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On a Nonlinear Heat Equation with Degeneracy on the Boundary

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Abstract. The paper studies the stability of weak solutions of a nonlinear heat equation with degenerate on the boundary. A new kind of weak solutions are introduced. By the new weak solution, the stability of weak solutions is proved only dependent on the initial value.

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1 Introduction

Consider the nonlinear heat equation

$$u_t = \operatorname{div}(k(u,x,t)\nabla u) + f(u,x,t), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with appropriately smooth boundary, the function $k(u,x,t)$ has the meaning of nonlinear thermal conductivity, which depends on the temperature $u = u(x,t)$. It is generally assume that the matrix $k(u,x,t)$ is semidefinite. If $k(u,x,t) = k(x)$, Eq. (1.1) becomes a linear parabolic equation, we would like to suggest that, for linear equations, any boundedness estimate is equivalent to a stability result (i.e., control of differences of solutions in terms of differences of data), but this is not the truth for nonlinear equations generally. One can see the well-known monographs or textbooks [1–7] and the references therein. However, in some special case, if we add some restrictions to $k(u,x,t)$, the character may be still true. For simplicity, the paper limits to consider

$$k(u,x,t) = ma(x)u^{m-1}, \qquad m > 0,$$

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with $a(x)>0$ when $x\in\Omega$, but $a(x)=0$ when $x\in\partial\Omega$. In other words, we will consider the following nonlinear equation

$$u_t=\mathrm{div}(a(x)\nabla u^m)+f(u,x,t),\qquad (x,t)\in Q_T. \tag{1.2}$$

From my own perspective, the initial value

$$u(x,0)=u_0(x),\qquad x\in\Omega, \tag{1.3}$$

is indispensable. While, the usual boundary value

$$u(x,t)=0,\qquad (x,t)\in\partial\Omega\times(0,T), \tag{1.4}$$

may be superfluous. To see that, if $f\equiv 0$ in (1.2), we suppose that u and v are two classical solutions of equation (1.2) with the initial values $u(x,0)$ and $v(x,0)$ respectively. Then we have

$$\begin{aligned}&\int_\Omega S_\eta(u^m-v^m)(u-v)_t\mathrm{d}x+\int_\Omega a(x)S'_\eta(u^m-v^m)|\nabla u^m-\nabla v^m|^2\mathrm{d}x\\ =&\int_{\partial\Omega} a(x)S_\eta(u^m-v^m)(\nabla u-\nabla v)\cdot\vec{n}\mathrm{d}\Sigma=0,\end{aligned}$$

where $\vec{n}$ is the outer unit normal vector of Ω, $S_\eta(s)$ is the approximate function of the sign function (the details are given (3.1)-(3.2) later). Then

$$\begin{aligned}&\int_\Omega S_\eta(u^m-v^m)(u-v)_t\mathrm{d}x\le 0,\\ &\lim_{\eta\to 0}\int_\Omega S_\eta(u^m-v^m)(u-v)_t\mathrm{d}x=\int_\Omega \mathrm{sign}(u^m-v^m)(u-v)_t\mathrm{d}x\\ =&\int_\Omega \mathrm{sign}(u-v)(u-v)_t\mathrm{d}x=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|u-v|\mathrm{d}x.\end{aligned}$$

Then, even without any boundary value condition (1.4), the classical solutions have the stability

$$\int_\Omega|u(x,t)-v(x,t)|\mathrm{d}x\le\int_\Omega|u_0(x)-v_0(x)|\mathrm{d}x. \tag{1.5}$$

Certainly, since $|\nabla u^m|$ may be singular or degenerate on $\overline{\Omega}$, equation (1.2) only has a weak solution generally.

Thus, to study the well-posedness of weak solutions to equation (1.2), or a more general reaction-diffusion equation with the type

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_i}\left(a^{ij}(u,x,t)\frac{\partial u}{\partial x_j}\right)+\mathrm{div}(b(u,x,t))+f(u,x,t),\ \ (x,t)\in\Omega\times(0,T), \tag{1.6}$$

the whole boundary value condition (1.4) is overdetermined. For the linear case, the problem had been completely solved by Fichera [8], Oleinik [9] et al., for nonlinear case,