JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **32**, No. 1, pp. 20-32

doi: 10.4208/jpde.v32.n1.2
March 2019

On a Nonlinear Heat Equation with Degeneracy on the Boundary

ZHAN Huashui*

School of Applied Mathematics, Xiamen University of Technology, Xiamen 361024, China.

Received 29 May 2017; Accepted 11 March 2019

Abstract. The paper studies the stability of weak solutions of a nonlinear heat equation with degenerate on the boundary. A new kind of weak solutions are introduced. By the new weak solution, the stability of weak solutions is proved only dependent on the initial value.

AMS Subject Classifications: 35L65 35R35

Chinese Library Classifications: O175.26

Key Words: Nonlinear heat equation; weak solution; stability, initial value.

1 Introduction

Consider the nonlinear heat equation

$$u_t=\mathrm{div}(k(u,x,t)\nabla u)+f(u,x,t),\quad (x,t)\in Q_T=\Omega\times(0,T), \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with appropriately smooth boundary, the function $k(u,x,t)$ has the meaning of nonlinear thermal conductivity, which depends on the temperature $u=u(x,t)$. It is generally assume that the matrix $k(u,x,t)$ is semidefinite. If $k(u,x,t)=k(x)$, Eq. (1.1) becomes a linear parabolic equation, we would like to suggest that, for linear equations, any boundedness estimate is equivalent to a stability result (i.e., control of differences of solutions in terms of differences of data), but this is not the truth for nonlinear equations generally. One can see the well-known monographs or textbooks [1–7] and the references therein. However, in some special case, if we add some restrictions to $k(u,x,t)$, the character may be still true. For simplicity, the paper limits to consider

$$k(u,x,t)=ma(x)u^{m-1},\qquad m>0,$$

*Corresponding author. *Email address:* 2012111007@xmut.edu.cn (H.S. Zhan)

with $a(x)>0$ when $x\in\Omega$, but $a(x)=0$ when $x\in\partial\Omega$. In other words, we will consider the following nonlinear equation

$$u_t=\mathrm{div}(a(x)\nabla u^m)+f(u,x,t), \qquad (x,t)\in Q_T. \tag{1.2}$$

From my own perspective, the initial value

$$u(x,0)=u_0(x), \qquad x\in\Omega, \tag{1.3}$$

is indispensable. While, the usual boundary value

$$u(x,t)=0, \qquad (x,t)\in\partial\Omega\times(0,T), \tag{1.4}$$

may be superfluous. To see that, if $f\equiv 0$ in (1.2), we suppose that u and v are two classical solutions of equation (1.2) with the initial values $u(x,0)$ and $v(x,0)$ respectively. Then we have

$$\begin{aligned}&\int_\Omega S_\eta(u^m-v^m)(u-v)_t\mathrm{d}x+\int_\Omega a(x)S_\eta'(u^m-v^m)|\nabla u^m-\nabla v^m|^2\mathrm{d}x\\ =&\int_{\partial\Omega} a(x)S_\eta(u^m-v^m)(\nabla u-\nabla v)\cdot\vec{n}\mathrm{d}\Sigma=0,\end{aligned}$$

where $\vec{n}$ is the outer unit normal vector of Ω, $S_\eta(s)$ is the approximate function of the sign function (the details are given (3.1)-(3.2) later). Then

$$\begin{aligned}&\int_\Omega S_\eta(u^m-v^m)(u-v)_t\mathrm{d}x\le 0,\\ &\lim_{\eta\to 0}\int_\Omega S_\eta(u^m-v^m)(u-v)_t\mathrm{d}x=\int_\Omega \mathrm{sign}(u^m-v^m)(u-v)_t\mathrm{d}x\\ =&\int_\Omega \mathrm{sign}(u-v)(u-v)_t\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u-v|\mathrm{d}x.\end{aligned}$$

Then, even without any boundary value condition (1.4), the classical solutions have the stability

$$\int_\Omega|u(x,t)-v(x,t)|\mathrm{d}x\le\int_\Omega|u_0(x)-v_0(x)|\mathrm{d}x. \tag{1.5}$$

Certainly, since $|\nabla u^m|$ may be singular or degenerate on $\overline{\Omega}$, equation (1.2) only has a weak solution generally.

Thus, to study the well-posedness of weak solutions to equation (1.2), or a more general reaction-diffusion equation with the type

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a^{ij}(u,x,t)\frac{\partial u}{\partial x_j}\right)+\mathrm{div}(b(u,x,t))+f(u,x,t),\ \ (x,t)\in\Omega\times(0,T), \tag{1.6}$$

the whole boundary value condition (1.4) is overdetermined. For the linear case, the problem had been completely solved by Fichera [8], Oleinik [9] et al., for nonlinear case, the situation is far from success. Many important papers, for examples, references [8,10–17] have been devoting to the corresponding problem. In our paper, we introduce a new kind of weak solutions for Eq. (1.6). By the new definition, we can deal with the stability, or the uniqueness of the weak solutions independent of the boundary value condition.

