Principal Eigenvalue for Cooperative \((p,q)\)-biharmonic Systems

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Abstract. In this article, we are interested in the simplicity and the existence of the first strictly principal eigenvalue or semitrivial principal eigenvalue of the \((p,q)\)-biharmonic systems with Navier boundary conditions.

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1 Introduction

Let \( \Omega \subseteq \mathbb{R}^N \) (with \( N \geq 1 \)) be a bounded domain with smooth boundary \( \partial \Omega \) and \( \alpha, \beta, p, q \) be constants such that \( \alpha \geq 0, \beta \geq 0, p > 1, q > 1 \) and \( \frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1 \).

Our aim is to study the following eigenvalue problem

\[
\begin{align*}
\Delta_p^2 u - \lambda m_1(x)|u|^p - 2u &= m(x)|v|^{\beta+1}|u|^{a-1}u \quad \text{in } \Omega, \\
\Delta_q^2 v - \lambda m_2(x)|v|^q - 2v &= m(x)|u|^{\alpha+1}|v|^{\beta-1}v \quad \text{in } \Omega, \\
u = \Delta u = v = \Delta v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

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where $\Delta^2_p u = \Delta(|\Delta u|^{p-2} \Delta u)$ is the $p$-biharmonic operator and $\lambda$ is a real parameter. The coefficients $m_1, m_2, m \in L^\infty(\Omega)$ are assumed to be nonnegatives in $\Omega$.

In [1], Talibi and Tsouli have investigated the scalar version of problem $(Q)$ with $m \equiv 0$, which reads

$$\begin{cases}
\Delta(p|\Delta u|^{p-2} \Delta u) = \lambda a(x)|u|^{p-2}u & \text{in } \Omega, \\
u = \Delta u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\rho \in C(\overline{\Omega})$ such that $\rho > 0$ and $a \in L^\infty(\Omega)$. They proved that $(P_{a,p,\rho})$ possesses at least one non-decreasing sequence of eigenvalues and studied $(P_{a,p,\rho})$ in the particular one dimensional case. The authors, in the same reference gave the first eigenvalue $\lambda_{1,p,\rho}(a)$ and showed that if $a \geq 0$ a.e. in $\Omega$, then $\lambda_{1,p,\rho}(a)$ is simple (i.e. the associated eigenfunctions are a constant multiple of one another) and principal i.e. the associated eigenfunction, denoted by $\varphi_{p,\rho,a}$ is positive or negative on $\Omega$ with

$$\lambda_{1,p,\rho}(a) = \inf_{u \in \mathcal{A}} \int_\Omega \rho |\Delta u|^p \, dx,$$

where

$$\mathcal{A} = \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \int_{\Omega} a |u|^p \, dx = 1 \right\}.$$

The problem $(P_{a,p,\rho})$ was considered by P. Drábek and M. Otani for $\rho \equiv 1$ and $a \equiv 1$ [2]. By using a transformation of the problem to a known Poisson problem, they showed that $(P_{a,p,\rho})$ has a principal positive eigenvalue which is simple and isolated. In the case $N = 1$ they gave a description of all eigenvalues and associated eigenfunctions.

El Khalil et al. [3] also considered problem $(P_{a,p,\rho})$ for $\rho \equiv 1, a \equiv 1$ with Dirichlet boundary conditions and showed that the spectrum contains at least one non-decreasing sequence of positive eigenvalues.

Benedikt [4] gave the spectrum of the p-biharmonic operator with Dirichlet and Neumann boundary conditions in the case $N = 1, \rho \equiv 1$ and $a \equiv 1$.

It is important to note that $(u, \lambda)$ is solution of problem $(P_{m_1,p,1})$ if and only if $[(u,0);\lambda]$ is solution of $(Q)$. This kind of solution is called "semitrivial solution" of $(Q)$. Furthermore if $[(u,0);\lambda]$ is solution of $(Q)$ with $u$ of one sign on $\Omega$, then $\lambda$ is called "semitrivial principal eigenvalue" of $(Q)$. Consequently, there are two forms of semitrivial solutions for problem $(Q)$: one of the type $[(u,0);\lambda]$ with $u \equiv 0$ and $(u,\lambda)$ solution of the problem $(P_{m_1,p,1})$ and the second of the type $[(0,v);\lambda]$ with $v \equiv 0$ and $(v,\lambda)$ solution of the problem $(P_{m_2,q,1})$. In particular $\lambda_{1,p,1}(m_1)$ and $\lambda_{1,q,1}(m_2)$ are semitrivial principal eigenvalues of $(Q)$.

This paper is organized as follows. We construct the eigencurve associated to problem $(Q)$ in Section 2. Section 3 is devoted to the study of strictly principal eigenvalue of $(Q)$. 

Throughout this work, the Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r, \forall r \in (1,\infty]$ and the norm in a normed space $X$ by $\|\cdot\|_X$. We denote by
\[
Y_{pq}(\Omega) = \left[ W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \right] \times \left[ W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \right],
\]
which is a reflexive Banach space endowed with the norm
\[
\|(u,v)\| = \|\Delta u\|_p + \|\Delta v\|_q
\]
(see, e.g., [5]). The weak convergence in $Y_{pq}(\Omega)$ is denoted by $\rightharpoonup$. The positive and negative part of a function $w$ are denoted by $w^+ = \max\{w,0\}$ and $w^- = \max\{-w,0\}$. Equalities (and inequalities) between two functions must be understood a.e..

For all $f \in L^r(\Omega)$, the Poisson equation associated with the Dirichlet problem
\[
\begin{align*}
-\Delta u &= f(x) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
is uniquely solvable in $X_r = W^{2,r}(\Omega) \cap W^{1,r}_0(\Omega)$ (see for example [6]). We denote by $\Lambda$ the inverse operator of $-\Delta: X_r \rightarrow L^r(\Omega)$. The following lemma gives us some properties of the operator $\Lambda$:

**Lemma 1.1.** ([1, 2]).

1. **(Continuity)** There exists a constant $c_r > 0$ such that
   \[
   \|\Lambda f\|_{W^{2,r}} \leq c_r \|f\|_r
   \]
   holds for all $r \in (1,\infty)$ and $f \in L^r(\Omega)$.

2. **(Continuity)** Given $k \in \mathbb{N}^*$, there exists a constant $c_{r,k} > 0$ such that
   \[
   \|\Lambda f\|_{W^{k,2r}} \leq c_{r,k} \|f\|_{W^{k,r}}
   \]
   holds for all $r \in (1,\infty)$ and $f \in W^{k,r}(\Omega)$.

3. **(Symmetry)** The identity
   \[
   \int_{\Omega} \Lambda u \cdot v dx = \int_{\Omega} u \cdot \Lambda v dx
   \]
   holds for $u \in L^r(\Omega)$ and $v \in L^{r'}(\Omega)$ with $r' = \frac{r}{r-1}$ and $r \in (1,\infty)$.

4. **(Regularity)** Given $f \in L^\infty(\Omega)$, we have $\Lambda f \in C^{1,\nu}(\overline{\Omega})$ for all $\nu \in (0,1)$. Moreover, there exists $c_\nu > 0$ such that
   \[
   \|\Lambda f\|_{C^{1,\nu}(\Omega)} \leq c_\nu \|f\|_\infty.
   \]

5. **(Regularity and Hopf-type maximum principle)** Let $f \in C(\overline{\Omega})$ and $f \geq 0$ then $w = \Lambda f \in C^{1,\nu}(\overline{\Omega})$, for all $\nu \in (0,1)$ and $w$ satisfies: $w > 0$ in $\Omega$, $\frac{\partial w}{\partial n} < 0$ on $\partial \Omega$.

6. **(Order preserving property)** Given $f, g \in L^r(\Omega)$ if $f \leq g$ in $\Omega$, then $\Lambda f \leq \Lambda g$ in $\Omega$. 

