solutions independent of the boundary value condition.

**Definition 1.1.** A function \( u(x,t) \) is said to be a weak solution of Eq. (1.2) with the initial value (1.3), if

\[
\begin{aligned}
u &\in L^\infty(Q_T), \quad u_t \in L^2(Q_T), \quad a(x)|\nabla u_m|^2 \in L^1(Q_T), \\
\int\int_{Q_T} [u_t g(\varphi_1 \varphi_2) + a(x) \nabla u_m \cdot \nabla g(\varphi_1 \varphi_2) - f(u,x,t) g(\varphi_1 \varphi_2)] \, dx \, dt = 0,
\end{aligned}
\]

and the initial value is satisfies in the sense of that

\[
\lim_{t \to 0} \int_\Omega |u(x,t) - u_0(x)| \, dx = 0.
\]

In what follows, we suppose that \( m > 0 \) and

\[
u_0 \in L^\infty(\Omega), \quad a(x) \nabla u_0^m \in L^2(\Omega).
\]

In the first place, we would like to suggest that, the existence of Eq. (1.2) in the sense of Definition 1.1 is easily to be obtained, we will give a basic result and the skeleton of the proof in the second section of the paper. The paper mainly follows with interest in the stability of weak solutions without any boundary value condition.

**Theorem 1.1.** Let \( u,v \) be two solutions of Eq. (1.2) with the different initial value \( u_0(x), v_0(x) \) respectively. If

\[
\int_\Omega a^{-1}(x) \, dx < \infty, \quad |f(u,x,t) - f(v,x,t)| \leq c |u - v|,
\]

then

\[
\int_\Omega |u(x,t) - v(x,t)| \leq c \int_\Omega |u_0(x) - v_0(x)| \, dx, \, a.e. \, t \in [0,T).
\]

**Theorem 1.2.** Let \( u,v \) be two solutions of Eq. (1.2) with the different initial value \( u_0(x), v_0(x) \) respectively. If \( f \) satisfies (1.12), \( a(x) \) satisfies

\[
\frac{1}{\eta^2} \int_{\Omega \setminus \Omega_\eta} a(x) \, dx \leq c,
\]

then the stability (1.13) is true. Here \( \Omega_\eta = \{ x \in \Omega : d(x, \partial \Omega) > \eta \} \) for small \( \eta > 0 \).
We give a brief explanation of the conditions (1.11) and (1.14). Let \( a(x) = d^\alpha(x), d(x) = \text{dist}(x, \partial \Omega), \alpha > 0 \) be a constant. Then the condition (1.11) implies that \( 0 < \alpha < 1 \); while the condition (1.14) means that \( \alpha > 1 \). Thus, Theorem 1.3 is the complement of Theorem 1.1. It is astonished that, if \( a(x) = d^\alpha(x) \), our results exclude the case \( \alpha = 1 \).

At last, we would like to suggest that if \( \{ x \in \Omega : a(x) = 0 \} \) has the interior point in \( \Omega \), then the equation is strongly degenerate. In such case, only in the sense of the entropy solutions, the stability (or the uniqueness) of the weak solutions can be proved. One can refer to the references [8, 10–17] etc. In the last section of our paper (Section 5), a special case of equation (1.6) is considered.

So, the main results listed above (Theorems 1.1-1.2) not only clarify that, if the equation is weakly degenerate (i.e. there is not the interior point in the set \( \{ x \in \Omega : a(x) = 0 \} \subset \Omega \)), then the usual boundary value condition (1.3) is overdetermined, but also show that the stability of weak solutions may be proved without any boundary value condition. Certainly, if the equation is strongly degenerate, whether the stability of the solutions can be obtained without any boundary value condition is still an open problem.

2 The existence of the solution

In general, the weak solution is defined as follows.

**Definition 2.1.** A nonnegative function \( u(x, t) \) is said to be a weak solution of equation (1.2) with the initial value (1.3), if \( u \) satisfies (1.7) and for any function \( \varphi \in C^1_0(Q_T), \)
\[
\iint_{Q_T} (u_t \varphi + a(x) \nabla u \cdot \nabla \varphi) dxdt = \iint_{Q_T} f(u, x, t) \varphi(x, t) dxdt, \tag{2.1}
\]
the initial value (1.3) is true in the sense of (1.9).