Definition 1.1. *A function $u(x,t)$ is said to be a weak solution of Eq.* (1.2) *with the initial value* (1.3), *if*

$$u\in L^{\infty}(Q_T),\ u_t\in L^2(Q_T),\ a(x)|\nabla u^m|^2\in L^1(Q_T), \tag{1.7}$$

for any function $g(s)\in C^1(\mathbb{R})$, $g(0)=0$, $\varphi_1\in C_0^1(\Omega)$, $\varphi_2\in L^{\infty}(0,T;W^{1,2}_{loc}(\Omega))$,

$$\iint_{Q_T}[u_t g(\varphi_1\varphi_2)+a(x)\nabla u^m\cdot\nabla g(\varphi_1\varphi_2)-f(u,x,t)g(\varphi_1\varphi_2)]\mathrm{d}x\mathrm{d}t=0, \tag{1.8}$$

and the initial value is satisfies in the sense of that

$$\lim_{t\to 0}\int_{\Omega}|u(x,t)-u_0(x)|\mathrm{d}x=0. \tag{1.9}$$

In what follows, we suppose that $m>0$ and

$$u_0\in L^{\infty}(\Omega),\quad a(x)\nabla u_0^m\in L^2(\Omega). \tag{1.10}$$

In the first place, we would like to suggest that, the existence of Eq. (1.2) in the sense of Definition 1.1 is easily to be obtained, we will give a basic result and the skeleton of the proof in the second section of the paper. The paper mainly follows with interest in the stability of weak solutions without any boundary value condition.

Theorem 1.1. *Let u,v be two solutions of Eq.* (1.2) *with the different initial value $u_0(x),v_0(x)$ respectively. If*

$$\int_{\Omega}a^{-1}(x)\mathrm{d}x<\infty, \tag{1.11}$$

$$|f(u,x,t)-f(v,x,t)|\le c|u-v|, \tag{1.12}$$

then

$$\int_{\Omega}|u(x,t)-v(x,t)|\le c\int_{\Omega}|u_0(x)-v_0(x)|\mathrm{d}x,\ a.e.\ t\in[0,T). \tag{1.13}$$

Theorem 1.2. *Let u,v be two solutions of Eq.* (1.2) *with the different initial value $u_0(x),v_0(x)$ respectively. If f satisfies* (1.12), *$a(x)$ satisfies*

$$\frac{1}{\eta^2}\int_{\Omega\setminus\Omega_\eta}a(x)\mathrm{d}x\le c, \tag{1.14}$$

then the stability (1.13) *is true. Here $\Omega_\eta=\{x\in\Omega:d(x,\partial\Omega)>\eta\}$ for small $\eta>0$.*

We give a brief explanation of the conditions (1.11) and (1.14). Let $a(x)=d^{\alpha}(x)$, $d(x)=\mathrm{dist}(x,\partial\Omega)$, $\alpha>0$ be a constant. Then the condition (1.11) implies that $0<\alpha<1$; while the condition (1.14) means that $\alpha>1$. Thus, Theorem 1.3 is the complement of Theorem 1.1. It is astonished that, if $a(x)=d^{\alpha}(x)$, our results exclude the case $\alpha=1$.

At last, we would like to suggest that if $\{x\in\Omega: a(x)=0\}$ has the interior point in Ω, then the equation is strongly degenerate. In such case, only in the sense of the entropy solutions, the stability (or the uniqueness) of the weak solutions can be proved. One can refer to the references [8, 10–17] etc. In the last section of our paper (Section 5), a special case of equation (1.6) is considered.

So, the main results listed above (Theorems 1.1-1.2) not only clarify that, if the equation is weakly degenerate (i.e. there is not the interior point in the set $\{x\in\Omega: a(x)=0\}\subset\Omega$), then the usual boundary value condition (1.3) is overdetermined, but also show that the stability of weak solutions may be proved without any boundary value condition. Certainly, if the equation is strongly degenerate, whether the stability of the solutions can be obtained without any boundary value condition is still an open problem.

2 The existence of the solution

In general, the weak solution is defined as follows.

Definition 2.1. *A nonnegative function $u(x,t)$ is said to be a weak solution of equation* (1.2) *with the initial value* (1.3)*, if u satisfies* (1.7) *and for any function $\varphi\in C_0^1(Q_T)$,*

$$\iint_{Q_T}(u_t\varphi+a(x)\nabla u^m\nabla\varphi)\mathrm{d}x\mathrm{d}t=\iint_{Q_T}f(u,x,t)\varphi(x,t)\mathrm{d}x\mathrm{d}t, \tag{2.1}$$

the initial value (1.3) *is true in the sense of* (1.9).