2 An eigenvalue curve associated to problem \((Q)\)

It is well established that (see, e.g., [7–11]), in order to prove the existence of strictly principal eigenvalue or semitrivial principal eigenvalue of \((Q)\), one fixes \(\lambda\) and embeds the problem into the new eigenvalue problem of parameter \(\mu \in \mathbb{R}\):

\[
\begin{cases}
    \Delta_p^2 u - m(x)|v|^{\beta+1}|u|^\gamma u - \lambda m_1(x)|u|^p - u = \mu |u|^p u & \text{in } \Omega, \\
    \Delta_q v - m(x)|v|^{\beta+1}v - \lambda m_2(x)|v|^\gamma - v = \mu |v|^\gamma v & \text{in } \Omega, \\
    u = \Delta u = v = \Delta v = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

\((Q_\lambda)\)

**Definition 2.1.**

1. \([u,v];\mu \in Y_{p,q}(\Omega) \times \mathbb{R}\) is a (weak) solution to problem \((Q_\lambda)\) if

\[
\int_\Omega |\Delta u|^p - |\Delta v|^q + u \varphi_1 dx - \int_\Omega m |v|^{\beta+1} |u|^\gamma \varphi_1 dx - \lambda \int_\Omega m_1 |u|^p - u \varphi_1 dx = \mu \int_\Omega |u|^p u \varphi_1 dx,
\]

\((2.2)\)

2. \([u,v];\lambda \in Y_{p,q}(\Omega) \times \mathbb{R}\) is a (weak) solution to problem \((Q)\) if

\[
\int_\Omega |\Delta u|^p - |\Delta v|^q + u \varphi_1 dx - \int_\Omega m |v|^{\beta+1} |u|^\gamma \varphi_1 dx = \lambda \int_\Omega m_1 |u|^p - u \varphi_1 dx,
\]

\((2.4)\)

for all \((\varphi_1, \varphi_2) \in Y_{p,q}(\Omega).

3. If \([u,v];\lambda \in Y_{p,q}(\Omega) \times \mathbb{R}\) (resp. \([u,v];\mu \in Y_{p,q}(\Omega) \times \mathbb{R}\)) is a (weak) solution to problem \((Q)\) (resp. \((Q_\lambda)\)), \((u,v)\) shall be called an eigenfunction of the problem \((Q)\) (resp. \((Q_\lambda)\)) associated to the eigenvalue \(\lambda\) (resp. \(\mu(\lambda)\)). Let us agree to say that an eigenvalue of \((Q)\) or \((Q_\lambda)\) is strictly principal (resp. semitrivial principal) if it is associated to an eigenfunction \((u,v)\) such that \(u > 0\) or \(u < 0\) and \(v > 0\) or \(v < 0\) (resp. \([u > 0\) and \(v \equiv 0\) or \(u < 0\) and \(v \equiv 0\)) or \([u \equiv 0\) and \(v > 0\) or \(u \equiv 0\) and \(v < 0\)).

We are going to consider the smallest eigenvalue \(\mu \in \mathbb{R}\) of problem \((Q_\lambda)\). In order to do so, we define the energy functional

\[
J_\lambda: Y_{p,q}(\Omega) \rightarrow \mathbb{R}
\]
\[(u,v) \mapsto J_\lambda(u,v) = \frac{\alpha+1}{p} \|\Delta u\|_p^p + \frac{\beta+1}{q} \|\Delta v\|_q^q - V(u,v) - \lambda M(u,v),\]

where

\[V(u,v) = \int_\Omega m|u|^{\alpha+1}|v|^{\beta+1}dx, \quad M(u,v) = \frac{\alpha+1}{p} M_1(u) + \frac{\beta+1}{q} M_2(v)\]

with

\[M_1(u) = \int_\Omega m_1 |u|^p dx, \quad M_2(v) = \int_\Omega m_2 |v|^q dx, \quad \forall (u,v) \in \mathcal{Y}_{pq}(\Omega).\]

Equalities (2.2) and (2.3) are equivalent to

\[\nabla J_\lambda(u,v) = \mu \nabla I(u,v)\]

where

\[I(u,v) = \frac{\alpha+1}{p} \|u\|_p^p + \frac{\beta+1}{q} \|v\|_q^q, \quad \forall (u,v) \in \mathcal{Y}_{pq}(\Omega).\]

**Lemma 2.1.** Let \((\omega_1, \omega_2) \in L^\infty(\Omega) \times L^\infty(\Omega).\) If \(\omega_1, \omega_2 > 0\) on \(\Omega\) then there exist three positive constants \(c_1, c_2, c_3\) such that

\[\|\Delta u\|_p^p + \|\Delta v\|_q^q \leq c_1 J_\lambda(u,v) + c_2 \int_\Omega \omega_1 |u|^p dx + c_3 \int_\Omega \omega_2 |v|^q dx\]

for every \((u,v) \in \mathcal{Y}_{pq}(\Omega).\)

**Proof.** We only sketch it since it is adapted from [10] in \((p,q)\)-laplacian systems case. First, note that

\[M_1(u) \leq \|m_1\|_\infty \|u\|_p^p, \quad M_2(u) \leq \|m_2\|_\infty \|u\|_q^q.\]

Since \(\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1\), it is well known by Young inequality that:

\[V(u,v) \leq \|m\|_\infty \int_\Omega \left[ \frac{\alpha+1}{p} |u|^p + \frac{\beta+1}{q} |v|^q \right] dx.\]

We set \(k_3 = \max\{k_1, k_2\}\) with:

\[k_1 = \|m\|_\infty \max \left\{ \frac{\alpha+1}{p}, \frac{\beta+1}{q} \right\}, \quad k_2 = |\lambda| \max \left\{ \frac{\alpha+1}{p} \|m_1\|_\infty, \frac{\beta+1}{q} \|m_2\|_\infty \right\}.\]

Then, one has:

\[V(u,v) \leq k_1 (\|u\|_p^p + \|v\|_q^q), \quad |\lambda M(u,v)| \leq k_2 (\|u\|_p^p + \|v\|_q^q).\]
On the other hand according to the proof of [9, Lemma 2] in p-Laplacian case, for $\varepsilon > 0$ there exist $M_\varepsilon > 0$ and $M'_\varepsilon > 0$ such that:

$$\|u\|_p^p \leq \varepsilon \|\Delta u\|_p^p + M_\varepsilon \int_\Omega \omega_1 |u|^p \, dx,$$

$$\|v\|_q^q \leq \varepsilon \|\Delta v\|_q^q + M'_\varepsilon \frac{1}{\varepsilon} \int_\Omega \omega_2 |v|^q \, dx.$$ 

Now, we have

$$\frac{\alpha + 1}{p} \|\Delta u\|_p^p + \frac{\beta + 1}{q} \|\Delta v\|_q^q = J_\lambda(u,v) - V(u,v) + \lambda M(u,v).$$

Then, one has:

$$\frac{\alpha + 1}{p} \|\Delta u\|_p^p + \frac{\beta + 1}{q} \|\Delta v\|_q^q \leq J_\lambda(u,v) + 2k_3 \left( \|u\|_p^p + \|v\|_q^q \right)$$

$$\leq J_\lambda(u,v) + 2ek_3 \left( \|\Delta u\|_p^p + \|\Delta v\|_q^q \right) + 2k_3 M_\varepsilon \int_\Omega \omega_1 |u|^p \, dx + 2k_3 M'_\varepsilon \int_\Omega \omega_2 |v|^q \, dx.$$ 

Let $\varepsilon > 0$ be such that $k_4 = \min \left\{ \frac{\alpha + 1}{p} - 2ek_3, \frac{\beta + 1}{q} - 2ek_3 \right\} > 0$. Thus, one has:

$$k_4 (\|\Delta u\|_p^p + \|\Delta v\|_q^q) \leq J_\lambda(u,v) + 2k_3 M_\varepsilon \int_\Omega \omega_1 |u|^p \, dx + 2k_3 M'_\varepsilon \int_\Omega \omega_2 |v|^q \, dx.$$ 

We deduce

$$\|\Delta u\|_p^p + \|\Delta v\|_q^q \leq \frac{1}{k_4} J_\lambda(u,v) + \frac{2k_3 M_\varepsilon}{k_4} \int_\Omega \omega_1 |u|^p \, dx + \frac{2k_3 M'_\varepsilon}{k_4} \int_\Omega \omega_2 |v|^q \, dx.$$ 

We can take $c_1 = 1/k_4$, $c_2 = 2k_3 M_\varepsilon / k_4$ and $c_3 = 2k_3 M'_\varepsilon / k_4$. □

**Proposition 2.1.** The value

$$\mu_1(\lambda) := \inf \{ J_\lambda(u,v) : (u,v) \in \mathcal{M} \},$$

(2.8)

where

$$\mathcal{M} = \{(u,v) \in Y_{pq}(\Omega) : I(u,v) = 1 \},$$

is the smallest eigenvalue of $(Q_\lambda)$.

