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Principal Eigenvalue for Cooperative (p,q)-biharmonic Systems

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Abstract. In this article, we are interested in the simplicity and the existence of the first strictly principal eigenvalue or semitrivial principal eigenvalue of the (p,q)-biharmonic systems with Navier boundary conditions.

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1 Introduction

Let $\Omega\subset\mathbb{R}^N$ (with $N\geq 1$) be a bounded domain with smooth boundary $\partial\Omega$ and α, β, p, q be constants such that $\alpha\geq 0$, $\beta\geq 0$, $p>1$, $q>1$ and $\dfrac{\alpha+1}{p}+\dfrac{\beta+1}{q}=1$.

Our aim is to study the following eigenvalue problem

$$(Q):\begin{cases}\Delta_p^2 u-\lambda m_1(x)|u|^{p-2}u=m(x)|v|^{\beta+1}|u|^{\alpha-1}u & \text{in } \Omega,\\ \Delta_q^2 v-\lambda m_2(x)|v|^{q-2}v=m(x)|u|^{\alpha+1}|v|^{\beta-1}v & \text{in } \Omega,\\ u=\Delta u=v=\Delta v=0 & \text{on } \partial\Omega,\end{cases}$$

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where $\Delta_p^2 u = \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator and λ is a real parameter. The coefficients m_1, m_2, $m \in L^\infty(\Omega)$ are assumed to be nonnegatives in Ω.

In [1], Talbi and Tsouli have investigated the scalar version of problem (Q) with $m \equiv 0$, which reads

$$(P_{a,p,\rho}): \begin{cases} \Delta(\rho|\Delta u|^{p-2}\Delta u) = \lambda a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\rho \in C(\overline{\Omega})$ such that $\rho > 0$ and $a \in L^\infty(\Omega)$. They proved that $(P_{a,p,\rho})$ possesses at least one non-decreasing sequence of eigenvalues and studied $(P_{a,p,\rho})$ in the particular one dimensional case. The authors, in the same reference gave the first eigenvalue $\lambda_{1,p,\rho}(a)$ and showed that if $a \geq 0$ a.e. in Ω, then $\lambda_{1,p,\rho}(a)$ is simple (i.e. the associated eigenfunctions are a constant multiple of one another) and principal i.e. the associated eigenfunction, denoted by $\varphi_{p,\rho,a}$ is positive or negative on Ω with

$$\lambda_{1,p,\rho}(a) = \inf_{u \in \mathcal{A}} \int_\Omega \rho|\Delta u|^p \mathrm{d}x, \tag{1.1}$$

where

$$\mathcal{A} = \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \int_\Omega a|u|^p \mathrm{d}x = 1 \right\}. \tag{1.2}$$

The problem $(P_{a,p,\rho})$ was considered by P. Drábek and M. Ôtani for $\rho \equiv 1$ and $a \equiv 1$ [2]. By using a transformation of the problem to a known Poisson problem, they showed that $(P_{a,p,\rho})$ has a principal positive eigenvalue which is simple and isolated. In the case $N = 1$ they gave a description of all eigenvalues and associated eigenfunctions.

El Khalil et al. [3] also considered problem $(P_{a,p,\rho})$ for $\rho \equiv 1$, $a \equiv 1$ with Dirichlet boundary conditions and showed that the spectrum contains at least one non-decreasing sequence of positive eigenvalues.

Benedikt [4] gave the spectrum of the p-biharmonic operator with Dirichlet and Neumann boundary conditions in the case $N = 1$, $\rho \equiv 1$ and $a \equiv 1$.

It is important to note that (u,λ) is solution of problem $(P_{m_1,p,1})$ if and only if $[(u,0);\lambda]$ is solution of (Q). This kind of solution is called "semitrivial solution" of (Q). Furthermore if $[(u,0);\lambda]$ is solution of (Q) with u of one sign on Ω, then λ is called "semitrivial principal eigenvalue" of (Q). Consequently, there are two forms of semitrivial solutions for problem (Q): one of the type $[(u,0);\lambda]$ with $u \not\equiv 0$ and (u,λ) solution of the problem $(P_{m_1,p,1})$ and the second of the type $[(0,v);\lambda]$ with $v \not\equiv 0$ and (v,λ) solution of the problem $(P_{m_2,q,1})$. In particular $\lambda_{1,p,1}(m_1)$ and $\lambda_{1,q,1}(m_2)$ are semitrivial principal eigenvalues of (Q).

This paper is organized as follows. We construct the eigencurve associated to problem (Q) in Section 2. Section 3 is devoted to the study of strictly principal eigenvalue of (Q).

Throughout this work, the Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$, $\forall r\in(1,\infty]$ and the norm in a normed space X by $\|\cdot\|_X$. We denote by

$$Y_{pq}(\Omega)=\left[W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\right]\times\left[W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)\right],$$

which is a reflexive Banach space endowed with the norm

$$\|(u,v)\|=\|\Delta u\|_p+\|\Delta v\|_q$$

(see, e.g., [5]). The weak convergence in $Y_{pq}(\Omega)$ is denoted by $\rightharpoonup$. The positive and negative part of a function w are denoted by $w^+=\max\{w,0\}$ and $w^-=\max\{-w,0\}$. Equalities (and inequalities) between two functions must be understood a.e..

For all $f\in L^r(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases}-\Delta u=f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

is uniquely solvable in $X_r=W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega)$ (see for example [6]). We denote by Λ the inverse operator of $-\Delta: X_r\longmapsto L^r(\Omega)$. The following lemma gives us some properties of the operator Λ:

Lemma 1.1. ([1,2]).

1. *(Continuity) There exists a constant $c_r>0$ such that*
$$\|\Lambda f\|_{W^{2,r}}\le c_r\|f\|_r$$
holds for all $r\in(1,\infty)$ and $f\in L^r(\Omega)$.

2. *(Continuity) Given $k\in\mathbb{N}^*$, there exists a constant $c_{r,k}>0$ such that*
$$\|\Lambda f\|_{W^{k+2,r}}\le c_{r,k}\|f\|_{W^{k,r}}$$
holds for all $r\in(1,\infty)$ and $f\in W^{k,r}(\Omega)$.

3. *(Symmetry) The identity*
$$\int_\Omega \Lambda u\cdot v\mathrm{d}x=\int_\Omega u\cdot\Lambda v\mathrm{d}x$$
holds for $u\in L^r(\Omega)$ and $v\in L^{r'}(\Omega)$ with $r'=\frac{r}{r-1}$ and $r\in(1,\infty)$.

4. *(Regularity) Given $f\in L^\infty(\Omega)$, we have $\Lambda f\in C^{1,\nu}(\overline{\Omega})$ for all $\nu\in(0,1)$. Moreover, there exists $c_\nu>0$ such that*
$$\|\Lambda f\|_{C^{1,\nu}(\Omega)}\le c_\nu\|f\|_\infty.$$

5. *(Regularity and Hopf-type maximum principle) Let $f\in C(\overline{\Omega})$ and $f\ge0$ then $w=\Lambda f\in C^{1,\nu}(\overline{\Omega})$, for all $\nu\in(0,1)$ and w satisfies: $w>0$ in Ω, $\frac{\partial w}{\partial n}<0$ on $\partial\Omega$.*

6. *(Order preserving property) Given $f,g\in L^r(\Omega)$ if $f\le g$ in Ω, then $\Lambda f<\Lambda g$ in Ω.*

2 An eigenvalue curve associated to problem (Q)

It is well established that (see, e.g., [7–11]), in order to prove the existence of strictly principal eigenvalue or semitrivial principal eigenvalue of (Q), one fixes λ and embeds the problem into the new eigenvalue problem of parameter $\mu\in\mathbb{R}$:

$$(Q_\lambda):\begin{cases}\Delta_p^2 u-m(x)|v|^{\beta+1}|u|^{\alpha-1}u-\lambda m_1(x)|u|^{p-2}u=\mu|u|^{p-2}u & \text{in }\ \Omega,\\ \Delta_q^2 v-m(x)|u|^{\alpha+1}|v|^{\beta-1}v-\lambda m_2(x)|v|^{q-2}v=\mu|v|^{q-2}v & \text{in }\ \Omega,\\ u=\Delta u=v=\Delta v=0 & \text{on }\ \partial\Omega.\end{cases} \tag{2.1}$$

Definition 2.1. *1.* $[(u,v);\mu]\in Y_{p,q}(\Omega)\times\mathbb{R}$ *is a (weak) solution to problem* (Q_λ) *if*

$$\begin{aligned}&\int_\Omega|\Delta u|^{p-2}\Delta u\Delta\varphi_1\mathrm{d}x-\int_\Omega m|v|^{\beta+1}|u|^{\alpha-1}u\varphi_1\mathrm{d}x-\lambda\int_\Omega m_1|u|^{p-2}u\varphi_1\mathrm{d}x\\&\quad=\mu\int_\Omega|u|^{p-2}u\varphi_1\mathrm{d}x,\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\int_\Omega|\Delta v|^{q-2}\Delta v\Delta\varphi_2\mathrm{d}x-\int_\Omega m|u|^{\alpha+1}|v|^{\beta-1}v\varphi_2\mathrm{d}x-\lambda\int_\Omega m_2|v|^{q-2}v\varphi_2\mathrm{d}x\\&\quad=\mu\int_\Omega|v|^{q-2}v\varphi_1\mathrm{d}x,\end{aligned} \tag{2.3}$$

for all $(\varphi_1,\varphi_2)\in Y_{pq}(\Omega)$.