Theorem 2.1. *If $u_0(x)$ satisfies*

$$0\le u_0\in L^{\infty}(\Omega),\sqrt{a(x)}\nabla u_0^m\in L^2(\Omega), \tag{2.2}$$

$f(s,x,t)\ge 0$ when $s<0$, then there exists a nonnegative solution of equation (1.2) *with the initial value* (1.3) *in the sense of Definition* 2.1.

Proof. We consider the following regularization problem

$$\begin{cases} u_{nt}=\mathrm{div}\left((a(x)+\frac{1}{n})\nabla u_n^m\right)+f(u_n,x,t), & (x,t)\in Q_T,\\ u_n(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u_n(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{2.3}$$

By the monotone convergent method [18], the initial-boundary value problem (2.3) has a nonnegative weak solution $u_n(x,t)$,

$$u_n(x,t)\in L^\infty(\Omega),\ u_{nt}\in L^2(Q_T),\ u_n^m\in L^2(0,T;W_0^{1,2}(\Omega)),$$

$$\|u_n(x,t)\|_{L^\infty(Q_T)}\le c. \tag{2.4}$$

Let us multiply u_n^m on both sides of the equation, integrate it over Q_T. Then

$$\begin{aligned}&\frac{1}{m+1}\int_\Omega u_n^{m+1}\mathrm{d}x+\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)|\nabla u_n^m|^2\mathrm{d}x\mathrm{d}t+\iint_{Q_T}f(u_n,x,t)u_n^m\mathrm{d}x\mathrm{d}t\\ =&\frac{1}{m+1}\int_\Omega u_0^{m+1}\mathrm{d}x,\end{aligned}$$

accordingly,

$$\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)|\nabla u_n^m|^2\mathrm{d}x\mathrm{d}t\le c. \tag{2.5}$$

Let us multiply u_{nt} on both sides of the equation, integrate it over Q_T. Then

$$\begin{aligned}&\frac{1}{2}\iint_{Q_T}u_{nt}^2\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\mathrm{div}\left(\left(a(x)+\frac{1}{n}\right)\nabla u_n^m\right)u_{nt}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}f(u_n,x,t)u_{nt}\mathrm{d}x\mathrm{d}t,\\ &\iint_{Q_T}\mathrm{div}\left(\left(a(x)+\frac{1}{n}\right)\nabla u_n^m\right)u_{nt}\mathrm{d}x\mathrm{d}t=-\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)\nabla u_n^m\cdot\nabla u_{nt}\mathrm{d}x\mathrm{d}t\\ &\quad=-m\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)u_n^{m-1}\nabla u_n\cdot\nabla u_{nt}\mathrm{d}x\mathrm{d}t\\ &\quad=-\frac{m}{2}\iint_{Q_T}\left(a(x)+\frac{1}{n}\right)u_n^{m-1}\frac{d}{dt}|\nabla u_n|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{2.6}$$

$$\left|\iint_{Q_T}f(u_n,x,t)u_{nt}\mathrm{d}x\mathrm{d}t\right|\le c+\frac{1}{4}\iint_{Q_T}u_{nt}^2\mathrm{d}x\mathrm{d}t. \tag{2.7}$$

By (2.6)-(2.7), we have

$$\iint_{Q_T}u_{nt}^2\mathrm{d}x\mathrm{d}t\le c. \tag{2.8}$$

Thus, by (2.4), (2.5) and (2.8), there exists a function $u(x,t)\in L^\infty(Q_T)$ and a $n-$dimensional vector $\overrightarrow{\zeta}=(\zeta_1,\cdots,\zeta_n)$ such that $u_\varepsilon\to u$ $a.e.\in Q_T$, and

$$\begin{aligned}&u\in L^\infty(Q_T),\quad \frac{\partial u}{\partial t}\in L^2(Q_T),\quad \left|\overrightarrow{\zeta}\right|\in L^2(Q_T),\\ &u_\varepsilon\rightharpoonup u,\ \text{weakly star in } L^\infty(Q_T),\qquad u_\varepsilon\to u,\ \text{in } L^2_{loc}(Q_T),\\ &\frac{\partial u_\varepsilon}{\partial t}\rightharpoonup\frac{\partial u}{\partial t}\ \text{ in } L^2(Q_T),\qquad \sqrt{a(x)+\frac{1}{n}}\nabla u_\varepsilon^m\rightharpoonup\overrightarrow{\zeta}\ \text{ in } L^2(Q_T).\end{aligned}$$

As the usual porous medium equation [5], we can prove that

$$\zeta=\sqrt{a(x)}\nabla u^m.$$

Also, we can prove that (1.9) is true. Thus, u is a solution of equation (1.2) with the initial value (1.3) in the sense of Definition 2.1. $\square$

Theorem 2.2. *If $u_0(x)$ satisfies* (2.3), $f(s,x,t)\geq 0$ *when* $s<0$, *then there exists a nonnegative solution of equation* (1.2) *with the initial value* (1.3) *in the sense of Definition* 1.1.