**Proof.** By Lemma 2.1, one has for $\omega_1 = \omega_2 \equiv 1,$

$$0 \leq \|\Delta u\|_p^p + \|\Delta v\|_q^q \leq c_1 J_\lambda(u,v) + c_2 \int_\Omega |u|^p \, dx + c_3 \int_\Omega |v|^q \, dx$$

$$\leq c_1 J_\lambda(u,v) + c_2 3 \left[ \frac{\alpha + 1}{p} \int_\Omega |u|^p \, dx + \frac{\beta + 1}{q} \int_\Omega |v|^q \, dx \right]$$

$$= c_1 J_\lambda(u,v) + c_2 3, \quad \forall (u,v) \in \mathcal{M},$$
where $c_{2,3} = \max\left\{ \frac{pc_2}{\alpha + \Gamma}, \frac{qc_3}{\beta + \Gamma} \right\}$, so that $f_\lambda$ is bounded below on $\mathcal{M}$. Furthermore any sequence $(u_n,v_n)$ that minimizes $f_\lambda$ on $\mathcal{M}$ is bounded in $Y_{pq}(\Omega)$.

Thus there exists $(u_0,v_0) \in Y_{pq}(\Omega)$ such that, up to a subsequence, $(u_n,v_n)$ converges weakly to $(u_0,v_0)$ in $Y_{pq}(\Omega)$ and strongly in $L^p(\Omega) \times L^q(\Omega)$. Hence

$$J_\lambda(u_0,v_0) \leq \lim_{n \to \infty} J_\lambda(u_n,v_n) = \mu_1(\lambda), \quad (u_0,v_0) \in \mathcal{M}$$

and consequently $J_\lambda(u_0,v_0) = \mu_1(\lambda)$. By the Lagrange multipliers rule, $\mu_1(\lambda)$ is an eigenvalue for $(Q_\lambda)$ and $(u_0,v_0)$ is an associated eigenfunction. Moreover for any eigenvalue $\mu(\lambda)$ associated to $(u_\lambda,v_\lambda) \in Y_{pq}(\Omega) \setminus \{(0,0)\}$, $J_\lambda(u_\lambda,v_\lambda) = \mu(\lambda) I(u_\lambda,v_\lambda)$ with $I(u_\lambda,v_\lambda) > 0$.

Consequently

$$\mu_1(\lambda) \leq J_\lambda \left( \frac{u_\lambda}{I(u_\lambda,v_\lambda)^{\frac{1}{2}}}, \frac{v_\lambda}{I(u_\lambda,v_\lambda)^{\frac{1}{2}}} \right) = \frac{J_\lambda(u_\lambda,v_\lambda)}{I(u_\lambda,v_\lambda)} = \mu(\lambda).$$

We conclude that $\mu_1(\lambda)$ is the smallest eigenvalue of $(Q_\lambda)$. $\square$

For $m = m_1 = m_2 \equiv 0$, we denote by

$$\mu_0 = \mu_1(\lambda) = \inf \left\{ \frac{\alpha + 1}{p} \| \Delta u \|_p^p + \frac{\beta + 1}{q} \| \Delta v \|_q^q : (u,v) \in \mathcal{M} \right\}.$$ 

Since the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with $r \in \{p,q\}$ does not contain any constant non trivial function, one has $\mu_0 > 0$.

\begin{proposition}
The following results hold
\begin{enumerate}
\item $\mu_1$ is concave and differentiable with $\mu_1'(\lambda) = -M(u_0,v_0)$ where $(u_0,v_0)$ is some eigenfunction of $(Q_\lambda)$ associated to $\mu_1(\lambda)$ for all $\lambda \in \mathbb{R}$.
\item $\lim_{\lambda \to +\infty} \mu_1(\lambda) = -\infty$.
\item $\mu_1$ is strictly decreasing.
\end{enumerate}
\end{proposition}

\begin{proof}
We provide the proof in the following steps.
\begin{enumerate}
\item The concavity of $\mu_1$ follows from the concavity of the mapping $\lambda \mapsto f_\lambda(u,v)$, for a fixed $(u,v) \in Y_{pq}(\Omega)$. In particular $\mu_1$ is continuous. Now let $\lambda_n \to \lambda$ and $(u_n,v_n)$, $(u_\lambda,v_\lambda)$ be the $I$-normalized eigenfunctions related to $\mu_1(\lambda_n)$, $\mu_1(\lambda)$ respectively. We apply Lemma 2.1 with $\omega_1 = \omega_2 = 1$ to get

$$\| \Delta u_n \|_p^p + \| \Delta v_n \|_q^q \leq c_1 J_\lambda(u_n,v_n) + c_2 \int_\Omega |u_n|^p dx + c_3 \int_\Omega |v_n|^q dx,$$

$$\leq c_1 J_\lambda(u_n,v_n) + \max \left\{ \frac{pc_2}{\alpha + \Gamma}, \frac{qc_3}{\beta + \Gamma} \right\}$$

\end{enumerate}


This completes the proof of the proposition.

Moreover
\[ \lim_{n \to \infty} c_1 \mu_1(\lambda_n) + \max \left\{ \frac{pc_2}{\alpha+1}, \frac{qc_3}{\beta+1} \right\} = c_1 \mu_1(\lambda) + \max \left\{ \frac{pc_2}{\alpha+1}, \frac{qc_3}{\beta+1} \right\}. \]

So we conclude that \((u_n,v_n)_n\) is a bounded sequence in \(Y_{pq}(\Omega)\). Hence there exists \((u_0,v_0)\) such that, up to a subsequence, \((u_n,v_n) \rightharpoonup (u_0,v_0)\) in \(Y_{pq}(\Omega)\), strongly in \(L^p(\Omega) \times L^q(\Omega)\). Then \((u_0,v_0) \in \mathcal{M}\) and from \[ J_\lambda(u_0,v_0) \leq \lim_{n \to \infty} J_\lambda(u_n,v_n) = \mu_1(\lambda) \]
we infer that \(\mu_1(\lambda) = J_\lambda(u_0,v_0) = J_\lambda(u_\lambda,v_\lambda)\) and \((u_0,v_0)\) is an eigenfunction of \((Q_\lambda)\) associated to \(\mu_1(\lambda)\). Furthermore
\[
\begin{cases}
\mu_1(\lambda_n) - \mu_1(\lambda) \geq -(\lambda_n - \lambda)M(u_n,v_n), \\
\mu_1(\lambda_n) - \mu_1(\lambda) \leq -(\lambda_n - \lambda)M(u_0,v_0).
\end{cases}
\]

Hence
\[
\begin{cases}
-M(u_n,v_n) \leq \frac{\mu_1(\lambda_n) - \mu_1(\lambda)}{\lambda_n - \lambda} \leq -M(u_0,v_0), & \text{if } \lambda_n > \lambda, \\
-M(u_0,v_0) \leq \frac{\mu_1(\lambda_n) - \mu_1(\lambda)}{\lambda_n - \lambda} \leq -M(u_n,v_n), & \text{if } \lambda_n < \lambda.
\end{cases}
\]

Passing to the limit we get \(\mu_1'(\lambda) = -M(u_0,v_0)\).