2. $[(u,v);\lambda]\in Y_{p,q}(\Omega)\times\mathbb{R}$ *is a (weak) solution to problem* (Q) *if*

$$\int_\Omega|\Delta u|^{p-2}\Delta u\Delta\varphi_1\mathrm{d}x-\int_\Omega m|v|^{\beta+1}|u|^{\alpha-1}u\varphi_1\mathrm{d}x=\lambda\int_\Omega m_1|u|^{p-2}u\varphi_1\mathrm{d}x, \tag{2.4}$$

$$\int_\Omega|\Delta v|^{q-2}\Delta v\Delta\varphi_2\mathrm{d}x-\int_\Omega m|u|^{\alpha+1}|v|^{\beta-1}v\varphi_2\mathrm{d}x=\lambda\int_\Omega m_2|v|^{q-2}v\varphi_2\mathrm{d}x, \tag{2.5}$$

for all $(\varphi_1,\varphi_2)\in Y_{pq}(\Omega)$.

3. *If* $[(u,v);\lambda]\in Y_{p,q}(\Omega)\times\mathbb{R}$ *(resp.* $[(u,v);\mu]\in Y_{p,q}(\Omega)\times\mathbb{R}$*) is a (weak) solution to problem* (Q) *(resp.* (Q_λ)*),* (u,v) *shall be called an eigenfunction of the problem* (Q) *(resp.* (Q_λ)*) associated to the eigenvalue* λ *(resp.* $\mu(\lambda)$*). Let us agree to say that an eigenvalue of* (Q) *or* (Q_λ) *is strictly principal (resp. semitrivial principal) if it is associated to an eigenfunction* (u,v) *such that* $u>0$ *or* $u<0$ *and* $v>0$ *or* $v<0$ *(resp. [*$u>0$ *and* $v\equiv0$ *or* $u<0$ *and* $v\equiv0$*] or [*$u\equiv0$ *and* $v>0$ *or* $u\equiv0$ *and* $v<0$*]).*

We are going to consider the smallest eigenvalue $\mu\in\mathbb{R}$ of problem (Q_λ). In order to do so, we define the energy functional

$$J_\lambda:\ Y_{p,q}(\Omega)\ \longrightarrow\ \mathbb{R}$$

$$(u,v) \quad \longmapsto \quad J_\lambda(u,v) = \frac{\alpha+1}{p}\|\Delta u\|_p^p + \frac{\beta+1}{q}\|\Delta v\|_q^q - V(u,v) - \lambda M(u,v),$$

where

$$V(u,v) = \int_\Omega m|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x, \qquad M(u,v) = \frac{\alpha+1}{p}M_1(u) + \frac{\beta+1}{q}M_2(v)$$

with

$$M_1(u) = \int_\Omega m_1|u|^p\mathrm{d}x, \qquad M_2(v) = \int_\Omega m_2|v|^q\mathrm{d}x, \quad \forall (u,v) \in Y_{pq}(\Omega).$$

Equalities (2.2) and (2.3) are equivalent to

$$\nabla J_\lambda(u,v) = \mu \nabla I(u,v)$$

where

$$I(u,v) = \frac{\alpha+1}{p}\|u\|_p^p + \frac{\beta+1}{q}\|v\|_q^q, \qquad \forall (u,v) \in Y_{pq}(\Omega).$$

Lemma 2.1. *Let $(\omega_1, \omega_2) \in L^\infty(\Omega) \times L^\infty(\Omega)$. If $\omega_1, \omega_2 > 0$ on Ω then there exist three positive constants c_1, c_2, c_3 such that*

$$\|\Delta u\|_p^p + \|\Delta v\|_q^q \le c_1 J_\lambda(u,v) + c_2\int_\Omega \omega_1|u|^p\mathrm{d}x + c_3\int_\Omega \omega_2|v|^q\mathrm{d}x, \tag{2.6}$$

for every $(u,v) \in Y_{pq}(\Omega)$.

Proof. We only sketch it since it is adapted from [10] in (p,q)-laplacian systems case. First, note that

$$M_1(u) \le \|m_1\|_\infty \|u\|_p^p, \qquad M_2(u) \le \|m_2\|_\infty \|u\|_q^q.$$

Since $\dfrac{\alpha+1}{p} + \dfrac{\beta+1}{q} = 1$, it is well known by Young inequality that:

$$V(u,v) \le \|m\|_\infty \int_\Omega \left[\frac{\alpha+1}{p}|u|^p + \frac{\beta+1}{q}|v|^q\right]\mathrm{d}x. \tag{2.7}$$

We set $k_3 = \max\{k_1, k_2\}$ with:

$$k_1 = \|m\|_\infty \max\left\{\frac{\alpha+1}{p}, \frac{\beta+1}{q}\right\}, \qquad k_2 = |\lambda| \max\left\{\frac{\alpha+1}{p}\|m_1\|_\infty, \frac{\beta+1}{q}\|m_2\|_\infty\right\}.$$

Then, one has:

$$V(u,v) \le k_1\left(\|u\|_p^p + \|v\|_q^q\right), \qquad |\lambda M(u,v)| \le k_2\left(\|u\|_p^p + \|v\|_q^q\right).$$

On the other hand according to the proof of [9, Lemma 2] in p-Laplacian case, for $\varepsilon>0$ there exist $M_\varepsilon>0$ and $M_\varepsilon^{'}>0$ such that:

$$\|u\|_p^p\leq\varepsilon\|\Delta u\|_p^p+M_\varepsilon\int_\Omega\omega_1|u|^p\mathrm{d}x,\qquad \|v\|_q^q\leq\varepsilon\|\Delta v\|_q^q+M_\varepsilon^{'}\int_\Omega\omega_2|v|^q\mathrm{d}x.$$

Now, we have

$$\frac{\alpha+1}{p}\|\Delta u\|_p^p+\frac{\beta+1}{q}\|\Delta v\|_q^q=J_\lambda(u,v)-V(u,v)+\lambda M(u,v).$$

Then, one has:

$$\begin{aligned}&\frac{\alpha+1}{p}\|\Delta u\|_p^p+\frac{\beta+1}{q}\|\Delta v\|_q^q\leq J_\lambda(u,v)+2k_3\left(\|u\|_p^p+\|v\|_q^q\right)\\ \leq& J_\lambda(u,v)+2\varepsilon k_3\left(\|\Delta u\|_p^p+\|\Delta v\|_q^q\right)+2k_3M_\varepsilon\int_\Omega\omega_1|u|^p\mathrm{d}x+2k_3M_\varepsilon^{'}\int_\Omega\omega_2|v|^q\mathrm{d}x.\end{aligned}$$

Let $\varepsilon>0$ be such that $k_4=\min\left\{\frac{\alpha+1}{p}-2\varepsilon k_3,\frac{\beta+1}{q}-2\varepsilon k_3\right\}>0$. Thus, one has:

$$k_4(\|\Delta u\|_p^p+\|\Delta v\|_q^q)\leq J_\lambda(u,v)+2k_3M_\varepsilon\int_\Omega\omega_1|u|^p\mathrm{d}x+2k_3M_\varepsilon^{'}\int_\Omega\omega_2|v|^q\mathrm{d}x.$$

We deduce

$$\|\Delta u\|_p^p+\|\Delta v\|_q^q\leq\frac{1}{k_4}J_\lambda(u,v)+\frac{2k_3M_\varepsilon}{k_4}\int_\Omega\omega_1|u|^p\mathrm{d}x+\frac{2k_3M_\varepsilon^{'}}{k_4}\int_\Omega\omega_2|v|^q\mathrm{d}x.$$

We can take $c_1=1/k_4$, $c_2=2k_3M_\varepsilon/k_4$ and $c_3=2k_3M_\varepsilon^{'}/k_4$. □

Proposition 2.1. *The value*

$$\mu_1(\lambda):=\inf\{J_\lambda(u,v):(u,v)\in\mathcal{M}\},\tag{2.8}$$

where

$$\mathcal{M}=\{(u,v)\in Y_{pq}(\Omega):I(u,v)=1\},$$

is the smallest eigenvalue of (Q_λ).