**Theorem 2.1.** If \( u_0(x) \) satisfies
\[
0 \leq u_0 \in L^\infty(\Omega), a(x) \nabla u_0 \in L^2(\Omega), \tag{2.2}
\]
f \( f(s, x, t) \geq 0 \) when \( s < 0 \), then there exists a nonnegative solution of equation (1.2) with the initial value (1.3) in the sense of Definition 2.1.

**Proof.** We consider the following regularization problem
\[
\begin{align*}
    u_{mt} &= \text{div} \left( (a(x) + \frac{1}{n}) \nabla u_m \right) + f(u_m, x, t), & (x, t) &\in Q_T, \\
    u_0(x, t) &= 0, & (x, t) &\in \partial \Omega \times (0, T), \\
    u_0(x, 0) &= u_0(x), & x &\in \Omega.
\end{align*} \tag{2.3}
\]
By the monotone convergent method [18], the initial-boundary value problem (2.3) has a nonnegative weak solution $u_n(x,t)$,

$$u_n(x,t) \in L^\infty(\Omega), \quad u_{nt} \in L^2(Q_T), \quad u_{nt}^m \in L^2(0,T;W_0^{1,2}(\Omega)),$$

$$\|u_n(x,t)\|_{L^\infty(Q_T)} \leq c. \quad (2.4)$$

Let us multiply $u_{nt}^m$ on both sides of the equation, integrate it over $Q_T$. Then

$$\frac{1}{m+1} \int \int_{Q_T} u_{nt}^{m+1} dx + \int \int_{Q_T} \left( a(x) + \frac{1}{n} \right) |\nabla u_{nt}^m|^2 dx dt + \int \int_{Q_T} f(u_n,x,t) u_{nt}^m dx dt = \frac{1}{m+1} \int \int_{\Omega} u_{nt}^m dx,$$

accordingly,

$$\int \int_{Q_T} \left( a(x) + \frac{1}{n} \right) |\nabla u_{nt}^m|^2 dx dt \leq c. \quad (2.5)$$

Let us multiply $u_{nt}$ on both sides of the equation, integrate it over $Q_T$. Then

$$\frac{1}{2} \int \int_{Q_T} u_{nt}^2 dx dt = \int \int_{Q_T} \text{div} \left( \left( a(x) + \frac{1}{n} \right) \nabla u_{nt}^m \right) u_{nt} dx dt + \int \int_{Q_T} f(u_n,x,t) u_{nt} dx dt,$$

$$\int \int_{Q_T} \text{div} \left( \left( a(x) + \frac{1}{n} \right) \nabla u_{nt}^m \right) u_{nt} dx dt = - \int \int_{Q_T} \left( a(x) + \frac{1}{n} \right) \nabla u_{nt}^m \cdot \nabla u_{nt} dx dt$$

$$= - \frac{m}{2} \int \int_{Q_T} \left( a(x) + \frac{1}{n} \right) u_{nt}^{m-1} \frac{d}{dt} |\nabla u_n|^2 dx dt \quad (2.6)$$

$$\left| \int \int_{Q_T} f(u_n,x,t) u_{nt} dx dt \right| \leq c + \frac{1}{4} \int \int_{Q_T} u_{nt}^2 dx dt \quad (2.7)$$

By (2.6)-(2.7), we have

$$\int \int_{Q_T} u_{nt}^2 dx dt \leq c. \quad (2.8)$$

Thus, by (2.4), (2.5) and (2.8), there exists a function $u(x,t) \in L^\infty(Q_T)$ and a $n$-dimensional vector $\zeta = (\zeta_1, \cdots, \zeta_n)$ such that $u_n \rightarrow u$ a.e. in $Q_T$, and

$$u \in L^\infty(Q_T), \quad \frac{\partial u}{\partial t} \in L^2(Q_T), \quad |\zeta| \in L^2(Q_T),$$

$u_n \rightarrow u$, weakly star in $L^\infty(Q_T)$, \quad $u_{nt} \rightarrow u$, in $L^2_{0_{loc}}(Q_T),$

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in} \quad L^2\left(Q_T, \sqrt{a(x) + \frac{1}{n} \nabla u_n^m} \rightarrow \zeta \right) \quad \text{in} \quad L^2\left(Q_T\right).$$
As the usual porous medium equation [5], we can prove that
\[ \zeta = \sqrt{a(x) \nabla u^m}. \]
Also, we can prove that (1.9) is true. Thus, \( u \) is a solution of equation (1.2) with the initial value (1.3) in the sense of Definition 2.1.