2. We know that \(m_1\) is nonnegative, then there exists a function \(u \in X_p\) such that \(M_1(u) > 0\) and \(I(u,0) = 1\). Then, for all \(\lambda \in \mathbb{R}^+\), \(\mu_1(\lambda) \leq J_\lambda(u,0)\). We deduce that
\[
\lim_{\lambda \to +\infty} J_\lambda(u,0) = \lim_{\lambda \to +\infty} E_m(u,0) - \lambda M(u,0) = -\infty,
\]
where
\[
E_m(u,v) = \frac{\alpha+1}{p} \|\Delta u\|^p_p + \frac{\beta+1}{q} \|\Delta v\|^q_q - \int_\Omega m|u|^\alpha|v|^\beta+1 dx.
\]

Thus \(\lim_{\lambda \to +\infty} \mu_1(\lambda) = -\infty\).

3. The result is clear from the fact that \(M(u_\lambda,v_\lambda) > 0\) for any \(\lambda \in \mathbb{R}\). Indeed, if \(\lambda_1 < \lambda_2\) then
\[
\mu_1(\lambda_1) = E_m(u_{\lambda_1},v_{\lambda_1}) - \lambda_1 M(u_{\lambda_1},v_{\lambda_1}) \geq E_m(u_{\lambda_1},v_{\lambda_1}) - \lambda_2 M(u_{\lambda_1},v_{\lambda_1}) \geq \mu_1(\lambda_2).
\]

This completes the proof of the proposition. \(\square\)
3 Strictly or semitrivial principal eigenvalues

Note that, if $\mu_1(\lambda) = 0$ then $\lambda$ is an eigenvalue of problem $(Q)$. Our purpose is to find a reasonable assumption on $m$ so that there exists at least one $\lambda \in (0, \infty)$ such that $\mu_1(\lambda) = 0$.

**Lemma 3.1.** If $\|m\|_\infty < \mu_0$ then, $\mu_1(0) > 0$ and $\mu_1(\lambda) = 0$ has a unique positive solution $\lambda$ (eigenvalue of $(Q)$).

**Proof.** Assume that $\|m\|_\infty < \mu_0$. By (2.7), we have $V(u, v) \leq \|m\|_\infty I(u, v)$, $\forall (u, v) \in Y_{pq}(\Omega)$. Then, one has

$$
\frac{\alpha + 1}{p} \|\Delta u\|_p^p + \frac{\beta + 1}{q} \|\Delta v\|_q^q - \|m\|_\infty I(u, v) \leq E_m(u, v), \quad \forall (u, v) \in Y_{pq}(\Omega).
$$

We deduce that:

$$
\mu_0 \leq E_m(u, v) + \|m\|_\infty, \quad \forall (u, v) \in M,
$$

$$
\mu_0 - \|m\|_\infty \leq \inf \{E_m(u, v), (u, v) \in M\} \leq \mu_1(0).
$$

Consequently, $\mu_1(0) > 0$. Moreover, from Proposition 2.1, $\mu_1$ is strictly decreasing. We deduce that, $\mu_1(\lambda) = 0$ has a unique positive solution $\lambda$ and $\lambda$ is an eigenvalue of $(Q)$. \qed

We will denote by

$$
L(\Omega) := ([L^p(\Omega) \times L^q(\Omega)] \setminus \{(0, 0)\}) \times \mathbb{R},
$$

$$
L_0(\Omega) := ([L^p(\Omega) \times L^q(\Omega)] \setminus \{(0, 0)\}) \times \{0\}.
$$

We apply some results proved by Drábek and Ótani [2] and some ideas used by Talbi and Tsouli [1].

**Remark 3.1.**

1. $\forall u \in X_r$, $\forall v \in L^r(\Omega)$ with $r \in (1, \infty)$: $v = -\Delta u \iff u = \Lambda v$.

2. Let $N_r$ be the Nemyskii operator with $r \in (1, \infty)$, defined by

$$
N_r(u)(x) = \begin{cases} |u(x)|^{r-2}u(x) & \text{if } u(x) \neq 0, \\
0 & \text{if } u(x) = 0.
\end{cases}
$$

We have

$$
\forall v \in L^r(\Omega), \quad \forall w \in L^{r'}(\Omega): \quad N_r(v) = w \iff v = N'_r(w)
$$

with $r' = \frac{r}{r-1}$. 


Lemma 3.2. If \([(u,v);\mu(\lambda)]\) is a solution of \((Q_\lambda)\) then \(-\Delta u, -\Delta v \in C(\overline{\Omega})\) and \(u, v \in C^{1,\nu}(\overline{\Omega})\), for all \(\nu \in (0,1)\).
Proof. Without loss of generality, one can assume that $p \leq q$. Let $p_0 \in [p, \infty)$, $q_0 \in [q, \infty)$ such that $p_0 = q_0$ if $p = q$. Suppose that $\varphi = N_{p'}(\Lambda \theta_1) \in L^{p_0}(\Omega)$, $w = N_{q'}(\Lambda \theta_2) \in L^{q_0}(\Omega)$ with

$$
\begin{cases}
\theta_1 = \omega_1 N_p(\Lambda \varphi) + m|\Lambda w|^{\beta+1}|\Lambda \varphi|^{\alpha-1} \Lambda \varphi, \\
\theta_2 = \omega_2 N_q(\Lambda w) + m|\Lambda \varphi|^{\alpha+1}|\Lambda w|^{\beta-1} \Lambda w,
\end{cases}
$$

where $\omega_1 \in L^\infty(\Omega)$, $\omega_2 \in L^\infty(\Omega)$. It is easy to see that:

1. If $p = q$, then:
   
   - (a) $\varphi$, $w \in L^{p_1}(\Omega)$, with $\frac{1}{p_1} = \frac{1}{p_0} - \frac{2p'}{N}$, if $p_0 < \frac{N}{2p'}$
   
   - (b) $\varphi$, $w \in L^{\frac{N}{p}}(\Omega)$, $\forall k \in (1, +\infty)$, if $p_0 = \frac{N}{2p'}$
   
   - (c) $\varphi$, $w \in C(\Omega)$, if $p_0 > \frac{N}{2p'}$. Indeed, one have
     - i. $\varphi$, $w \in C(\Omega)$, if $\frac{N}{2} < p_0$.
     - ii. If $\frac{N}{2} = p_0$, then $\theta_1$, $\theta_2 \in L^{\frac{N}{p}}(\Omega)$, for all $k \in (1, +\infty)$. We can take $k$ such that $\frac{k}{p-1} > \frac{N}{2}$. Thus $\varphi$, $w \in C(\Omega)$.
     - iii. If $\frac{N}{2p'} < p_0 < \frac{N}{2}$, then: $\theta_1$, $\theta_2 \in L^{\frac{N}{p}}(\Omega)$ with $r_0 = \frac{Np_0}{N-2p_0}$ and $\frac{r_0}{p-1} > \frac{N}{2}$. Then $\varphi$, $w \in C(\Omega)$.