Proof. By Lemma 2.1, one has for $\omega_1=\omega_2\equiv 1$,

$$\begin{aligned}0\leq\|\Delta u\|_p^p+\|\Delta v\|_q^q\leq& c_1J_\lambda(u,v)+c_2\int_\Omega|u|^p\mathrm{d}x+c_3\int_\Omega|v|^q\mathrm{d}x\\ \leq& c_1J_\lambda(u,v)+c_{2,3}\left[\frac{\alpha+1}{p}\int_\Omega|u|^p\mathrm{d}x+\frac{\beta+1}{q}\int_\Omega|v|^q\mathrm{d}x\right]\\ =&c_1J_\lambda(u,v)+c_{2,3},\qquad\forall(u,v)\in\mathcal{M},\end{aligned}$$

where $c_{2,3}=\max\{\frac{pc_2}{\alpha+1},\frac{qc_3}{\beta+1}\}$, so that J_λ is bounded below on $\mathcal{M}$. Furthermore any sequence (u_n,v_n) that minimizes J_λ on $\mathcal{M}$ is bounded in $Y_{pq}(\Omega)$.

Thus there exists $(u_0,v_0)\in Y_{pq}(\Omega)$ such that, up to a subsequence, (u_n,v_n) converges weakly to (u_0,v_0) in $Y_{pq}(\Omega)$ and strongly in $L^p(\Omega)\times L^q(\Omega)$. Hence

$$J_\lambda(u_0,v_0)\leq \lim_{n\longrightarrow\infty} J_\lambda(u_n,v_n)=\mu_1(\lambda),\quad (u_0,v_0)\in\mathcal{M}$$

and consequently $J_\lambda(u_0,v_0)=\mu_1(\lambda)$. By the Lagrange multipliers rule, $\mu_1(\lambda)$ is an eigenvalue for (Q_λ) and (u_0,v_0) is an associated eigenfunction. Moreover for any eigenvalue $\mu(\lambda)$ associated to $(u_\lambda,v_\lambda)\in Y_{pq}(\Omega)\setminus\{(0,0)\}$, $J_\lambda(u_\lambda,v_\lambda)=\mu(\lambda)I(u_\lambda,v_\lambda)$ with $I(u_\lambda,v_\lambda)>0$. Consequently

$$\mu_1(\lambda)\leq J_\lambda\left(\frac{u_\lambda}{I(u_\lambda,v_\lambda)^{\frac{1}{p}}},\frac{v_\lambda}{I(u_\lambda,v_\lambda)^{\frac{1}{q}}}\right)=\frac{J_\lambda(u_\lambda,v_\lambda)}{I(u_\lambda,v_\lambda)}=\mu(\lambda).$$

We conclude that $\mu_1(\lambda)$ is the smallest eigenvalue of (Q_λ). □

For $m=m_1=m_2\equiv 0$, we denote by

$$\mu_0=\mu_1(\lambda)=\inf\left\{\frac{\alpha+1}{p}\|\Delta u\|_p^p+\frac{\beta+1}{q}\|\Delta v\|_q^q:(u,v)\in\mathcal{M}\right\}.$$

Since the space $W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega)$ with $r\in\{p,q\}$ does not contain any constant non trivial function, one has $\mu_0>0$.

Proposition 2.2. *The following results hold*

1. *μ_1 is concave and differentiable with $\mu_1'(\lambda)=-M(u_0,v_0)$ where (u_0,v_0) is some eigenfunction of (Q_λ) associated to $\mu_1(\lambda)$ for all $\lambda\in\mathbb{R}$.*
2. $\lim_{\lambda\to+\infty}\mu_1(\lambda)=-\infty$.
3. *μ_1 is strictly decreasing.*

Proof. We provide the proof in the following steps.

1. The concavity of μ_1 follows from the concavity of the mapping $\lambda\mapsto J_\lambda(u,v)$, for a fixed $(u,v)\in Y_{pq}(\Omega)$. In particular μ_1 is continuous. Now let $\lambda_n\longrightarrow\lambda$ and (u_n,v_n), (u_λ,v_λ) be the I-normalized eigenfunctions related to $\mu_1(\lambda_n)$, $\mu_1(\lambda)$ respectively. We apply Lemma 2.1 with $\omega_1=\omega_2=1$ to get

$$\begin{aligned}\|\Delta u_n\|_p^p+\|\Delta v_n\|_q^q&\leq c_1J_\lambda(u_n,v_n)+c_2\int_\Omega|u_n|^p\mathrm{d}x+c_3\int_\Omega|v_n|^q\mathrm{d}x,\\&\leq c_1J_\lambda(u_n,v_n)+\max\left\{\frac{pc_2}{\alpha+1},\frac{qc_3}{\beta+1}\right\}\end{aligned}$$

$$=c_1\mu_1(\lambda_n)+\max\left\{\frac{pc_2}{\alpha+1},\frac{qc_3}{\beta+1}\right\}.$$

Moreover

$$\lim_{n\to\infty} c_1\mu_1(\lambda_n)+\max\left\{\frac{pc_2}{\alpha+1},\frac{qc_3}{\beta+1}\right\}=c_1\mu_1(\lambda)+\max\left\{\frac{pc_2}{\alpha+1},\frac{qc_3}{\beta+1}\right\}.$$

So we conclude that $(u_n,v_n)_n$ is a bounded sequence in $Y_{pq}(\Omega)$. Hence there exists (u_0,v_0) such that, up to a subsequence, $(u_n,v_n)\rightharpoonup(u_0,v_0)$ in $Y_{pq}(\Omega)$, strongly in $L^p(\Omega)\times L^q(\Omega)$. Then $(u_0,v_0)\in\mathcal{M}$ and from

$$J_\lambda(u_0,v_0)\le\lim_{n\longrightarrow\infty} J_\lambda(u_n,v_n)=\mu_1(\lambda)$$

we infer that $\mu_1(\lambda)=J_\lambda(u_0,v_0)=J_\lambda(u_\lambda,v_\lambda)$ and (u_0,v_0) is an eigenfunction of (Q_λ) associated to $\mu_1(\lambda)$. Furthermore

$$\begin{cases}\mu_1(\lambda_n)-\mu_1(\lambda)\ge-(\lambda_n-\lambda)M(u_n,v_n),\\ \mu_1(\lambda_n)-\mu_1(\lambda)\le-(\lambda_n-\lambda)M(u_0,v_0).\end{cases}$$

Hence

$$\begin{cases}-M(u_n,v_n)\le\dfrac{\mu_1(\lambda_n)-\mu_1(\lambda)}{\lambda_n-\lambda}\le-M(u_0,v_0), & \text{if } \lambda_n>\lambda,\\ -M(u_0,v_0)\le\dfrac{\mu_1(\lambda_n)-\mu_1(\lambda)}{\lambda_n-\lambda}\le-M(u_n,v_n), & \text{if } \lambda_n<\lambda.\end{cases}$$

Passing to the limit we get $\mu_1'(\lambda)=-M(u_0,v_0)$.