Proof. According to Theorem 2.1, for any function $\varphi\in C_0^1(Q_T)$ and for any function $g(s)\in C^1(\mathbb{R})$, $g(0)=0$, we have

$$\iint_{Q_T}[u_t g(\varphi)+a(x)\nabla u^m\nabla g(\varphi)]\mathrm{d}x\mathrm{d}t=\iint_{Q_T} f(u,x,t)g(\varphi)\mathrm{d}x\mathrm{d}t.$$

If we denote $\Omega_\varphi=\mathrm{supp}\varphi$, then

$$\int_0^T\int_{\Omega_\varphi}[u_t g(\varphi)+a(x)\nabla u^m\nabla g(\varphi)-f(u,x,t)g(\varphi)]\mathrm{d}x\mathrm{d}t=0.$$

Now, for any function $\varphi_1\in C_0^1(\Omega)$, $\varphi_2\in L^\infty(0,T;W_{loc}^{1,2}(\Omega))$, it is clearly that $\varphi_2\in W^{1,2}(\Omega_{\varphi_1})$. By the fact of that $C_0^\infty(\Omega_{\varphi_1})$ is dense in $W^{1,2}(\Omega_{\varphi_1})$, by a process of limit, we have

$$\int_0^T\int_{\Omega_{\varphi_1}}[u_t g(\varphi_1\varphi_2)+a(x)\nabla u^m\nabla g(\varphi_1\varphi_2)-f(u,x,t)g(\varphi_1\varphi_2)]\mathrm{d}x\mathrm{d}t=0.$$

which implies that

$$\int_0^T\int_{\Omega}[u_t g(\varphi_1\varphi_2)+a(x)\nabla u^m\nabla g(\varphi_1\varphi_2)-f(u,x,t)g(\varphi_1\varphi_2)]\mathrm{d}x\mathrm{d}t=0.$$

Thus, we have the conclusion. $\square$

3 Proof of Theorem 1.1

For small $\eta>0$, let

$$S_\eta(s)=\int_0^s h_\eta(\tau)\mathrm{d}\tau,\ h_\eta(s)=\frac{2}{\eta}\left(1-\frac{|s|}{\eta}\right)_+. \tag{3.1}$$

Obviously $h_\eta(s)\in C(\mathbb{R})$, and

$$h_\eta(s)\geq 0,\ |sh_\eta(s)|\leq 1, |S_\eta(s)|\leq 1; \lim_{\eta\to 0}S_\eta(s)=\mathrm{sgn}s,\ \lim_{\eta\to 0}sS'_\eta(s)=0. \tag{3.2}$$

Proof of Theorem 1.1. Let u, v be two solutions of equation (1.2) with the initial values $u_0(x), v_0(x)$. We can choose $S_\eta(a^\beta(u^m-v^m))$ as the test function. Then

$$\begin{aligned}&\int_\Omega S_\eta(a^\beta(u^m-v^m))\frac{\partial(u-v)}{\partial t}\mathrm{d}x+\int_\Omega a^{\beta+1}(x)(\nabla u^m-\nabla v^m)\cdot\nabla(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\mathrm{d}x\\&\quad+\int_\Omega a(x)(\nabla u^m-\nabla v^m)\cdot\nabla a^\beta(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\mathrm{d}x\\&\quad+\int_\Omega[f(u,x,t)-f(v,x,t)]S_\eta(a^\beta(u^m-v^m))\mathrm{d}x=0.\end{aligned}\tag{3.3}$$

Thus

$$\begin{aligned}&\lim_{\eta\to0}\int_\Omega S_\eta(a^\beta(u^m-v^m))\frac{\partial(u-v)}{\partial t}\mathrm{d}x=\int_\Omega \mathrm{sign}(a^\beta(u^m-v^m))\frac{\partial(u-v)}{\partial t}\mathrm{d}x\\&\qquad=\int_\Omega \mathrm{sign}(u-v)\frac{\partial(u-v)}{\partial t}\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u-v|\mathrm{d}x,\end{aligned}\tag{3.4}$$

$$\int_\Omega a^{\beta+1}(x)|\nabla u^m-\nabla v^m|^2S_\eta'(a^\beta(u^m-v^m))\mathrm{d}x\geq0.\tag{3.5}$$