2. If $p < q$, then:
   
   - (a) If $p_0 < \frac{N}{2p'}$, then:
     - i. $\theta_1 \in L^{\frac{N_0}{p}}(\Omega)$ and $\varphi \in L^{p_1}(\Omega)$ with $r_0 = \frac{Np_0}{N-2p_0}$, $p_1 = \frac{Np_0(p-1)-2pp_0}{N(p-1)-2pp_0}$, if $q_0 \geq \frac{N}{2}$.
     - ii. If $q_0 < \frac{N}{2}$, then:
       - A. $\theta_1 \in L^{\frac{N_0}{p}}(\Omega)$ and $\varphi \in L^{p_1}(\Omega)$ with $s_0 = \frac{N_0}{N-2p_0}$, if $ps_0 > qr_0$.
       - B. $\theta_2 \in L^{\frac{N_0}{q}}(\Omega)$, if $ps_0 < qr_0$.
       - C. $\theta_1 \in L^{\frac{N_0}{p}}(\Omega)$, $\theta_2 \in L^{\frac{N_0}{q}}(\Omega)$ and $\varphi \in L^{p_1}(\Omega)$, if $ps_0 = qr_0$.
   
   - (b) If $p_0 = \frac{N}{2p'}$, then:
     - i. $\varphi \in L^{\frac{N_0}{p-1}}(\Omega)$, $\forall k \in (1, +\infty)$, if $q_0 \geq \frac{N}{2}$.
     - ii. If $q_0 < \frac{N}{2}$, then:
       - A. $\varphi \in L^{\frac{N_0}{p-1}}(\Omega)$, $\forall k \in (1, +\infty)$, if $ps_0 > qr_0$.
       - B. $\theta_2 \in L^{\frac{N_0}{q}}(\Omega)$, if $ps_0 < qr_0$.
       - C. $\theta_2 \in L^{\frac{N_0}{q}}(\Omega)$ and $\varphi \in L^{\frac{N_0}{p-1}}(\Omega)$, $\forall k \in (1, +\infty)$, if $ps_0 = qr_0$. 

(c) if \( p_0 > \frac{N}{2p'}, \) then

i. \( \varphi, w \in C(\Omega), \) if \( \frac{N}{2} \leq p_0 < q_0. \)

ii. if \( \frac{N}{2p'} \leq p_0 < \frac{N}{2}, \) then:
   A. \( \varphi \in C(\Omega), \) if \( \frac{N}{2} \leq q_0. \)
   B. \( \theta_1 \in L^{\frac{m}{p}}(\Omega) \) or \( \theta_2 \in L^{\frac{m}{p'}}(\Omega) \) if \( \frac{N}{2} > q_0. \)

Let \([u, v); \mu(\lambda)] \in Y_{pq}(\Omega) \times \mathbb{R} \) be a solution of \((Q_\lambda)\), then \([\varphi, w); \mu(\lambda)] \) is a solution of \((Q_\lambda')\) with \( \varphi = -\Delta u = N_{p'}(\Lambda \theta_1) \in L^p(\Omega), w = -\Delta v = N_q(\Lambda \theta_2) \in L^q(\Omega) \) with

\[
\begin{cases}
\theta_1 = \omega_1 N_p(\Lambda \varphi) + m|\Lambda w|^\beta + |\Lambda \varphi|^{\alpha - 1} \Lambda \varphi, \\
\theta_2 = \omega_2 N_q(\Lambda w) + m|\Lambda \varphi|\beta + |\Lambda \varphi|^{\alpha - 1} \Lambda \varphi,
\end{cases}
\]

where \( \omega_1 = \mu(\lambda) + \lambda m_1 \in L^\infty(\Omega), \omega_2 = \mu(\lambda) + \lambda m_2 \in L^\infty(\Omega). \)

**Case (1):** \( p = q \)

We easily see that \( \varphi, w \in C(\Omega) \) from assertion 1c, if \( p > \frac{N}{2p'} \).

Now take suitable \( (p_n), p = p_0 \) and \( k \in \mathbb{N} \) such that \( p_{k-1} < \frac{N}{2p'} < p_k \) with \( \frac{1}{p_k} = \frac{1}{p_0} - 2\frac{k'}{N}. \)

Then applying assertion 1a with \( p_0 = p_0, p_1, ..., p_{k-1}, \) we deduce \( \varphi, w \in L^{p_k}(\Omega). \) Hence from assertion 1c, \( \varphi, w \in C(\Omega) \) follows.

**Case (2):** \( p < q \) and \( \frac{N}{2p'} \leq p. \)

1. We deduce \( \varphi, w \in C(\Omega) \) from assertion 2b and 2c, if \( \frac{N}{2} \leq q. \)

2. If \( \frac{N}{2} > q, \) take suitable \( s_n = \frac{N\theta_n}{N-2q_n} \) with \( \frac{1}{q_n} = \frac{1}{q_0} - 2\frac{n'}{N}, q_0 = q \) and \( k \in \mathbb{N} \) such that \( \frac{N}{2} > \frac{N}{2} \).

Then applying assertion 2b and 2c with \( q_0 = q_0, q_1, ..., q_k, p_0 = p, \) we deduce \( \theta_2 \in L^{\frac{m}{p}}(\Omega). \) Hence \( \Lambda \theta_2 \in C(\Omega) \) and \( \varphi, w \in C(\Omega) \) follows.

**Case (3):** \( p < q \) and \( \frac{N}{2q'} > p. \)

1. If \( \frac{N}{2q'} \leq q, \) take suitable \( (p_n), p = p_0 \) and \( k \in \mathbb{N} \) such that \( p_{k-1} < \frac{N}{2p'} < p_k \) with \( \frac{1}{p_k} = \frac{1}{p_0} - 2\frac{k'}{N}. \)

Then applying assertion 2a with \( p_0 = p_0, p_1, ..., p_{k-1}, q_0 = q, \) we deduce \( \varphi \in L^{p_k}(\Omega) \) and \( \varphi, w \in C(\Omega) \) follows.

2. If \( \frac{N}{2q'} > q, \) take suitable \( (p_n), (q_n) \) and \( k \in \mathbb{N} \) such that \( p = p_0, q = q_0, p_{k-1} < \frac{N}{2p'} < p_k, q_{j-1} < \frac{N}{2q'} < q_j \) with \( \frac{1}{p_k} = \frac{1}{p_0} - 2\frac{k'}{N} \) and \( \frac{1}{q_j} = \frac{1}{q_0} - 2\frac{j'}{N}. \)

Then applying assertion 2a with \( p_0 = p_0, p_1, ..., p_{k-1}, q_0 = q_0, q_1, ..., q_{j-1}, \) we deduce \( \varphi \in L^{p_k}(\Omega), w \in L^{q_j}(\Omega) \) and \( \varphi, w \in C(\Omega) \) follows.
Hence we deduce that $\phi, w \in L^\infty(\Omega)$ and from the assertion in Lemma 1.1 that $u = \Lambda \phi$, $v = \Lambda w \in C^1(\overline{\Omega})$ for all $v \in (0,1)$. \hfill \Box

**Lemma 3.3.** $[(\phi_1, w_1); \mu_1(\lambda)] \in L(\Omega)$ is a solution of problem $(Q'_\lambda)$, if and only if

$$G_\lambda(\phi_1, w_1) = 0 = \min_{(\phi, w) \in L^*(\Omega)} G_\lambda(\phi, w)$$

(3.6)

where

$$G_\lambda(\phi, w) = F_\lambda(\phi, w) - \mu_1(\lambda) R(\phi, w), \quad L^*(\Omega) = [L^p(\Omega) \times L^q(\Omega)] \setminus \{(0,0)\}.$$

*Proof.* Assume that $[(\phi_1, w_1); \mu_1(\lambda)] \in L(\Omega)$ is a solution of problem $(Q'_\lambda)$. Then $F_\lambda(\phi_1, w_1) = \mu_1(\lambda) R(\phi_1, w_1)$. Hence $G_\lambda(\phi_1, w_1) = F_\lambda(\phi_1, w_1) - \mu_1(\lambda) R(\phi_1, w_1) = 0$. Put

$$\bar{\phi} = \frac{\phi}{|R(\phi, w)|^\frac{1}{p}}, \quad \bar{w} = \frac{w}{|R(\phi, w)|^\frac{1}{q}}$$

for every $(\phi, w) \in L^*(\Omega)$. Then $R(\bar{\phi}, \bar{w}) = 1$. We deduce that

$$\mu_1(\lambda) \leq F_\lambda(\bar{\phi}, \bar{w}) = \frac{F_\lambda(\phi, w)}{R(\phi, w)},$$

$$G_\lambda(\phi, w) = F_\lambda(\phi, w) - \mu_1(\lambda) R(\phi, w) \geq 0$$

(3.7) (3.8)

for all $(\phi, w) \in L^*(\Omega)$. We claim that (3.6) holds.