**Theorem 2.2.** If \( u_0(x) \) satisfies (2.3), \( f(s,x,t) \geq 0 \) when \( s < 0 \), then there exists a nonnegative solution of equation (1.2) with the initial value (1.3) in the sense of Definition 1.1.

**Proof.** According to Theorem 2.1, for any function \( \phi \in C_0^1(Q_T) \) and for any function \( g(s) \in C^1(\mathbb{R}), g(0) = 0 \), we have
\[
\iint_{Q_T} [u_t g(\phi) + a(x) \nabla u^m \nabla g(\phi)] \, dx \, dt = \iint_{Q_T} f(u,x,t) g(\phi) \, dx \, dt.
\]
If we denote \( \Omega_\phi = \text{supp} \phi \), then
\[
\int_0^T \int_{\Omega_\phi} \left[ u_t g(\phi) + a(x) \nabla u^m \nabla g(\phi) - f(u,x,t) g(\phi) \right] \, dx \, dt = 0.
\]
Now, for any function \( \phi_1 \in C_0^1(\Omega), \phi_2 \in L^\infty(0,T;W_{loc}^{1,2}(\Omega)) \), it is clearly that \( \phi_2 \in W^{1,2}(\Omega_{\phi_1}) \). By the fact of that \( C_0^\infty(\Omega_{\phi_1}) \) is dense in \( W^{1,2}(\Omega_{\phi_1}) \), by a process of limit, we have
\[
\int_0^T \int_{\Omega_{\phi_1}} \left[ u_t g(\phi_1 \phi_2) + a(x) \nabla u^m \nabla g(\phi_1 \phi_2) - f(u,x,t) g(\phi_1 \phi_2) \right] \, dx \, dt = 0.
\]
which implies that
\[
\int_0^T \int_{\Omega} \left[ u_t g(\phi_1 \phi_2) + a(x) \nabla u^m \nabla g(\phi_1 \phi_2) - f(u,x,t) g(\phi_1 \phi_2) \right] \, dx \, dt = 0.
\]
Thus, we have the conclusion. \( \square \)

### 3 Proof of Theorem 1.1

For small \( \eta > 0 \), let
\[
S_\eta(s) = \int_0^s h_\eta(\tau) \, d\tau, \quad h_\eta(s) = \frac{2}{\eta} \left( 1 - \frac{|s|}{\eta} \right)^+.
\] (3.1)

Obviously \( h_\eta(s) \in C(\mathbb{R}) \), and
\[
h_\eta(s) \geq 0, \quad |sh_\eta(s)| \leq 1, \quad |S_\eta(s)| \leq 1; \lim_{\eta \to 0} S_\eta(s) = \text{sgn}s, \lim_{\eta \to 0} S_\eta'(s) = 0.
\] (3.2)
Proof of Theorem 1.1. Let $u$, $v$ be two solutions of equation (1.2) with the initial values $u_0(x), v_0(x)$. We can choose $S_\eta(a^\beta(u^m - v^m))$ as the test function. Then

$$
\int \Omega S_\eta(a^\beta(u^m - v^m)) \frac{\partial(u-v)}{\partial t} \, dx + \int \Omega a^{\beta+1}(x)(\nabla u^m - \nabla v^m) \cdot \nabla (u^m - v^m) S_\eta'(a^\beta(u^m - v^m)) \, dx \\
+ \int \Omega a(x)(\nabla u^m - \nabla v^m) \cdot \nabla a^\beta(u^m - v^m) S_\eta'(a^\beta(u^m - v^m)) \, dx \\
+ \int \Omega [f(u,x,t) - f(v,x,t)] S_\eta(a^\beta(u^m - v^m)) \, dx = 0. \quad (3.3)
$$

Thus

$$
\lim_{\eta \to 0} \int \Omega S_\eta(a^\beta(u^m - v^m)) \frac{\partial(u-v)}{\partial t} \, dx = \int \Omega \text{sign}(a^\beta(u^m - v^m)) a \frac{\partial(u-v)}{\partial t} \, dx \\
= \int \Omega \text{sign}(u-v) a \frac{\partial(u-v)}{\partial t} \, dx = \frac{d}{dt} \int \Omega |u-v| \, dx, \quad (3.4)
$$

$$
\int \Omega a^{\beta+1}(x) |\nabla u^m - \nabla v^m| |\nabla u^m - \nabla v^m| S_\eta'(a^\beta(u^m - v^m)) \, dx \geq 0. \quad (3.5)
$$