2. We know that m_1 is nonnegative, then there exists a function $u\in X_p$ such that $M_1(u)>0$ and $I(u,0)=1$. Then, for all $\lambda\in\mathbb{R}_+^*$, $\mu_1(\lambda)\le J_\lambda(u,0)$. We deduce that

$$\lim_{\lambda\longrightarrow+\infty} J_\lambda(u,0)=\lim_{\lambda\longrightarrow+\infty} E_m(u,0)-\lambda M(u,0)=-\infty,$$

where

$$E_m(u,v)=\frac{\alpha+1}{p}\|\Delta u\|_p^p+\frac{\beta+1}{q}\|\Delta v\|_q^q-\int_\Omega m|u|^{\alpha+1}|v|^{\beta+1}\mathrm{d}x.$$

Thus $\lim_{\lambda\longrightarrow+\infty}\mu_1(\lambda)=-\infty$.

3. The result is clear from the fact that $M(u_\lambda,v_\lambda)>0$ for any $\lambda\in\mathbb{R}$. Indeed, if $\lambda_1<\lambda_2$ then

$$\mu_1(\lambda_1)=E_m(u_{\lambda_1},v_{\lambda_1})-\lambda_1M(u_{\lambda_1},v_{\lambda_1})\ge E_m(u_{\lambda_1},v_{\lambda_1})-\lambda_2M(u_{\lambda_1},v_{\lambda_1})\ge\mu_1(\lambda_2).$$

This completes the proof of the proposition. □

3 Strictly or semitrivial principal eigenvalues

Note that, if $\mu_1(\lambda)=0$ then λ is an eigenvalue of problem (Q). Our purpose is to find a reasonable assumption on m so that there exists at least one $\lambda\in(0,\infty)$ such that $\mu_1(\lambda)=0$.

Lemma 3.1. *If $\|m\|_\infty<\mu_0$ then, $\mu_1(0)>0$ and $\mu_1(\lambda)=0$ has a unique positive solution λ (eigenvalue of (Q)).*

Proof. Assume that $\|m\|_\infty<\mu_0$. By (2.7), we have $V(u,v)\le\|m\|_\infty I(u,v)$, $\forall(u,v)\in Y_{pq}(\Omega)$. Then, one has

$$\frac{\alpha+1}{p}\|\Delta u\|_p^p+\frac{\beta+1}{q}\|\Delta v\|_q^q-\|m\|_\infty I(u,v)\le E_m(u,v),\quad \forall(u,v)\in Y_{pq}(\Omega).$$

We deduce that:

$$\begin{aligned}&\mu_0\le E_m(u,v)+\|m\|_\infty,\quad \forall(u,v)\in\mathcal{M},\\ &\mu_0-\|m\|_\infty\le\inf\{E_m(u,v),(u,v)\in\mathcal{M}\}\le\mu_1(0).\end{aligned}$$

Consequently, $\mu_1(0)>0$. Moreover, from Propoaition 2.1, μ_1 is strictly decreasing. We deduce that, $\mu_1(\lambda)=0$ has a unique positive solution λ and λ is an eigenvalue of (Q). □

We will denote by

$$L(\Omega):=([L^p(\Omega)\times L^q(\Omega)]\setminus\{(0,0)\})\times\mathbb{R},\tag{3.1}$$
$$L_0(\Omega):=([L^p(\Omega)\times L^q(\Omega)]\setminus\{(0,0)\})\times\{0\}.\tag{3.2}$$

We apply some results proved by Drábek and Ôtani [2] and some ideas used by Talbi and Tsouli [1].

Remark 3.1.

1. $\forall u\in X_r$, $\forall v\in L^r(\Omega)$ with $r\in(1,\infty)$: $v=-\Delta u\Longleftrightarrow u=\Lambda v$.
2. Let N_r be the Nemytskii operator with $r\in(1,\infty)$, defined by

$$N_r(u)(x)=\begin{cases}|u(x)|^{r-2}u(x) & \text{if } u(x)\neq0,\\ 0 & \text{if } u(x)=0.\end{cases}$$

   We have
$$\forall v\in L^r(\Omega),\quad \forall w\in L^{r'}(\Omega):\quad N_r(v)=w\Longleftrightarrow v=N_{r'}(w)\tag{3.3}$$
   with $r'=\frac{r}{r-1}$.

3. If (u,v) is an eigenfunction of (Q_λ) associated with μ then $\varphi=-\Delta u$, $w=-\Delta v$ satisfy:

$$\begin{cases} N_p(\varphi)=\Lambda([\mu(\lambda)+\lambda m_1]N_p(\Lambda\varphi)+m|\Lambda w|^{\beta+1}|\Lambda\varphi|^{\alpha-1}\Lambda\varphi), \\ N_q(w)=\Lambda([\mu(\lambda)+\lambda m_2]N_q(\Lambda w)+m|\Lambda\varphi|^{\alpha+1}|\Lambda w|^{\beta-1}\Lambda w). \end{cases}$$

Hence:

(a) $[(u_0,v_0);\mu(\lambda)]$ is a solution of (Q_λ) if and only if $[(\varphi_0,w_0);\mu(\lambda)]$ is a solution of problem

$$(Q_\lambda^{'}):\begin{cases} \text{Find } [(\varphi,w);\mu(\lambda)]\in L(\Omega) \text{ such that} \\ N_p(\varphi)=\Lambda([\mu(\lambda)+\lambda m_1]N_p(\Lambda\varphi)+m|\Lambda w|^{\beta+1}|\Lambda\varphi|^{\alpha-1}\Lambda\varphi) \\ N_q(w)=\Lambda([\mu(\lambda)+\lambda m_2]N_q(\Lambda w)+m|\Lambda\varphi|^{\alpha+1}|\Lambda w|^{\beta-1}\Lambda w) \end{cases}$$

with $\varphi_0=-\Delta u_0$ and $w_0=-\Delta v_0$.

(b) $[(\varphi_0,w_0);\mu(\lambda)]\in L_0(\Omega)$ is a solution of $(Q_\lambda^{'})$ if and only if $[(\varphi_0,w_0);\lambda]\in L(\Omega)$ is a solution of problem

$$(Q^{'}):\begin{cases} \text{Find } [(\varphi,w);\lambda]\in L(\Omega) \text{ such that} \\ N_p(\varphi)=\Lambda(\lambda m_1 N_p(\Lambda\varphi)+m|\Lambda w|^{\beta+1}|\Lambda\varphi|^{\alpha-1}\Lambda\varphi) \\ N_q(w)=\Lambda(\lambda m_2 N_q(\Lambda w)+m|\Lambda\varphi|^{\alpha+1}|\Lambda w|^{\beta-1}\Lambda w) \end{cases}$$

with $\varphi_0=-\Delta u_0$ and $w_0=-\Delta v_0$.

(c)

$$\mu_1(\lambda):=\inf\{F_\lambda(\varphi,w):(\varphi,w)\in L^p(\Omega)\times L^q(\Omega), R(\varphi,w)=1\} \tag{3.4}$$

where

$$\begin{aligned} F_\lambda(\varphi,w)=&\frac{\alpha+1}{p}\left[\int_\Omega|\varphi|^p\mathrm{d}x-\lambda\int_\Omega m_1|\Lambda\varphi|^p\mathrm{d}x\right] \\ &+\frac{\beta+1}{q}\left[\int_\Omega|w|^q\mathrm{d}x-\lambda\int_\Omega m_2|\Lambda w|^q\mathrm{d}x\right]-\int_\Omega m|\Lambda\varphi|^{\alpha+1}|\Lambda w|^{\beta+1}\mathrm{d}x, \\ R(\varphi,w)=&\frac{\alpha+1}{p}\|\Lambda\varphi\|_p^p+\frac{\beta+1}{q}\|\Lambda w\|_q^q. \end{aligned}$$

We may now assume the following condition:

$$(H_m):\|m\|_\infty<\mu_0. \tag{3.5}$$

Lemma 3.2. *If $[(u,v);\mu(\lambda)]$ is a solution of (Q_λ) then $-\Delta u$, $-\Delta v\in C(\overline{\Omega})$ and u, $v\in C^{1,\nu}(\overline{\Omega})$, for all $\nu\in(0,1)$.*

Proof. Without loss of generality, one can assume that $p \leq q$. Let $p_0 \in [p,\infty)$, $q_0 \in [q,\infty)$ such that $p_0 = q_0$ if p=q. Suppose that $\varphi = N_{p'}(\Lambda\theta_1) \in L^{p_0}(\Omega)$, $w = N_{q'}(\Lambda\theta_2) \in L^{q_0}(\Omega)$ with

$$\begin{cases} \theta_1 = \omega_1 N_p(\Lambda\varphi) + m|\Lambda w|^{\beta+1}|\Lambda\varphi|^{\alpha-1}\Lambda\varphi, \\ \theta_2 = \omega_2 N_q(\Lambda w) + m|\Lambda\varphi|^{\alpha+1}|\Lambda w|^{\beta-1}\Lambda w, \end{cases}$$

where $\omega_1 \in L^\infty(\Omega)$, $\omega_2 \in L^\infty(\Omega)$. It is easy to see that:

1. if $p = q$, then
   (a) φ, $w \in L^{p_1}(\Omega)$, with $\frac{1}{p_1} = \frac{1}{p_0} - \frac{2p'}{N}$, if $p_0 < \frac{N}{2p'}$.
   (b) φ, $w \in L^{\frac{k}{p'-1}}(\Omega)$, $\forall k \in (1,+\infty)$, if $p_0 = \frac{N}{2p'}$.
   (c) φ, $w \in C(\overline{\Omega})$, if $p_0 > \frac{N}{2p'}$. Indeed, one have
      i. φ, $w \in C(\overline{\Omega})$, if $\frac{N}{2} < p_0$.
      ii. if $\frac{N}{2} = p_0$, then θ_1, $\theta_2 \in L^{\frac{k}{p-1}}(\Omega)$, for all $k \in (1,+\infty)$. We can take k such that $\frac{k}{p-1} > \frac{N}{2}$. Thus φ, $w \in C(\overline{\Omega})$.
      iii. if $\frac{N}{2p'} < p_0 < \frac{N}{2}$, then: θ_1, $\theta_2 \in L^{\frac{r_0}{p-1}}(\Omega)$ with $r_0 = \frac{Np_0}{N-2p_0}$ and $\frac{r_0}{p-1} > \frac{N}{2}$. Then φ, $w \in C(\overline{\Omega})$.
2. if $p < q$, then :
   (a) if $p_0 < \frac{N}{2p'}$, then
      i. $\theta_1 \in L^{\frac{r_0}{p-1}}(\Omega)$ and $\varphi \in L^{p_1}(\Omega)$ with $r_0 = \frac{Np_0}{N-2p_0}$, $p_1 = \frac{Np_0(p-1)}{N(p-1)-2pp_0}$, if $q_0 \geq \frac{N}{2}$.
      ii. if $q_0 < \frac{N}{2}$, then:
         A. $\theta_1 \in L^{\frac{r_0}{p-1}}(\Omega)$ and $\varphi \in L^{p_1}(\Omega)$ with $s_0 = \frac{Nq_0}{N-2q_0}$, if $ps_0 > qr_0$.
         B. $\theta_2 \in L^{\frac{s_0}{q-1}}(\Omega)$, if $ps_0 < qr_0$.
         C. $\theta_1 \in L^{\frac{r_0}{p-1}}(\Omega)$, $\theta_2 \in L^{\frac{s_0}{q-1}}(\Omega)$ and $\varphi \in L^{p_1}(\Omega)$, if $ps_0 = qr_0$.
   (b) if $p_0 = \frac{N}{2p'}$, then
      i. $\varphi \in L^{\frac{k}{p'-1}}(\Omega)$, $\forall k \in (1,+\infty)$, if $q_0 \geq \frac{N}{2}$.
      ii. if $q_0 < \frac{N}{2}$, then:
         A. $\varphi \in L^{\frac{k}{p'-1}}(\Omega)$, $\forall k \in (1,+\infty)$, if $ps_0 > qr_0$.
         B. $\theta_2 \in L^{\frac{s_0}{q-1}}(\Omega)$, if $ps_0 < qr_0$.
         C. $\theta_2 \in L^{\frac{s_0}{q-1}}(\Omega)$ and $\varphi \in L^{\frac{k}{p'-1}}(\Omega)$, $\forall k \in (1,+\infty)$, if $ps_0 = qr_0$.

(c) if $p_0 > \frac{N}{2p'}$, then

i. $\varphi, w \in C(\overline{\Omega})$, if $\frac{N}{2} \le p_0 < q_0$.

ii. if $\frac{N}{2p'} \le p_0 < \frac{N}{2}$, then:

A. $\varphi \in C(\overline{\Omega})$, if $\frac{N}{2} \le q_0$.

B. $\theta_1 \in L^{\frac{r_0}{p-1}}(\Omega)$ or $\theta_2 \in L^{\frac{s_0}{q-1}}(\Omega)$ if $\frac{N}{2} > q_0$.

Let $[(u,v);\mu(\lambda)] \in Y_{pq}(\Omega) \times \mathbb{R}$ be a solution of (Q_λ), then $[(\varphi,w);\mu(\lambda)]$ is a solution of $(Q_\lambda^{'})$ with $\varphi = -\Delta u = N_{p'}(\Lambda\theta_1) \in L^p(\Omega)$, $w = -\Delta v = N_{q'}(\Lambda\theta_2) \in L^q(\Omega)$ with

$$\begin{cases} \theta_1 = \omega_1 N_p(\Lambda\varphi) + m|\Lambda w|^{\beta+1}|\Lambda\varphi|^{\alpha-1}\Lambda\varphi, \\ \theta_2 = \omega_2 N_q(\Lambda w) + m|\Lambda\varphi|^{\alpha+1}|\Lambda w|^{\beta-1}\Lambda w, \end{cases}$$

where $\omega_1 = \mu(\lambda) + \lambda m_1 \in L^\infty(\Omega)$, $\omega_2 = \mu(\lambda) + \lambda m_2 \in L^\infty(\Omega)$.

Case (1): $p = q$

We easily see that $\varphi, w \in C(\overline{\Omega})$ from assertion 1c, if $p > \frac{N}{2p'}$.

Now take suitable (p_n), $p = p_0$ and $k \in \mathbb{N}$ such that $p_{k-1} < \frac{N}{2p'} < p_k$ with $\frac{1}{p_k} = \frac{1}{p_0} - \frac{2kp^{'}}{N}$. Then applying assertion 1a with $p_0 = p_0$, $p_1,\dots, p_{k-1}$, we deduce $\varphi, w \in L^{p_k}(\Omega)$. Hence from assertion 1c, $\varphi, w \in C(\overline{\Omega})$ follows.

Case (2): $p < q$ and $\frac{N}{2p'} \le p$.

1. We deduce $\varphi, w \in C(\overline{\Omega})$ from assertion 2b and 2c, if $\frac{N}{2} \le q$.

2. If $\frac{N}{2} > q$, take suitable $s_n = \frac{Nq_n}{N-2q_n}$ with $\frac{1}{q_n} = \frac{1}{q_0} - \frac{2nq^{'}}{N}$, $q_0 = q$ and $k \in \mathbb{N}$ such that $\frac{s_k}{q-1} > \frac{N}{2}$. Then applying assertion 2b and 2c with $q_0 = q_0$, $q_1,\dots, q_k$, $p_0 = p$, we deduce $\theta_2 \in L^{\frac{s_k}{q-1}}(\Omega)$. Hence $\Lambda\theta_2 \in C(\overline{\Omega})$ and $\varphi, w \in C(\overline{\Omega})$ follows.

Case (3): $p < q$ and $\frac{N}{2p'} > p$.

1. If $\frac{N}{2q'} \le q$, take suitable (p_n), $p = p_0$ and $k \in \mathbb{N}$ such that $p_{k-1} < \frac{N}{2p'} < p_k$ with $\frac{1}{p_k} = \frac{1}{p_0} - \frac{2kp^{'}}{N}$. Then applying assertion 2a with $p_0 = p_0$, $p_1,\dots, p_{k-1}$, $q_0 = q$, we deduce $\varphi \in L^{p_k}(\Omega)$ and $\varphi, w \in C(\overline{\Omega})$ follows.

2. If $\frac{N}{2q'} > q$, take suitables (p_n), (q_n) and $k, j \in \mathbb{N}$ such that $p = p_0$, $q = q_0$, $p_{k-1} < \frac{N}{2p'} < p_k$, $q_{j-1} < \frac{N}{2q'} < q_j$ with $\frac{1}{p_k} = \frac{1}{p_0} - \frac{2kp^{'}}{N}$ and $\frac{1}{q_j} = \frac{1}{q_0} - \frac{2jq^{'}}{N}$. Then applying assertion 2a with $p_0 = p_0$, $p_1,\dots, p_{k-1}$, and $q_0 = q_0$, $q_1,\dots, q_{j-1}$, we deduce $\varphi \in L^{p_k}(\Omega)$, $w \in L^{q_j}(\Omega)$ and φ, $w \in C(\overline{\Omega})$ follows.