Now suppose that (3.6) holds. We deduce that $\nabla G_\lambda(\phi_1, w_1) = (0,0)$. Then

$$\langle \frac{\partial G_\lambda}{\partial \phi}(\phi_1, w_1), \Psi \rangle = \langle \frac{\partial G_\lambda}{\partial w}(\phi_1, w_1), \theta \rangle = 0,$$

(3.9)

for all $(\Psi, \theta) \in [L^p(\Omega) \times L^q(\Omega)]$. Hence, $[(\phi_1, w_1); \mu_1(\lambda)] \in L(\Omega)$ is a solution of $(Q'_\lambda)$. \hfill \Box

**Lemma 3.4.** If $(H_m)$ holds and $[(\phi_1, w_1); \mu_1(\lambda)] \in L_0(\Omega)$ is a solution of problem $(Q'_\lambda)$ then $[(|\phi_1|, |w_1|); \mu_1(\lambda)] \in L_0(\Omega)$ is a solution of problem $(Q'_\lambda)$.

*Proof.* Assume that $(H_m)$ holds and $[(\phi_1, w_1); \mu_1(\lambda)] \in L_0(\Omega)$ is a solution of problem $(Q'_\lambda)$. Then $G_\lambda(\phi_1, w_1) = 0$, $\mu_1(\lambda) = 0$, $\lambda > 0$ and $[(|\phi_1|, |w_1|)] \in [L^p(\Omega) \times L^q(\Omega)] \setminus \{(0,0)\}$. Hence $G_\lambda(|\phi_1|, |w_1|) \geq 0$.

Additionally, one has $|\Lambda(|\phi_1|)|^r \geq |\Lambda \phi_1|^r$ and $|\Lambda(|w_1|)|^r \geq |\Lambda w_1|^r$, for all $r \in (1;\infty)$. We deduce that:

$$-\lambda \int_{\Omega} m_1 |\Lambda(|\phi_1|)|^p d\alpha \leq -\lambda \int_{\Omega} m_1 |\Lambda \phi_1|^p d\alpha,$$

$$-\lambda \int_{\Omega} m_2 |\Lambda(|w_1|)|^q d\alpha \leq -\lambda \int_{\Omega} m_2 |\Lambda w_1|^q d\alpha,$$

$$-m |\Lambda(|\phi_1|)|^{n+1} |\Lambda(|w_1|)|^{\beta+1} d\alpha \leq -m |\Lambda \phi_1|^{n+1} |\Lambda w_1|^{|\beta+1|} d\alpha.$$

Consequently, $F_\lambda(|\phi_1|, |w_1|) \leq F_\lambda(\phi_1, w_1)$ and $G_\lambda(|\phi_1|, |w_1|) \leq G_\lambda(\phi_1, w_1) = 0$. Thus $G_\lambda(|\phi_1|, |w_1|) = 0$ and $[(|\phi_1|, |w_1|); \mu_1(\lambda)]$ is solution of $(Q'_\lambda)$.
**Proposition 3.1.** Assume that \((H_m)\) holds and \(\mu_1(\lambda) = 0\). Then \(\lambda\) is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem \((Q)\).

**Proof.** Assume that \((H_m)\) holds and \(\mu_1(\lambda) = 0\). Then \(\lambda\) is an eigenvalue of problem \((Q)\) associated with \((u,v) \in Y_{pq}(\Omega) \setminus \{(0,0)\}\).

If \(u \not\equiv 0\) and \(v \not\equiv 0\), then \([\{(\varphi,w);\mu_1(\lambda)\},\{(|\varphi|,|w|);\mu_1(\lambda)\}]\in L_0(\Omega)\) are solutions of problem \((Q_\lambda)\) with \(\varphi = -\Delta u \not\equiv 0\) and \(w = -\Delta v \not\equiv 0\). Since \(|\varphi| \geq 0\) and \(|w| \geq 0\), then \(\Lambda(|\varphi|) > 0\), \(\Lambda(|w|) > 0\). Therefore

\[
N_p(\Lambda|\varphi|) > 0, \quad N_q(\Lambda|w|) > 0, \quad |\Lambda(|w|)|^{\beta+1}|\Lambda(|\varphi|)|^a > 0, \quad |\Lambda(|\varphi|)|^{\alpha+1}|\Lambda(|w|)|^{\beta} > 0
\]

and

\[
\begin{align*}
|\varphi| &= N_p' \left( \Lambda \left[ \lambda m_1 N_p(\Lambda|\varphi|) + m |\Lambda(|w|)|^{\beta+1}|\Lambda(|\varphi|)|^a - \Lambda(|\varphi|) \right] \right) > 0, \\
|w| &= N_q' \left( \Lambda \left[ \lambda m_2 N_q(\Lambda|w|) + m |\Lambda(|\varphi|)|^{\alpha+1}|\Lambda(|w|)|^{\beta} - \Lambda(|w|) \right] \right) > 0.
\end{align*}
\]

We then conclude that \([\{(\varphi,w);\mu_1(\lambda)\}]\) is solution of problem \((Q_\lambda)\) with \(\varphi\) positive in \(\Omega\) or negative in \(\Omega\) and \(w\) is positive in \(\Omega\) or negative in \(\Omega\).

Since by Lemma 3.2, \(\varphi, w \in C(\overline{\Omega})\), we deduce that \(u = \Lambda \varphi\) positive in \(\Omega\) or negative in \(\Omega\) and \(v = \Lambda w\) positive in \(\Omega\) or negative in \(\Omega\), from the Lemma 1.1. Then \(\lambda\) is strictly principal eigenvalue of \((Q)\).

If \([u \equiv 0\) and \(v \equiv 0]\) or \([u \equiv 0\) and \(v \equiv 0]\), then we also prove that \([u \equiv 0\) in \(\Omega\) or \(v < 0\) in \(\Omega\)\) or \([u > 0\) in \(\Omega\) or \(v < 0\) in \(\Omega\) and \(v \equiv 0]\). Then \(\lambda\) is a semitrivial principal eigenvalue of \((Q)\). \(\square\)

**Lemma 3.5.** Let \(A, B, C\) and \(r\) be real numbers satisfying \(A \geq 0, B \geq 0, C \geq \max\{B-A,0\}\) and \(r \in [1, +\infty)\). Then

\[|A+C|^r + |B-C|^r \geq A^r + B^r.\]

**Proof.** See the proof of [2, Lemme 2.5] if \(r \in (1, +\infty)\). Assume that \(r = 1\), then

\[
\begin{align*}
|A+C| + |B-C| &= A+C + B-C = A+B, \quad \text{if} \quad B-C \geq 0 \\
|A+C| + |B-C| &= A-B + 2C > A+B, \quad \text{if} \quad B-C < 0.
\end{align*}
\]

Thus \(|A+C| + |B-C| \geq A+B.\) \(\square\)