By that $|\nabla a(x)| \leq c$ in $\Omega$, we have

$$
\left| \int \Omega a(x)(u^m - v^m) S_\eta'(a^\beta(u^m - v^m))(\nabla u^m - \nabla v^m) \cdot \nabla a^\beta \, dx \right| \\
\leq c \int \Omega a^\beta |u^m - v^m| S_\eta'(a^\beta(u^m - v^m)) |\nabla u^m - \nabla v^m| \, dx \\
= c \int \{\Omega a^\beta |u^m - v^m| < \eta\} a^{-\frac{1}{2}} a^\beta |u^m - v^m| S_\eta'(a^\beta(u^m - v^m)) a^{\frac{1}{2}} |\nabla u^m - \nabla v^m| \, dx \\
\leq c \left( \int \{\Omega a^\beta |u^m - v^m| < \eta\} \left[ a^{-\frac{1}{2}} a^\beta |u^m - v^m| S_\eta'(a^\beta(u^m - v^m)) \right]^2 \, dx \right)^{\frac{1}{2}} \\
- \cdot \left( \int \{\Omega a^\beta |u^m - v^m| < \eta\} a(x)(|\nabla u^m|^2 + |\nabla v^m|^2) \, dx \right)^{\frac{1}{2}}.
$$

If $\{x \in \Omega: u^m - v^m = 0\}$ has 0 measure, since

$$
\int \Omega a^{-1}(x) \, dx < \infty,
$$

consequently

$$
\int \{\Omega a^\beta |u^m - v^m| < \eta\} \left[ a^{-\frac{1}{2}} a^\beta (u^m - v^m) S_\eta'(a^\beta(u^m - v^m)) \right]^2 \, dx \\
\leq \int \{\Omega a^\beta |u^m - v^m| < \eta\} a^{-1}(x) \, dx < \infty. \quad (3.6)
$$
Let \( \eta \) be positive measure, obviously,

\[
\lim_{\eta \to 0} \left( \int_{\{ \Omega : a^\beta |u^m - v^m| < \eta \}} a(x) \left( |\nabla u^m|^2 + |\nabla v^m|^2 \right) \right)^{\frac{1}{2}} = 0.
\]

Using Lebesgue dominated convergence theorem, in both cases, we have

\[
\lim_{\eta \to 0} \left( \int_{\{ \Omega : |u^m - v^m| = 0 \}} a(x) \left( |\nabla u^m|^2 + |\nabla v^m|^2 \right) \right)^{\frac{1}{2}} = 0.
\]

If \( \{ x \in \Omega : u^m - v^m = 0 \} \) has a positive measure, obviously,

\[
\lim_{\eta \to 0} \left( \int_{\{ \Omega : a^\beta |u^m - v^m| < \eta \}} \left| a^{-\frac{1}{2}} a^\beta (u^m - v^m) S'_\eta (a^\beta (u^m - v^m)) \right|^2 \right)^{\frac{1}{2}} = 0.
\]

In addition,

\[
\lim_{\eta \to 0} \left| \int_{\Omega} a(u^m - v^m) S'_\eta (a^\beta (u^m - v^m)) \left( \nabla u^m - \nabla v^m \right) \cdot a^\beta \right| = 0. \tag{3.7}
\]

Now, let \( \eta \to 0 \) in (3.3). By (3.4)-(3.8), we have

\[
\int_{\Omega} |u(x,t) - v(x,t)| \, dx \leq c \int_{\Omega} |u_0 - v_0| \, dx, \quad \forall t \in [0,T).
\]

Theorem 1.1 is proved. \( \square \)

4 The proof of Theorem 1.2

Proof of Theorem 1.2. Let \( u, v \) be two solutions of equation (1.1) with the initial values \( u_0(x), v_0(x) \) respectively. Denote \( \Omega_\eta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \eta \} \), let \( \xi_\eta \in C^\infty_0(\Omega) \) such that \( \xi_\eta = 1 \) on \( \Omega_\eta \), \( 0 \leq \xi_\eta \leq 1 \) and

\[
|\nabla \xi_\eta| \leq \frac{c}{\eta}.
\]