By that $|\nabla a(x)|\leq c$ in Ω, we have

$$\begin{aligned}&\left|\int_\Omega a(x)(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))(\nabla u^m-\nabla v^m)\cdot\nabla a^\beta\mathrm{d}x\right|\\&\leq c\int_\Omega a^\beta|u^m-v^m|S_\eta'(a^\beta(u^m-v^m))|\nabla u^m-\nabla v^m|\mathrm{d}x\\&=c\int_{\{\Omega:a^\beta|u^m-v^m|<\eta\}}a^{-\frac12}a^\beta|u^m-v^m|S_\eta'(a^\beta(u^m-v^m))a^{\frac12}|\nabla u^m-\nabla v^m|\mathrm{d}x\\&\leq c\left(\int_{\{\Omega:a^\beta|u^m-v^m|<\eta\}}\left[a^{-\frac12}a^\beta|u^m-v^m|S_\eta'(a^\beta(u^m-v^m))\right]^2\mathrm{d}x\right)^{\frac12}\\&\qquad\cdot\left(\int_{\{\Omega:a^\beta|u^m-v^m|<\eta\}}a(x)(|\nabla u^m|^2+|\nabla v^m|^2)\mathrm{d}x\right)^{\frac12}.\end{aligned}$$

If $\{x\in\Omega:u^m-v^m=0\}$ has 0 measure, since

$$\int_\Omega a^{-1}(x)\mathrm{d}x<\infty,$$

consequently

$$\begin{aligned}&\int_{\{\Omega:a^\beta|u^m-v^m|<\eta\}}\left|a^{-\frac12}a^\beta(u^m-v^m)S_\eta'(a^\beta(u^m-v^m))\right|^2\mathrm{d}x\\&\leq\int_{\{\Omega:a^\beta|u^m-v^m|<\eta\}}a^{-1}(x)\mathrm{d}x<\infty.\end{aligned}\tag{3.6}$$

Then

$$\lim_{\eta\to 0}\left(\int_{\{\Omega:a^\beta|u^m-v^m|<\beta\}} a(x)(|\nabla u^m|^2+|\nabla v^m|^2)\mathrm{d}x\right)^{\frac{1}{2}}$$
$$=\left(\int_{\{\Omega:|u^m-v^m|=0\}} a(x)(|\nabla u^m|^2+|\nabla v^m|^2)\mathrm{d}x\right)^{\frac{1}{2}}=0.$$

If $\{x\in\Omega: u^m-v^m=0\}$ has a positive measure, obviously,

$$\lim_{\eta\to 0}\left(\int_{\{\Omega:a^\beta|u^m-v^m|<\eta\}}\left|a^{-\frac{1}{2}}a^\beta(u^m-v^m)S'_\eta(a^\beta(u^m-v^m))\right|^2\mathrm{d}x\right)^{\frac{1}{p}}$$
$$=\left(\int_{\{\Omega:|u^m-v^m|=0\}}\lim_{\eta\to 0}\left|a^{-\frac{1}{2}}a^\beta(u^m-v^m)S'_\eta(a^\beta(u^m-v^m))\right|^2\mathrm{d}x\right)^{\frac{1}{2}}=0.$$

Using Lebesgue dominated convergence theorem, in both cases, we have

$$\lim_{\eta\to 0}\left|\int_\Omega a(u^m-v^m)S'_\eta(a^\beta(u^m-v^m))(\nabla u^m-\nabla v^m)\cdot a^\beta\mathrm{d}x\right|=0. \tag{3.7}$$

In addition,

$$\lim_{\eta\to 0}\left|\int_\Omega[f(u,x,t)-f(v,x,t)]S_\eta(a^\beta(u^m-v^m))\mathrm{d}x\right|\le c\int_\Omega|u-v|\mathrm{d}x. \tag{3.8}$$

Now, let $\eta\to 0$ in (3.3). By(3.4)-(3.8), we have

$$\int_\Omega|u(x,t)-v(x,t)|\mathrm{d}x\leqslant c\int_\Omega|u_0-v_0|\mathrm{d}x,\quad \forall t\in[0,T).$$

Theorem 1.1 is proved. □

4 The proof of Theorem 1.2

Proof of Theorem 1.2. Let u, v be two solutions of equation (1.1) with the initial values $u_0(x),v_0(x)$ respectively. Denote $\Omega_\eta=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>\eta\}$, let $\xi_\eta\in C_0^\infty(\Omega)$ such that $\xi_\eta=1$ on Ω_η, $0\le\xi_\eta\le 1$ and

$$|\nabla\xi_\eta|\le\frac{c}{\eta}.$$

We can choose $g(s)\equiv s$, $\varphi_1=\chi_{[\tau,s]}\xi_\eta$, $\varphi_2=(u^m-v^m)$, and choose $\chi_{[\tau,s]}S_\eta(u^m-v^m)\xi_\eta$ as a test function. Here $\chi_{[\tau,s]}$ is the characteristic function of $[\tau,s)\subseteq[0,T)$. Then

$$\iint_{Q_{\tau s}} S_\eta(u^m-v^m)\xi_\eta\frac{\partial(u-v)}{\partial t}\mathrm{d}x\mathrm{d}t$$

$$\begin{aligned}&=-\iint_{Q_{\tau s}} a(x)(\nabla u^m-\nabla v^m)\nabla[S_\eta(u^m-v^m)\xi_\eta]\mathrm{d}x\mathrm{d}t\\&\quad-\iint_{Q_{\tau s}}[f(u,x,t)-f(v,x,t)]S_\eta(u^m-v^m)\xi_\eta\mathrm{d}x\mathrm{d}t. \qquad (4.1)\end{aligned}$$