Hence we deduce that φ, $w\in L^\infty(\Omega)$ and from the assertion in Lemma 1.1 that $u=\Lambda\varphi$, $v=\Lambda w\in C^{1,\nu}(\overline{\Omega})$ for all $\nu\in(0,1)$. □

Lemma 3.3. *$[(\varphi_1,w_1);\mu_1(\lambda)]\in L(\Omega)$ is a solution of problem $(Q_\lambda^{'})$, if and only if*

$$G_\lambda(\varphi_1,w_1)=0=\min_{(\varphi,w)\in L^*(\Omega)} G_\lambda(\varphi,w) \tag{3.6}$$

where

$$G_\lambda(\varphi,w)=F_\lambda(\varphi,w)-\mu_1(\lambda)R(\varphi,w), \qquad L^*(\Omega)=[L^p(\Omega)\times L^q(\Omega)]\setminus\{(0,0)\}.$$

Proof. Assume that $[(\varphi_1,w_1);\mu_1(\lambda)]\in L(\Omega)$ is a solution of problem $(Q_\lambda^{'})$. Then $F_\lambda(\varphi_1,w_1)=\mu_1(\lambda)R(\varphi_1,w_1)$. Hence $G_\lambda(\varphi_1,w_1)=F_\lambda(\varphi_1,w_1)-\mu_1(\lambda)R(\varphi_1,w_1)=0$. Put

$$\overline{\varphi}=\frac{\varphi}{[R(\varphi,w)]^{\frac{1}{p}}},\ \overline{w}=\frac{w}{[R(\varphi,w)]^{\frac{1}{q}}} \ \text{ for every } (\varphi,w)\in L^*(\Omega).$$

Then $R(\overline{\varphi},\overline{w})=1$. We deduce that

$$\mu_1(\lambda)\le F_\lambda(\overline{\varphi},\overline{w})=\frac{F_\lambda(\varphi,w)}{R(\varphi,w)}, \tag{3.7}$$

$$G_\lambda(\varphi,w)=F_\lambda(\varphi,w)-\mu_1(\lambda)R(\varphi,w)\ge 0 \tag{3.8}$$

for all $(\varphi,w)\in L^*(\Omega)$. We claim that (3.6) holds.

Now suppose that (3.6) holds. We deduce that $\nabla G_\lambda(\varphi_1,w_1)=(0,0)$. Then

$$\langle\frac{\partial G_\lambda}{\partial\varphi}(\varphi_1,w_1),\Psi\rangle=\langle\frac{\partial G_\lambda}{\partial w}(\varphi_1,w_1),\theta\rangle=0, \tag{3.9}$$

for all $(\Psi,\theta)\in[L^p(\Omega)\times L^q(\Omega)]$. Hence, $[(\varphi_1,w_1);\mu_1(\lambda)]\in L(\Omega)$ is a solution of $(Q_\lambda^{'})$. □

Lemma 3.4. *If (H_m) holds and $[(\varphi_1,w_1);\mu_1(\lambda)]\in L_0(\Omega)$ is a solution of problem $(Q_\lambda^{'})$ then $[(|\varphi_1|,|w_1|);\mu_1(\lambda)]\in L_0(\Omega)$ is a solution of problem $(Q_\lambda^{'})$.*

Proof. Assume that (H_m) holds and $[(\varphi_1,w_1);\mu_1(\lambda)]\in L_0(\Omega)$ is a solution of problem $(Q_\lambda^{'})$. Then $G_\lambda(\varphi_1,w_1)=0$, $\mu_1(\lambda)=0$, $\lambda>0$ and $(|\varphi_1|,|w_1|)\in[L^p(\Omega)\times L^q(\Omega)]\setminus\{(0,0)\}$. Hence $G_\lambda(|\varphi_1|,|w_1|)\ge 0$.

Additionally, one has $|\Lambda(|\varphi_1|)|^r\ge|\Lambda\varphi_1|^r$ and $|\Lambda(|w_1|)|^r\ge|\Lambda w_1|^r$, for all $r\in(1;\infty)$. We deduce that:

$$\begin{aligned}
&-\lambda\int_\Omega m_1|\Lambda(|\varphi_1|)|^p\mathrm{d}x\le-\lambda\int_\Omega m_1|\Lambda\varphi_1|^p\mathrm{d}x,\\
&-\lambda\int_\Omega m_2|\Lambda(|w_1|)|^q\mathrm{d}x\le-\lambda\int_\Omega m_2|\Lambda w_1|^q\mathrm{d}x,\\
&-\int_\Omega m|\Lambda(|\varphi_1|)|^{\alpha+1}|\Lambda(|w_1|)|^{\beta+1}\mathrm{d}x\le-\int_\Omega m|\Lambda\varphi_1|^{\alpha+1}|\Lambda w_1|^{\beta+1}\mathrm{d}x.
\end{aligned}$$

Consequently, $F_\lambda(|\varphi_1|,|w_1|)\le F_\lambda(\varphi_1,w_1)$ and $G_\lambda(|\varphi_1|,|w_1|)\le G_\lambda(\varphi_1,w_1)=0$. Thus $G_\lambda(|\varphi_1|,|w_1|)=0$ and $[(|\varphi_1|,|w_1|);\mu_1(\lambda)]$ is solution of $(Q_\lambda^{'})$. □

Proposition 3.1. *Assume that (H_m) holds and $\mu_1(\lambda)=0$. Then λ is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem (Q).*

Proof. Assume that (H_m) holds and $\mu_1(\lambda)=0$. Then λ is an eigenvalue of problem (Q) associated with $(u,v)\in Y_{pq}(\Omega)\setminus\{(0,0)\}$.

If $u\not\equiv 0$ and $v\not\equiv 0$, then $[(\varphi,w);\mu_1(\lambda)]$, $[(|\varphi|,|w|);\mu_1(\lambda)]\in L_0(\Omega)$ are solutions of problem $(Q_\lambda^{'})$ with $\varphi=-\Delta u\not\equiv 0$ and $w=-\Delta v\not\equiv 0$. Since $|\varphi|\geq 0$ and $|w|\geq 0$, then $\Lambda(|\varphi|)>0$, $\Lambda(|w|)>0$. Therefore

$$N_p(\Lambda|\varphi|)>0,\ N_q(\Lambda|w|)>0,\ |\Lambda(|w|)|^{\beta+1}|\Lambda(|\varphi|)|^{\alpha}>0,\ |\Lambda(|\varphi|)|^{\alpha+1}|\Lambda(|w|)|^{\beta}>0$$

and

$$\begin{cases} |\varphi|=N_{p'}\left(\Lambda\left[\lambda m_1 N_p(\Lambda|\varphi|)+m|\Lambda(|w|)|^{\beta+1}|\Lambda(|\varphi|)|^{\alpha-1}\Lambda(|\varphi|)\right]\right)>0,\\ |w|=N_{q'}\left(\Lambda\left[\lambda m_2 N_q(\Lambda|w|)+m|\Lambda(|\varphi|)|^{\alpha+1}|\Lambda(|w|)|^{\beta-1}\Lambda(|w|)\right]\right)>0. \end{cases}$$

We then conclude that $[(\varphi,w);\mu_1(\lambda)]$ is solution of problem $(Q_\lambda^{'})$ with φ positive in Ω or negative in Ω and w is positive in Ω or negative in Ω.

Since by Lemma 3.2, φ, $w\in C(\overline{\Omega})$, we deduce that $u=\Lambda\varphi$ positive in Ω or negative in Ω and $v=\Lambda w$ positive in Ω or negative in Ω, from the Lemma 1.1. Then λ is strictly principal eigenvalue of (Q).