We can choose \( g(s) = s, \phi_1 = \chi_{[\tau, T]} \xi_\eta \), \( \phi_2 = (u^m - v^m) \), and choose \( \chi_{[\tau, T]} S_\eta (u^m - v^m) \xi_\eta \) as a test function. Here \( \chi_{[\tau, T]} \) is the characteristic function of \( [\tau, T] \subseteq [0, T) \). Then

\[
\int_{Q_T} S_\eta (u^m - v^m) \xi_\eta \frac{\partial (u - v)}{\partial t} \, dx \, dt
\]
\[-\iint_{Q_{\tau s}} a(x)(\nabla u^m - \nabla v^m) \nabla \left[ S_\eta (u^m - v^m) \xi \right] \, dx \, dt \\
- \iint_{Q_{\tau s}} [f(u,x,t) - f(v,x,t)] S_\eta (u^m - v^m) \xi \, dx \, dt. \quad (4.1)\]

We have
\[\iint_{Q_{\tau s}} a_\eta \xi |\nabla (u^m - v^m)|^2 S_\eta (u^m - v^m) \, dx \, dt \geq 0, \quad (4.2)\]

and
\[\left| \iint_{Q_{\tau s}} S_\eta (u^m - v^m) a(x)(\nabla u^m - \nabla v^m) \nabla \xi \, dx \, dt \right| \]
\[\leq \iint_{Q_{\tau s}} a(x)(|\nabla u^m|^2 + |\nabla v^m|^2) |\nabla \xi| \, dx \, dt \]
\[\leq \int_0^s \int_{\Omega \setminus \Omega_\eta} \left[ \frac{1}{2} a(x)(|\nabla u^m|^2 + |\nabla v^m|^2) + \frac{1}{2} a(x)|\nabla \xi|^2 \right] \, dx \, dt \]
\[\leq \int_0^s \int_{\Omega \setminus \Omega_\eta} \left[ \frac{1}{2} a(x)(|\nabla u^m|^2 + |\nabla v^m|^2) + c a(x) \eta^{-2} \right] \, dx \, dt, \quad (4.3)\]

which goes to zero when \( \eta \to 0 \), due to the assumption
\[\frac{1}{\eta^2} \int_{\Omega \setminus \Omega_\eta} a(x) \, dx \leq c.\]

At the same time,
\[\lim_{\eta \to 0} \int_{Q_{\tau s}} S_\eta (u^m - v^m) \xi \frac{\partial (u - v)}{\partial t} \, dx \, dt = \iint_{Q_{\tau s}} \text{sign}(u^m - v^m) \frac{\partial (u - v)}{\partial t} \, dx \, dt \]
\[= \iint_{Q_{\tau s}} \text{sign}(u - v) \frac{\partial (u - v)}{\partial t} \, dx \, dt. \quad (4.4)\]

\[\lim_{\eta \to 0} \left| \int_{Q_{\tau s}} [f(u,x,t) - f(v,x,t)] S_\eta (u^m - v^m) \xi \, dx \, dt \right| \leq c \iint_{Q_{\tau s}} |u - v| \, dx \, dt. \quad (4.5)\]

Let \( \eta \to 0 \) in (4.1). By (4.2)-(4.5), we have
\[\int \Omega |u(x,s) - v(x,s)| \, dx - \int \Omega |u(x,\tau) - v(x,\tau)| \, dx \leq c \int_\tau^s \int \Omega |u(x,t) - v(x,t)| \, dx \, dt. \quad (4.6)\]

By (4.6), using Growall’s inequality, we are able to obtain the stability (1.13). The proof is complete. \( \square \)
The existence of BV solution

At the end of the paper, we generalize equation (1.2) to the following degenerate parabolic equation

\[
\frac{\partial u}{\partial t} = \sum_{i=1}^N a_{x_i} \left( a(u,x,t) \frac{\partial u}{\partial x_i} \right) + f(u,x,t), \quad (x,t) \in Q_T = \Omega \times (0,T),
\]

with

\[
\begin{align*}
  u(x,0) &= u_0(x), \quad x \in \Omega, \\
  u(x,t) &= 0, \quad (x,t) \in \partial \Omega \times (0,T),
\end{align*}
\]

and study the existence of the entropy solution in BV space. However, we would like to suggest that the concepts of the entropy solution (Definition 5.1), the existence of the solution (Theorem 5.1) and the stability of solutions (Theorem 5.2), are just a minor version of our previous work [17].