We have

$$\iint_{Q_{\tau s}} a\xi_\eta|\nabla(u^m-v^m)|^2S'_\eta(u^m-v^m)\mathrm{d}x\mathrm{d}t\geq 0, \qquad (4.2)$$

and

$$\begin{aligned}&\left|\iint_{Q_{\tau s}} S_\eta(u^m-v^m)a(x)(\nabla u^m-\nabla v^m)\nabla\xi_\eta\mathrm{d}x\mathrm{d}t\right|\\&\leq\iint_{Q_{\tau s}} a(x)(|\nabla u^m|+|\nabla v^m|)|\nabla\xi_\eta|\mathrm{d}x\mathrm{d}t\\&\leq\int_0^s\int_{\Omega\setminus\Omega_\eta}\left[\frac{1}{2}a(x)(|\nabla u^m|^2+|\nabla v^m|^2)+\frac{1}{2}a(x)|\nabla\xi_\eta|^2\right]\mathrm{d}x\mathrm{d}t\\&\leq\int_0^s\int_{\Omega\setminus\Omega_\eta}\left[\frac{1}{2}a(x)(|\nabla u^m|^2+|\nabla v^m|^2)+ca(x)\eta^{-2}\right]\mathrm{d}x\mathrm{d}t, \qquad (4.3)\end{aligned}$$

which goes to zero when $\eta\to 0$, due to the assumption

$$\frac{1}{\eta^2}\int_{\Omega\setminus\Omega_\eta} a(x)\mathrm{d}x\leq c.$$

At the sane time,

$$\begin{aligned}&\lim_{\eta\to 0}\iint_{Q_{\tau s}} S_\eta(u^m-v^m)\xi_\eta\frac{\partial(u-v)}{\partial t}\mathrm{d}x\mathrm{d}t=\iint_{Q_{\tau s}}\mathrm{sign}(u^m-v^m)\frac{\partial(u-v)}{\partial t}\mathrm{d}x\mathrm{d}t\\&\quad=\iint_{Q_{\tau s}}\mathrm{sign}(u-v)\frac{\partial(u-v)}{\partial t}\mathrm{d}x\mathrm{d}t. \qquad (4.4)\\&\lim_{\eta\to 0}\left|\iint_{Q_{\tau s}}[f(u,x,t)-f(v,x,t)]S_\eta(u^m-v^m)\xi_\eta\mathrm{d}x\mathrm{d}t\right|\leq c|\iint_{Q_{\tau s}}|u-v|\mathrm{d}x\mathrm{d}t. \qquad (4.5)\end{aligned}$$

Let $\eta\to 0$ in (4.1). By (4.2)-(4.5), we have

$$\int_\Omega|u(x,s)-v(x,s)|\mathrm{d}x-\int_\Omega|u(x,\tau)-v(x,\tau)|\mathrm{d}x\leq c\int_\tau^s\int_\Omega|u(x,t)-v(x,t)|\mathrm{d}x\mathrm{d}t. \qquad (4.6)$$

By (4.6), using Growall's inequality, we are able to obtain the stability (1.13). The proof is complete. □

5 The existence of BV solution

At the end of the paper, we generalize equation (1.2) to the following degenerate parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a(u,x,t)\frac{\partial u}{\partial x_i}\right)+f(u,x,t),\quad (x,t)\in Q_T=\Omega\times(0,T),\tag{5.1}$$

with

$$u(x,0)=u_0(x),\qquad x\in\Omega,\tag{5.2}$$

$$u(x,t)=0,\qquad (x,t)\in\partial\Omega\times(0,T),\tag{5.3}$$

and study the existence of the entropy solution in BV space. However, we would like to suggest that the concepts of the entropy solution (Definition 5.1), the existence of the solution (Theorem 5.1) and the stability of solutions (Theorem 5.2), are just a minor version of our previous work [17].

Definition 5.1. *A function $u\in BV(Q_T)\cap L^\infty(Q_T)$ is said to be the entropy solution of equation* (5.1) *with the initial value* (5.2) *and with the boundary value condition* (5.3), *if*

1. *There exists $g^i\in L^2(Q_T)$, $i=1,2,\cdots,N$, such that for any $\varphi(x,t)\in C_0^1(Q_T)$,*

$$\iint_{Q_T}\varphi(x,t)g^i(x,t)\mathrm{d}x\mathrm{d}t=\iint_{Q_T}\varphi(x,t)\sqrt{\widehat{a(u,x,t)}}\frac{\partial u}{\partial x_i}\mathrm{d}x\mathrm{d}t,\tag{5.4}$$

where

$$\sqrt{\widehat{a(u,x,t)}(u,x,t)}=\int_0^1\sqrt{a(\tau u^+ +(1-\tau)u^-,x,t)}\mathrm{d}\tau,$$

which is called the composite mean value of $\sqrt{a(u,x,t)}$.