If $[u\equiv 0$ and $v\not\equiv 0]$ or $[u\not\equiv 0$ and $v\equiv 0]$, then we also prove that $[u\equiv 0$ and $v>0$ in Ω or $v<0$ in $\Omega]$ or $[u>0$ in Ω or $u<0$ in Ω and $v\equiv 0]$. Then λ is a semitrivial principal eigenvalue of (Q). □

Lemma 3.5. *Let A, B, C and r be real numbers satisfying $A\geq 0$, $B\geq 0$, $C\geq\max\{B-A,0\}$ and $r\in[1,+\infty)$. Then*

$$|A+C|^r+|B-C|^r\geq A^r+B^r.$$

Proof. See the proof of [2, Lemme 2.5] if $r\in(1,+\infty)$. Assume that $r=1$, then

$$\begin{cases} |A+C|+|B-C|=A+C+B-C=A+B, & \text{if}\quad B-C\geq 0\\ |A+C|+|B-C|=A-B+2C>A+B, & \text{if}\quad B-C<0. \end{cases}$$

Thus $|A+C|+|B-C|\geq A+B$. □

Lemma 3.6. *Suppose that (H_m) holds. If (φ_1,w_1) and (φ_2,w_2) are positive eigenfunctions of problem $(Q_\lambda^{'})$ associated with $\mu_1(\lambda)=0$, then (φ_{12},w_{12}), (φ_{12},w_{21}), (φ_{21},w_{12}) and (φ_{21},w_{21}) with*

$$\begin{cases} \varphi_{12}(x):=\max\{\varphi_1(x),\varphi_2(x)\}=\varphi_1(x)+(\varphi_2-\varphi_1)^+(x)\\ w_{12}(x):=\max\{w_1(x),w_2(x)\}=w_1(x)+(w_2-w_1)^+(x)\\ \varphi_{21}(x):=\min\{\varphi_1(x),\varphi_2(x)\}=\varphi_2(x)-(\varphi_2-\varphi_1)^+(x)\\ w_{21}(x):=\min\{w_1(x),w_2(x)\}=w_2(x)-(w_2-w_1)^+(x) \end{cases},$$

for all $x\in\Omega$, are eigenfunctions of $(Q_\lambda^{'})$ associated with $\mu_1(\lambda)=0$.

Proof. Assume that (H_m) holds and (φ_1,w_1), (φ_2,w_2) are positive eigenfunctions of problem $(Q_\lambda^{'})$ associated with $\mu_1(\lambda)=0$. By Lemma 3.5 we get

$$\begin{cases} |\Lambda\varphi_{12}|^p+|\Lambda\varphi_{21}|^p\geq|\Lambda\varphi_1|^p+|\Lambda\varphi_2|^p \\ |\Lambda w_{12}|^q+|\Lambda w_{21}|^q\geq|\Lambda w_1|^q+|\Lambda w_2|^q \\ |\Lambda\varphi_{12}|^{\alpha+1}+|\Lambda\varphi_{21}|^{\alpha+1}\geq|\Lambda\varphi_1|^{\alpha+1}+|\Lambda\varphi_2|^{\alpha+1} \\ |\Lambda w_{12}|^{\beta+1}+|\Lambda w_{21}|^{\beta+1}\geq|\Lambda w_1|^{\beta+1}+|\Lambda w_2|^{\beta+1}. \end{cases}$$

Then, one has:

$$-\lambda\int_\Omega m_1|\Lambda\varphi_{12}|^p\mathrm{d}x-\lambda\int_\Omega m_1|\Lambda\varphi_{21}|^p\mathrm{d}x\leq-\lambda\int_\Omega m_1|\Lambda\varphi_1|^p\mathrm{d}x-\lambda\int_\Omega m_1|\Lambda\varphi_2|^p\mathrm{d}x, \quad (3.10)$$

$$-\lambda\int_\Omega m_2|\Lambda w_{12}|^q\mathrm{d}x-\lambda\int_\Omega m_2|\Lambda w_{21}|^q\mathrm{d}x\leq-\lambda\int_\Omega m_2|\Lambda w_1|^q\mathrm{d}x-\lambda\int_\Omega m_2|\Lambda w_2|^q\mathrm{d}x. \quad (3.11)$$

Likewise, we have

$$Z_1(\varphi,w)\leq Z_2(\varphi,w)\leq-\int_\Omega m|\Lambda\varphi_1|^{\alpha+1}|\Lambda w_1|^{\beta+1}\mathrm{d}x-\int_\Omega m|\Lambda\varphi_2|^{\alpha+1}|\Lambda w_2|^{\beta+1}\mathrm{d}x, \quad (3.12)$$

with

$$\begin{aligned} Z_1(\varphi,w)=&-\int_\Omega m|\Lambda\varphi_{12}|^{\alpha+1}|\Lambda w_{12}|^{\beta+1}\mathrm{d}x-\int_\Omega m|\Lambda\varphi_{12}|^{\alpha+1}|\Lambda w_{21}|^{\beta+1}\mathrm{d}x \\ &-\int_\Omega m|\Lambda\varphi_{21}|^{\alpha+1}|\Lambda w_{12}|^{\beta+1}\mathrm{d}x-\int_\Omega m|\Lambda\varphi_{21}|^{\alpha+1}|\Lambda w_{21}|^{\beta+1}\mathrm{d}x, \\ Z_2(\varphi,w)=&-\int_\Omega m|\Lambda\varphi_1|^{\alpha+1}|\Lambda w_1|^{\beta+1}\mathrm{d}x-\int_\Omega m|\Lambda\varphi_1|^{\alpha+1}|\Lambda w_2|^{\beta+1}\mathrm{d}x \\ &-\int_\Omega m|\Lambda\varphi_2|^{\alpha+1}|\Lambda w_1|^{\beta+1}\mathrm{d}x-\int_\Omega m|\Lambda\varphi_2|^{\alpha+1}|\Lambda w_2|^{\beta+1}\mathrm{d}x. \end{aligned}$$

Additionally, we have:

$$\int_\Omega|\varphi_{12}|^p\mathrm{d}x+\int_\Omega|\varphi_{21}|^p\mathrm{d}x=\int_\Omega|\varphi_1|^p\mathrm{d}x+\int_\Omega|\varphi_2|^p\mathrm{d}x, \quad (3.13)$$

$$\int_\Omega|w_{12}|^q\mathrm{d}x+\int_\Omega|w_{21}|^q\mathrm{d}x=\int_\Omega|w_1|^q\mathrm{d}x+\int_\Omega|w_2|^q\mathrm{d}x. \quad (3.14)$$

By (3.10)–(3.14) we deduce that:

$$F_\lambda(\varphi_{12},w_{12})+F_\lambda(\varphi_{12},w_{21})+F_\lambda(\varphi_{21},w_{12})+F_\lambda(\varphi_{21},w_{21})\leq F_\lambda(\varphi_1,w_1)+F_\lambda(\varphi_2,w_2),$$
$$G_\lambda(\varphi_{12},w_{12})+G_\lambda(\varphi_{12},w_{21})+G_\lambda(\varphi_{21},w_{12})+G_\lambda(\varphi_{21},w_{21})\leq G_\lambda(\varphi_1,w_1)+G_\lambda(\varphi_2,w_2)=0.$$

It follows that

$$G_\lambda(\varphi_{12},w_{12})=G_\lambda(\varphi_{12},w_{21})=G_\lambda(\varphi_{21},w_{12})=G_\lambda(\varphi_{21},w_{21})=0.$$

Hence (φ_{12},w_{12}), (φ_{12},w_{21}), (φ_{21},w_{12}) and (φ_{21},w_{21}) are eigenfunctions of $(Q_\lambda^{'})$ associated with $\mu_1(\lambda)=0$. □

Proposition 3.2. *Assume that* (H_m) *holds and* $\mu_1(\lambda)=0$. *Then* λ *is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem* (Q) *and simple.*

Proof. Assume that (H_m) holds and $\mu_1(\lambda)=0$. Then λ is a semitrivial principal eigenvalue or strictly principal eigenvalue of problem (Q) from Proposition 3.1.

Case 1: Take λ as a strictly principal eigenvalue of (Q).

Let (u_{11},u_{12}) and (u_{21},u_{22}) be two positive eigenfunctions of (Q) associated with λ. Then, $[(v,w);0]$, $[(\varphi,\psi);0]$, $[(|v|,|w|);0]$, $[(|\varphi|,|\psi|);0]\in L_0(\Omega)$, are solutions of $(Q^{'}_{\lambda})$ with $v=-\Delta u_{11}>0$, $w=-\Delta u_{12}>0$, $\varphi=-\Delta u_{21}>0$ and $\psi=-\Delta u_{22}>0$.