**Lemma 3.6.** Suppose that \((H_m)\) holds. If \((\varphi_1,w_1)\) and \((\varphi_2,w_2)\) are positive eigenfunctions of problem \((Q_\lambda)\) associated with \(\mu_1(\lambda) = 0\), then \((\varphi_{12},w_{12}), (\varphi_{12},w_{21}), (\varphi_{21},w_{12})\) and \((\varphi_{21},w_{21})\) with

\[
\begin{align*}
\varphi_{12}(x) &:= \max\{\varphi_1(x),\varphi_2(x)\} = \varphi_1(x) + (\varphi_2 - \varphi_1)^+(x) \\
w_{12}(x) &:= \max\{w_1(x),w_2(x)\} = w_1(x) + (w_2 - w_1)^+(x) \\
\varphi_{21}(x) &:= \min\{\varphi_1(x),\varphi_2(x)\} = \varphi_2(x) - (\varphi_2 - \varphi_1)^+(x) \\
w_{21}(x) &:= \min\{w_1(x),w_2(x)\} = w_2(x) - (w_2 - w_1)^+(x)
\end{align*}
\]

for all \(x \in \Omega\), are eigenfunctions of \((Q_\lambda)\) associated with \(\mu_1(\lambda) = 0.\)
Proof. Assume that \((H_u)\) holds and \((\varphi_1, w_1), (\varphi_2, w_2)\) are positive eigenfunctions of problem \((Q_\lambda)\) associated with \(\mu_1(\lambda) = 0\). By Lemma 3.5 we get

\[
\begin{align*}
\left\{
\begin{array}{l}
|\Lambda \varphi_{12}|^p + |\Lambda \varphi_{21}|^p \geq |\Lambda \varphi_1|^p + |\Lambda \varphi_2|^p \\
|\Lambda w_{12}|^q + |\Lambda w_{21}|^q \geq |\Lambda w_1|^q + |\Lambda w_2|^q \\
|\Lambda \varphi_{12}|^{a+1} + |\Lambda \varphi_{21}|^{a+1} \geq |\Lambda \varphi_1|^{a+1} + |\Lambda \varphi_2|^{a+1} \\
|\Lambda w_{12}|^{\beta+1} + |\Lambda w_{21}|^{\beta+1} \geq |\Lambda w_1|^{\beta+1} + |\Lambda w_2|^{\beta+1}.
\end{array}
\right.
\end{align*}
\]

Then, one has:

\[
-\lambda \int_{\Omega} m_1 |\Lambda \varphi_{12}|^p dx - \lambda \int_{\Omega} m_1 |\Lambda \varphi_{21}|^p dx \leq -\lambda \int_{\Omega} m_1 |\Lambda \varphi_1|^p dx - \lambda \int_{\Omega} m_1 |\Lambda \varphi_2|^p dx,
\]

\[
-\lambda \int_{\Omega} m_2 |\Lambda w_{12}|^q dx - \lambda \int_{\Omega} m_2 |\Lambda w_{21}|^q dx \leq -\lambda \int_{\Omega} m_2 |\Lambda w_1|^q dx - \lambda \int_{\Omega} m_2 |\Lambda w_2|^q dx.
\]

Likewise, we have

\[
Z_1(\varphi, w) \leq Z_2(\varphi, w) \leq -\int_{\Omega} m |\Lambda \varphi_1|^{a+1} |\Lambda w_1|^{\beta+1} dx - \int_{\Omega} m |\Lambda \varphi_2|^{a+1} |\Lambda w_2|^{\beta+1} dx,
\]

with

\[
Z_1(\varphi, w) = -\int_{\Omega} m |\Lambda \varphi_{12}|^{a+1} |\Lambda w_{12}|^{\beta+1} dx - \int_{\Omega} m |\Lambda \varphi_{21}|^{a+1} |\Lambda w_{21}|^{\beta+1} dx
\]

\[
- \int_{\Omega} m |\Lambda \varphi_{21}|^{a+1} |\Lambda w_{12}|^{\beta+1} dx - \int_{\Omega} m |\Lambda \varphi_{21}|^{a+1} |\Lambda w_{21}|^{\beta+1} dx,
\]

\[
Z_2(\varphi, w) = -\int_{\Omega} m |\Lambda \varphi_1|^{a+1} |\Lambda w_1|^{\beta+1} dx - \int_{\Omega} m |\Lambda \varphi_2|^{a+1} |\Lambda w_2|^{\beta+1} dx
\]

\[
- \int_{\Omega} m |\Lambda \varphi_1|^{a+1} |\Lambda w_{12}|^{\beta+1} dx - \int_{\Omega} m |\Lambda \varphi_1|^{a+1} |\Lambda w_{21}|^{\beta+1} dx.
\]

Additionally, we have:

\[
\int_{\Omega} |\varphi_{12}|^p dx + \int_{\Omega} |\varphi_{21}|^p dx = \int_{\Omega} |\varphi_1|^p dx + \int_{\Omega} |\varphi_2|^p dx,
\]

\[
\int_{\Omega} |w_{12}|^q dx + \int_{\Omega} |w_{21}|^q dx = \int_{\Omega} |w_1|^q dx + \int_{\Omega} |w_2|^q dx.
\]

By (3.10)–(3.14) we deduce that:

\[
F_\lambda(\varphi_{12}, w_{12}) + F_\lambda(\varphi_{12}, w_{21}) + F_\lambda(\varphi_{21}, w_{12}) + F_\lambda(\varphi_{21}, w_{21}) \leq F_\lambda(\varphi_1, w_1) + F_\lambda(\varphi_2, w_2),
\]

\[
G_\lambda(\varphi_{12}, w_{12}) + G_\lambda(\varphi_{12}, w_{21}) + G_\lambda(\varphi_{21}, w_{12}) + G_\lambda(\varphi_{21}, w_{21}) \leq G_\lambda(\varphi_1, w_1) + G_\lambda(\varphi_2, w_2) = 0.
\]

It follows that

\[
G_\lambda(\varphi_{12}, w_{12}) = G_\lambda(\varphi_{12}, w_{21}) = G_\lambda(\varphi_{21}, w_{12}) = G_\lambda(\varphi_{21}, w_{21}) = 0.
\]

Hence \((\varphi_{12}, w_{12}), (\varphi_{12}, w_{21}), (\varphi_{21}, w_{12})\) and \((\varphi_{21}, w_{21})\) are eigenfunctions of \((Q_\lambda)\) associated with \(\mu_1(\lambda) = 0\).
Proposition 3.2. Assume that \((H_m)\) holds and \(\mu_1(\lambda) = 0\). Then \(\lambda\) is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem \((Q)\) and simple.

Proof. Assume that \((H_m)\) holds and \(\mu_1(\lambda) = 0\). Then \(\lambda\) is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem \((Q)\) from Proposition 3.1.

Case 1: Take \(\lambda\) as a strictly principal eigenvalue of \((Q)\).

Let \((u_{11}, u_{12})\) and \((u_{21}, u_{22})\) be two positive eigenfunctions of \((Q)\) associated with \(\lambda\).

Then, \([(v, w); 0], \((\varphi, \psi); 0\), \([(v), |w|]; 0\), \([(\varphi), |\psi|]; 0\) \(\in L_0(\Omega)\), are solutions of \((Q_1)\) with \(v = -\Delta u_{11} > 0\), \(w = -\Delta u_{12} > 0\), \(\varphi = -\Delta u_{21} > 0\) and \(\psi = -\Delta u_{22} > 0\).

For \(x_0 \in \Omega\), we set

\[
\begin{align*}
    k &= \frac{\varphi(x_0)}{v(x_0)}, \\
    \omega_1(x) &= \max\{\varphi(x), kv(x)\} \quad \text{and} \quad \omega_2(x) = \max\{\psi(x), k^\frac{p}{p-1}w(x)\},
\end{align*}
\]

for all \(x \in \Omega\).

From Lemma 3.6, \([(\omega_1, \omega_2); 0]\) is a solution of problem \((Q_1')\) because \([(kv, k^\frac{p}{p-1}w); 0]\) and \([(\varphi, \psi); 0]\) are solutions of \((Q_1)\). We deduce that \(N_p(v), N_q(w), N_p(\varphi), N_q(\psi), N_p(\omega_1), N_q(\omega_2) \in C^1(\Omega)\) and \(N_p(v), N_q(\psi) \in C^1(\Omega)\).