2. *For any $\varphi\in C_0^2(Q_T)$, $\varphi\geq 0$, for any $k\in\mathbb{R}$, for any small $\eta>0$, u satisfies*

$$\begin{aligned}&\iint_{Q_T}\left[I_\eta(u-k)\varphi_t+A_\eta(u,x,t,k)\Delta\varphi-\sum_{i=1}^N S'_\eta(u-k)\,|\,g^i\,|^2\,\varphi\right]\mathrm{d}x\mathrm{d}t\\&\quad+\iint_{Q_T}\int_k^u a_{x_i}(s,x,t)S_\eta(s-k)ds\varphi_{x_i}\mathrm{d}x\mathrm{d}t+\iint_{Q_T}f(u,x,t)\varphi S_\eta(u-k)\mathrm{d}x\mathrm{d}t\geq 0.\end{aligned}\tag{5.5}$$

3. *The homogeneous boundary value* (5.3) *is true in the sense of trace,*

$$\gamma u\,|_{\partial\Omega\times(0,T)}=0.\tag{5.6}$$

4. *The initial value is true in the sense of*

$$\lim_{t\to 0}\int_\Omega|\,u(x,t)-u_0(x)\,|\,\mathrm{d}x=0.\tag{5.7}$$

Here the double equal indices imply a summation from 1 up to N, and

$$A_\eta(u,x,t,k)=\int_k^u a(s,x,t)S_\eta(s-k)\mathrm{d}s,$$
$$A(u,x,t)=\int_0^u a(s,x,t)\mathrm{d}s, I_\eta(u-k)=\int_0^{u-k} S_\eta(s)\mathrm{d}s.$$

Theorem 5.1. *If $A(s,x,t)$ is C^3, $f(u,x,t)$ is C^1, $u_0(x)\in C^1(\Omega)$, and there is a constant $\delta>0$ such that*

$$a(u,x,t)-\delta\sum_{s=1}^{N+1}(a_{x_s})^2\geq 0, \tag{5.8}$$

then Eq. (5.1) *with the initial-boundary value conditions* (5.2)–(5.3) *has a entropy solution in the sense of Definition* 5.1. *Here $s=1,\cdots,N,N+1$, $x_{N+1}=t$.*

Theorem 5.2. *Suppose that $A(s,x,t)$ is C^2, $f(s,x,t)$ satisfies*

$$|f(s_1,x,t)-f(s_2,x,t)|\leq c|s_1-s_2|, \tag{5.9}$$

and there exists a constant $\delta>0$ such that

$$|\sqrt{a(\cdot,x,\cdot)}-\sqrt{a(\cdot,y,\cdot)}|\leq c|x-y|^{2+\delta}. \tag{5.10}$$

Let u,v be solutions of Eq. (1.1) *with the same homogeneous boundary value* (5.3) *and with the different initial values $u_0(x)$, $v_0(x)\in L^\infty(\Omega)$ respectively. Then*

$$\int_\Omega|u(x,t)-v(x,t)|\mathrm{d}x\leq\int_\Omega|u_0(x)-v_0(x)|\mathrm{d}x. \tag{5.11}$$

At last, let us come back to consider the main equation (1.2). If we compare (1.2) with the usual reaction-diffusion equation (5.1), then

$$a(u,x,t)=ma(x)u^{m-1},$$

the condition (5.8) becomes

$$ma(x)u^{m-1}-\delta m^2u^{2(m-1)}|\nabla a|^2=mu^{m-1}(a-\delta mu^{m-1}|\nabla a|^2).$$

By this observation, we have

Theorem 5.3. *If $m\geq 1$, and*

$$|\nabla a|^2\leq ca,$$

then equation (1.2) *has a BV solution in the sense of Definition* 5.1. *Moreover, if a satisfies the condition* (5.10)*, the solution is unique.*

Certainly, Theorem 5.1 and Theorem 5.2 show the well-posedness of the usual reaction-diffusion equation (5.1), including the case when the equation is strongly degenerate. Accordingly, they have their own independent significance.

Acknowledgement

The paper is supported by Natural Science Foundation of Fujian Province, supported by Science Foundation of Xiamen University of Technology, China.