For $x_0\in\Omega$, we set

$$k=\frac{\varphi(x_0)}{v(x_0)},\quad \omega_1(x)=\max\{\varphi(x),kv(x)\} \text{ and } \omega_2(x)=\max\left\{\psi(x),k^{\frac{p}{q}}w(x)\right\},$$

for all $x\in\Omega$.

From Lemma 3.6, $[(\omega_1,\omega_2);0]$ is a solution of problem $(Q^{'}_{\lambda})$ because $[(kv,k^{\frac{p}{q}}w);0]$ and $[(\varphi,\psi);0]$ are solutions of $(Q^{'}_{\lambda})$. We deduce that $N_p(v)$, $N_q(w)$, $N_p(\varphi)$, $N_q(\psi)$, $N_p(\omega_1)$, $N_q(\omega_2)\in C^{1,\nu}(\bar{\Omega})$ and $\dfrac{N_p(\varphi)}{N_p(v)}$, $\dfrac{N_q(\psi)}{N_q(w)}\in C^1(\Omega)$.

For any unit vector $e=(0,...,e_i,...,0)$ with $i\in\{1,...,N\}$ and $t\in\mathbb{R}$, we have

$$\begin{cases} N_p(\varphi)(x_0+te)-N_p(\varphi)(x_0)\le N_p(\omega_1)(x_0+te)-N_p(\omega_1)(x_0),\\ N_p(kv)(x_0+te)-N_p(kv)(x_0)\le N_p(\omega_1)(x_0+te)-N_p(\omega_1)(x_0).\end{cases}$$

Dividing these inequalities by $t>0$ and $t<0$ and letting t tend to $0^{\pm}$, we get

$$\begin{cases} \dfrac{\partial}{\partial x_i}[N_p(\varphi)](x_0)\le\dfrac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),\\ \dfrac{\partial}{\partial x_i}[N_p(kv)](x_0)\le\dfrac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),\end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial x_i}[N_p(\varphi)](x_0)\ge\dfrac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),\\ \dfrac{\partial}{\partial x_i}[N_p(kv)](x_0)\ge\dfrac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),\end{cases}$$

for all $i\in\{1,...,N\}$. Thus,

$$\begin{cases} \dfrac{\partial}{\partial x_i}[N_p(\varphi)](x_0)=\dfrac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),\\ \dfrac{\partial}{\partial x_i}[N_p(kv)](x_0)=\dfrac{\partial}{\partial x_i}[N_p(\omega_1)](x_0),\end{cases}$$

for all $i\in\{1,...,N\}$. Hence,

$$\nabla N_p(\varphi)(x_0)=\nabla N_p(\omega_1)(x_0)=\nabla N_p(kv)(x_0)=k^{p-1}\nabla N_p(v)(x_0).$$

Furthermore, we have

$$\nabla\left(\frac{N_p(\varphi)}{N_p(v)}\right)(x_0)=\frac{\nabla(N_p(\varphi))(x_0)N_p(v)(x_0)-N_p(\varphi)(x_0)\nabla(N_p(v))(x_0)}{\left[N_p(v)(x_0)\right]^2}$$
$$=\frac{[N_p(v)(x_0)-k^{1-p}N_p(\varphi)(x_0)]\nabla(N_p(\varphi))(x_0)}{\left[N_p(v)(x_0)\right]^2}=0.$$

We deduce that for all $x_0\in\Omega$, $\nabla\left(\frac{N_p(\varphi)}{N_p(v)}\right)(x_0)=0$. Consequently, $N_p(\frac{\varphi}{v})=\dfrac{N_p(\varphi)}{N_p(v)}=\text{const}=k^{p-1}$ in Ω. Then, $\varphi=kv$ in Ω.

It is easy to see all the same that $\psi=hw$ if for $x_0\in\Omega)$, we set

$$h=\frac{\psi(x_0)}{w(x_0)},\quad \overline{\omega}_1(x)=\max\{\psi(x),hw(x)\} \text{ and } \overline{\omega}_2(x)=\max\left\{\varphi(x),k^{\frac{q}{p}}v(x)\right\},$$

for all $x\in\Omega$.

Accordingly, $(\varphi,\psi)=(kv,hw)$ with $k=h^{\frac{q}{p}}$. We deduce that $(u_{21},u_{22})=(ku_{11},hu_{12})$ with $k=h^{\frac{q}{p}}$.

Let (u_{11},u_{12}) and (u_{21},u_{22}) be two eigenfunctions of (Q) associated with λ. If there exist $i,j\in\{1,2\}$ such that $u_{ij}<0$, then we can set $\overline{u}_{ij}=-u_{ij}$ and the result follows.

Case 2: Take λ as a semitrivial principal eigenvalue of (Q).

Let [$(u_{11},0)$ and $(u_{21},0)$] or [$(0,u_{12})$ and $(0,u_{22})$] be two eigenfunctions of (Q) associated with λ. It is easy to see that there exist [$k\neq 0$ real number] or [$h\neq 0$ real number] such that [$u_{11}=ku_{21}$] or [$u_{12}=hu_{22}$]. □

Theorem 3.1. *Assume that (H_m) holds. The lowest positive eigenvalue of problem (Q) is the value*

$$\lambda_1=\min_{(u,v)\in\mathcal{S}}E_m(u,v),\tag{3.15}$$

where

$$\mathcal{S}\;=\;\left\{(u,v)\in Y_{pq}(\Omega):\quad M(u,v)=1\right\}.$$

Moreover

1. $\lambda_1\leq\min\{\lambda_{1,p,1}(m_1),\lambda_{1,q,1}(m_2)\}$.
2. λ_1 *is semitrivial principal eigenvalue or strictly principal eigenvalue.*
3. λ_1 *is simple.*

Proof. Assume that (H_m) holds. Then from Proposition 2.2 and Lemma 3.1, there exists a unique real $\lambda_1 \in (0,\infty)$ solution of equation $\mu_1(\lambda)=0$, λ_1 is an eigenvalue of (Q) and

$$\mu_1^{'}(\lambda_1)=-M(u_0,v_0)<0=\mu_1(\lambda_1)=E_m(u_0,v_0)-\lambda_1 M(u_0,v_0)$$

with $(u_0,v_0)\in\mathcal{M}$. Then, $E_m(u_0,v_0)=\lambda_1 M(u_0,v_0)>0$ and we can set

$$\overline{u}_0=\frac{u_0}{[M(u_0,v_0)]^{\frac{1}{p}}},\quad \overline{v}_0=\frac{v_0}{[M(u_0,v_0)]^{\frac{1}{q}}}.$$

Thus, $(\overline{u}_0,\overline{v}_0)\in\mathcal{S}$ and $E_m(\overline{u}_0,\overline{v}_0)=\lambda_1$.

Additionally, for every $(u,v)\in\mathcal{S}$, one has

$$E_m\left(\frac{u}{[I(u,v)]^{\frac{1}{p}}},\frac{v}{[I(u,v)]^{\frac{1}{q}}}\right)\geq\lambda_1 M\left(\frac{u}{[I(u,v)]^{\frac{1}{p}}},\frac{v}{[I(u,v)]^{\frac{1}{q}}}\right),\ \text{ i.e. } E_m(u,v)\geq\lambda_1.$$

Consequently (3.15) holds. Moreover, from Proposition 3.2, λ_1 is a strictly principal eigenvalue or semitrivial principal eigenvalue and simple.

Set $\varphi_p=(\frac{p}{\alpha+1})^{\frac{1}{p}}\varphi_{p,1,m_1}$ and $\varphi_q=(\frac{q}{\beta+1})^{\frac{1}{q}}\varphi_{q,1,m_2}$. Then

$$\frac{\alpha+1}{p}M_1(\varphi_p)+\frac{\beta+1}{q}M_2(0)=1,\ \frac{\alpha+1}{p}M_1(0)+\frac{\beta+1}{q}M_2(\varphi_q)=1.$$

Thus

$$\begin{cases}\lambda_1\leq E_m(\varphi_p,0)=\frac{\alpha+1}{p}\|\Delta\varphi_p\|_p^p=\lambda_{1,p,1}(m_1),\\ \lambda_1\leq E_m(0,\varphi_q)=\frac{\beta+1}{q}\|\Delta\varphi_q\|_q^q=\lambda_{1,q,1}(m_2).\end{cases}$$

Consequently, $\lambda_1\leq\min\{\lambda_{1,p,1}(m_1),\lambda_{1,q,1}(m_2)\}$. □

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