For any unit vector \(e = (0, \ldots, e_i, \ldots, 0)\) with \(i \in \{1, \ldots, N\}\) and \(t \in \mathbb{R}\), we have

\[
\begin{align*}
    &N_p(\varphi)(x_0 + te) - N_p(\varphi)(x_0) \leq N_p(\omega_1)(x_0 + te) - N_p(\omega_1)(x_0), \\
    &N_p(kv)(x_0 + te) - N_p(kv)(x_0) \leq N_p(\omega_1)(x_0 + te) - N_p(\omega_1)(x_0).
\end{align*}
\]

Dividing these inequalities by \(t > 0\) and \(t < 0\) and letting \(t\) tend to \(0^\pm\), we get

\[
\begin{align*}
    \frac{\partial}{\partial x_i}[N_p(\varphi)](x_0) &\leq \frac{\partial}{\partial x_i}[N_p(\omega_1)](x_0), \\
    \frac{\partial}{\partial x_i}[N_p(kv)](x_0) &\leq \frac{\partial}{\partial x_i}[N_p(\omega_1)](x_0), \\
    \frac{\partial}{\partial x_i}[N_p(\varphi)](x_0) &\geq \frac{\partial}{\partial x_i}[N_p(\omega_1)](x_0), \\
    \frac{\partial}{\partial x_i}[N_p(kv)](x_0) &\geq \frac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),
\end{align*}
\]

for all \(i \in \{1, \ldots, N\}\). Thus,

\[
\begin{align*}
    \frac{\partial}{\partial x_i}[N_p(\varphi)](x_0) &= \frac{\partial}{\partial x_i}[N_p(\omega_1)](x_0), \\
    \frac{\partial}{\partial x_i}[N_p(kv)](x_0) &= \frac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),
\end{align*}
\]

for all \(i \in \{1, \ldots, N\}\). Hence,

\[
\nabla N_p(\varphi)(x_0) = \nabla N_p(\omega_1)(x_0) = \nabla N_p(kv)(x_0) = k^\frac{p-1}{p} \nabla N_p(v)(x_0).
\]
Furthermore, we have

$$\nabla \left( \frac{N_p(\varphi)}{N_p(v)} \right)(x_0) = \frac{\nabla (N_p(\varphi))(x_0)N_p(v)(x_0) - N_p(\varphi)(x_0)\nabla (N_p(v))(x_0)}{[N_p(v)(x_0)]^2}$$

$$= \frac{[N_p(v)(x_0) - k^{1-p}N_p(\varphi)(x_0)]\nabla (N_p(\varphi))(x_0)}{[N_p(v)(x_0)]^2} = 0.$$

We deduce that for all $x_0 \in \Omega$, $\nabla \left( \frac{N_p(\varphi)}{N_p(v)} \right)(x_0) = 0$. Consequently, $N_p(\varphi/v) = \text{const} = k^{p-1}$ in $\Omega$. Then, $\varphi = kv$ in $\Omega$.

It is easy to see all the same that $\psi = hw$. For $x_0 \in \Omega$, we set $h = \psi(x_0)/w(x_0)$, $\overline{\omega}_1(x) = \max \{ \psi(x), hw(x) \}$ and $\overline{\omega}_2(x) = \max \{ \varphi(x), k^{p}v(x) \}$, for all $x \in \Omega$.

Accordingly, $(\varphi, \psi) = (kv, hw)$ with $k = h^{\frac{1}{p}}$. We deduce that $(u_{21}, u_{22}) = (ku_{11}, hu_{12})$ with $k = h^{\frac{1}{p}}$.

Let $(u_{11}, u_{12})$ and $(u_{21}, u_{22})$ be two eigenfunctions of $(Q)$ associated with $\lambda$. If there exist $i, j \in \{1, 2\}$ such that $u_{ij} < 0$, then we can set $\overline{u}_{ij} = -u_{ij}$ and the result follows.

**Case 2:** Take $\lambda$ as a semitrivial principal eigenvalue of $(Q)$.

Let $[(u_{11}, 0)$ and $(u_{21}, 0)]$ or $[(0, u_{12})$ and $(0, u_{22})]$ be two eigenfunctions of $(Q)$ associated with $\lambda$. It is easy to see that there exist $[k \neq 0 \text{ real number}]$ or $[h \neq 0 \text{ real number}]$ such that $[u_{11} = ku_{21}]$ or $[u_{12} = hu_{22}]$.

**Theorem 3.1.** Assume that $(H_m)$ holds. The lowest positive eigenvalue of problem $(Q)$ is the value

$${\lambda}_1 = \min_{(u,v) \in \mathcal{S}} E_m(u,v),$$

where

$$\mathcal{S} = \{(u,v) \in Y_{pq}(\Omega): M(u,v) = 1 \}.$$

Moreover

1. $\lambda_1 \leq \min \{ \lambda_{1,p,1}(m_1), \lambda_{1,q,1}(m_2) \}$.

2. $\lambda_1$ is semitrivial principal eigenvalue or strictly principal eigenvalue.

3. $\lambda_1$ is simple.
Proof. Assume that \((H_m)\) holds. Then from Proposition 2.2 and Lemma 3.1, there exists a unique real \(\lambda_1 \in (0, \infty)\) solution of equation \(\mu_1(\lambda) = 0\), \(\lambda_1\) is an eigenvalue of \((Q)\) and
\[
\mu_1'(\lambda_1) = -M(u_0,v_0) \neq 0 = \mu_1(\lambda_1) = E_m(u_0,v_0) - \lambda_1 M(u_0,v_0)
\]
with \((u_0,v_0) \in M\). Then, \(E_m(u_0,v_0) = \lambda_1 M(u_0,v_0) > 0\) and we can set
\[
\overline{u}_0 = \frac{u_0}{|M(u_0,v_0)|^{\frac{1}{p}}}, \quad \overline{v}_0 = \frac{v_0}{|M(u_0,v_0)|^{\frac{1}{q}}}.
\]
Thus, \((\overline{u}_0,\overline{v}_0) \in \mathcal{S}\) and \(E_m(\overline{u}_0,\overline{v}_0) = \lambda_1\).

Additionally, for every \((u,v) \in \mathcal{S}\), one has
\[
E_m \left( \frac{u}{|I(u,v)|^{\frac{1}{p}}}, \frac{v}{|I(u,v)|^{\frac{1}{q}}} \right) \geq \lambda_1 M \left( \frac{u}{|I(u,v)|^{\frac{1}{p}}}, \frac{v}{|I(u,v)|^{\frac{1}{q}}} \right), \text{ i.e. } E_m(u,v) \geq \lambda_1.
\]
Consequently (3.15) holds. Moreover, from Proposition 3.2, \(\lambda_1\) is a strictly principal eigenvalue or semitrivial principal eigenvalue and simple.

Set \(\varphi_p = \left( \frac{p}{a+1} \right)^{\frac{1}{p}} \varphi_{p,1,m_1}\) and \(\varphi_q = \left( \frac{q}{b+1} \right)^{\frac{1}{q}} \varphi_{q,1,m_2}\). Then
\[
\frac{a+1}{p} M_1(\varphi_p) + \frac{b+1}{q} M_2(0) = 1, \quad \frac{a+1}{p} M_1(0) + \frac{b+1}{q} M_2(\varphi_q) = 1.
\]
Thus
\[
\begin{cases}
\lambda_1 \leq E_m(\varphi_p,0) = \frac{a+1}{p} \|\Delta \varphi_p\|_p = \lambda_{1,p,1}(m_1), \\
\lambda_1 \leq E_m(0,\varphi_q) = \frac{b+1}{q} \|\Delta \varphi_q\|_q = \lambda_{1,q,1}(m_2).
\end{cases}
\]
Consequently, \(\lambda_1 \leq \min\{\lambda_{1,p,1}(m_1),\lambda_{1,q,1}(m_2)\}\). \(\square\)

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References