References

[1] Friedman A., Partial Differential Equations of Parabolic Type. Prentice Hall, Englewood Cliffs, N.J., 1964.
[2] Ladyshenskaya O. A., Solounikov V.A. and Uraltseva N. N., Linear and quasilinear equations of parabolic type, Providence R.I: Am. Math. Soc., 1968.
[3] Smoller J., Shock Waves and Reaction Diffusion Equations, Academic Press, New York, 1983.
[4] Alexander A. S., Victor A. G., Sergei P.K. and Alexander P. M., Blow-up in Quasilinear Parabolic Equations, Walter de Gruyter. Berlin, New York, 1995.
[5] Wu Z., Zhao J., Yin J. and Li H., Nonlinear Diffusion Equations, Word Scientific Publishing, Singapore, 2001.
[6] DiBenedetto, E., Degenerate Parabolic Equations, Spring-Verlag, New York, 1993.
[7] Vázquez, J. L., Smoothing and Decay Estimates for Nonlinear Diffusion Equations, Oxford Lecture Series in Math. and its Appl., Oxford Univ. Press, 2005.
[8] Guarguaglini F.R., Milišić, V. and Terracina A., A discrete BGK approximation for strongly degenerate parabolic problems with boundary conditions, *J. Differential Equations*, **202** (2004), 183-207.
[9] Oleinik O. A. and Radkevic E. V., Second Order Differential Equations with Nonnegative Characteristic Form. Rhode Island: American Mathematical Society, and New York: Plenum Press, 1973.
[10] Lions P. L., Perthame B., and Tadmor E., A kinetic formation of multidimensional conservation laws and related equations, *J. Amer. Math. Soc.*, **7** (1994), 169-191.
[11] Kobayasi K. and Ohwa H., Uniqueness and existence for anisotropic degenerate parabolic equations with boundary conditions on a bounded rectangle, *J. Differential Equations*, **252** (2012), 137-167.
[12] Li Y. and Wang Q., Homogeneous Dirichlet problems for quasilinear anisotropic degenerate parabolic- hyperbolic equations, *J. Differential Equations*, **252** (2012), 4719-4741.
[13] Málek J., Necas, J., Rokyta M. and Ruzicka M., Weak and Measure-valued Solutions to Evolutionary PDES. Applied Mathematics and Mathematical Computation, 13, Chapman and HALL, London, 1996.
[14] Mascia C., Porretta, A. and Terracina, A., Nonhomogeneous Dirichlet problems for degenerate parabolic-hyperbolic equations, *Arch. Ration. Mech. Anal.*, **163(2)** (2) (2002), 87-124.
[15] Vallet, G., Dirichlet problem for a degenerated hyperbolic-parabolic equation, *Advances in Mathematical Sciences and Applications*, **15** (2005), 423-450.
[16] Zhan H., The solutions of a hyperbolic-parabolic mixed type equation on half-space domain, *J. Differential Equations*, **259** (2015), 1449-1481.
[17] Zhan H., The boundary degeneracy theory of a strongly degenerate parabolic equation, *Boundary Value Problems*, **15** (2016), DOI 10.1186/s13661-015-0516-0.
[18] Zhan H., Yuan H., Porous medium equation with boundary degeneracy, *J. Jilin Univ. (Science Edition)*, **53** (5) (2015), 829-834.

[19] Fichera G., Sulle equazioni differenziali lineari ellittico-paraboliche delsecondo ordine, Atti Accad, Naz. Lincei. *Mem. CI. Sci Fis. Mat. Nat. Sez.*, **1** (8) (1956), 1-30.

[20] Yin J., Wang C. Properties of the boundary flux of a singular diffusion process, *Chin. Ann. Math.*, **25B** (2) (2004), 175-182.

[21] Yin J. and Wang C., Evolutionary weighted p-Laplacian with boundary degeneracy, *J. Differential Equations*, **237** (2007), 421-445.

[22] Enrico G., Minimal Surfaces and Functions of Bounded Variation, Birkhauser, Bosten. Basel. Stuttgart, Switzerland, 1984.

[23] Volpert A. I., Hudjaev, S.I., On the problem for quasilinear degenerate parabolic equations of second order (Russian), *Mat.Sb.*, **3** (1967), 374-396.

[24] Volpert, A.I., BV space and quasilinear equations, *Mat.Sb.*, **73** (1967), 255-302.

[25] Volpert, A.I. and Hudjave, S.I., Cauchy's problem for degenerate second order quasilinear parabolic equation, *Transl. Math. USSR Sb.*, **7** (1967), 365-387.

[26] Evans L.C., Weak Convergence Methods for Nonlinear Partial Differential Equations, Conference Board of the Mathematical Sciences, Regional Conferences Series in Mathematics Number 74, 1998.

[27] Wu Z., Zhao J., The first boundary value problem for quasilinear degenerate parabolic equations of second order in several variables, *Chin. Ann. Math.*, **4B** (1) (1993), 57-76.

[28] Gu L., Second Order Parabolic Partial Differential Equations, the publishing company of Xiamen University, Xiamen, 2002.

[29] Zhan H., The Study of the Cauchy Problem of a Second Order Quasilinear Degenerate Parabolic Equation and the Parallelism of a Riemannian Manifold, Doctorial Thesis, Xiamen University, Xiamen, 2004.