**Definition 5.1.** A function \( u \in BV(Q_T) \cap L^\infty(Q_T) \) is said to be the entropy solution of equation (5.1) with the initial value (5.2) and with the boundary value condition (5.3), if

1. There exists \( g^i \in L^2(Q_T) \), \( i = 1, 2, \ldots, N \), such that for any \( \varphi(x,t) \in C^0_0(Q_T) \),

\[
\int_{Q_T} \varphi(x,t) g^i(x,t) \, dx \, dt = \int_{Q_T} \varphi(x,t) \sqrt{a(u,x,t)} \frac{\partial u}{\partial x_i} \, dx \, dt,
\]

where

\[
\sqrt{a(u,x,t)}(u,x,t) = \int_0^1 \sqrt{a(\tau u^+ + (1-\tau)u^-)} \, d\tau,
\]

which is called the composite mean value of \( \sqrt{a(u,x,t)} \).

2. For any \( \varphi \in C^2_0(Q_T) \), \( \varphi \geq 0 \), for any \( k \in \mathbb{R} \), for any small \( \eta > 0 \), \( u \) satisfies

\[
\int_{Q_T} \left[ I_\eta(u-k) \varphi_t + A_\eta(u,x,t,k) \Delta \varphi + \sum_{i=1}^N S'_{\eta}(u-k) |g^i|^2 \varphi \right] \, dx \, dt + \int_{Q_T} \int_k^u a_{x_i}(s,x,t) S_{\eta}(s-k) \, ds \, \varphi_{x_i} \, dx \, dt + \int_{Q_T} f(u,x,t) \varphi S_{\eta}(u-k) \, dx \, dt \geq 0.
\]

3. The homogeneous boundary value (5.3) is true in the sense of trace,

\[
\gamma u \big|_{\partial \Omega \times (0,T)} = 0.
\]

4. The initial value is true in the sense of

\[
\lim_{t \to 0} \int_{\Omega} |u(x,t) - u_0(x)| \, dx = 0.
\]
Here the double equal indices imply a summation from 1 up to $N$, and
\[
A_\eta(u,x,t,k) = \int_k^u a(s,x,t)S_\eta(s-k)ds,
\]
\[
A(u,x,t) = \int_0^u a(s,x,t)ds, I_\eta(u-k) = \int_0^{u-k} S_\eta(s)ds.
\]

**Theorem 5.1.** If $A(s,x,t)$ is $C^3$, $f(u,x,t)$ is $C^1$, $u_0(x) \in C^1(\Omega)$, and there is a constant $\delta > 0$ such that
\[
a(u,x,t) - \delta \sum_{s=1}^{N+1} (a_{x_s})^2 \geq 0,
\]
then Eq. (5.1) with the initial-boundary value conditions (5.2)–(5.3) has an entropy solution in the sense of Definition 5.1. Here $s = 1, \cdots, N, N+1, x_{N+1} = t$.

**Theorem 5.2.** Suppose that $A(s,x,t)$ is $C^2$, $f(s,x,t)$ satisfies
\[
|f(s_1,x,t) - f(s_2,x,t)| \leq c|s_1 - s_2|,
\]
and there exists a constant $\delta > 0$ such that
\[
|\sqrt{a(\cdot,x,\cdot)} - \sqrt{a(\cdot,y,\cdot)}| \leq c|x-y|^{2+\delta}.
\]

Let $u,v$ be solutions of Eq. (1.1) with the same homogeneous boundary value (5.3) and with the different initial values $u_0(x), v_0(x) \in L^\infty(\Omega)$ respectively. Then
\[
\int_\Omega |u(x,t) - v(x,t)| \, dx \leq \int_\Omega |u_0(x) - v_0(x)| \, dx.
\]

At last, let us come back to consider the main equation (1.2). If we compare (1.2) with the usual reaction-diffusion equation (5.1), then
\[
a(u,x,t) = ma(x)u^{m-1},
\]
the condition (5.8) becomes
\[
ma(x)u^{m-1} - \delta m^2 u^{2(m-1)} |\nabla a|^2 = mu^{m-1}(a - \delta mu^{m-1}|\nabla a|^2).
\]

By this observation, we have

**Theorem 5.3.** If $m \geq 1$, and
\[
|\nabla a|^2 \leq ca,
\]
then equation (1.2) has a BV solution in the sense of Definition 5.1. Moreover, if $a$ satisfies the condition (5.10), the solution is unique.

Certainly, Theorem 5.1 and Theorem 5.2 show the well-posedness of the usual reaction-diffusion equation (5.1), including the case when the equation is strongly degenerate. Accordingly, they have their own independent significance.
Acknowledgement

The paper is supported by Natural Science Foundation of Fujian Province, supported by Science Foundation of Xiamen University of Technology, China.

References