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# Properties for Nonlinear Fractional SubLaplace Equations on the Heisenberg Group

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**Abstract.** The aim of the paper is to study properties of solutions to the nonlinear fractional subLaplace equations on the Heisenberg group. Based on the method of moving planes to the Heisenberg group, we prove the Liouville property of solutions on a half space and the symmetry and monotonicity of the solutions on the whole group respectively.

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**Key Words**: Heisenberg group; fractional subLaplace equation; method of moving planes.

## 1 Introduction

In this paper, we consider the fractional subLaplacian

$$(-\Delta_{\mathbb{H}})^s u(\xi) = C_{Q,s} PV \int_{\mathbb{H}^n} \frac{u(\xi)-u(\eta)}{|\eta^{-1}\circ\xi|_{\mathbb{H}}^{Q+2s}} \mathrm{d}\eta \tag{1.1}$$

on the Heisenberg group $\mathbb{H}^n$, where $0<s<1$, $Q=2n+2$, $C_{Q,s}$ is a positive constant and $PV$ is the Cauchy principle value, and study properties of cylindrical solutions to the equation

$$(-\Delta_{\mathbb{H}})^s u(\xi) = u^p(\xi) \quad \text{in } \mathbb{H}^n_+ = \{\xi \in \mathbb{H}^n | t>0\}, \tag{1.2}$$

where $1<p<\infty$, and another equation

$$(-\Delta_{\mathbb{H}})^s u(\xi) = f(u) \quad \text{in } \mathbb{H}^n, \tag{1.3}$$

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where $f$ is Lipschitz continuous.

Recall that the fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudodifferential operator defined by

$$(-\Delta)^{\alpha}u(x)=C_{n,\alpha}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B_{\varepsilon}(x)}\frac{u(x)-u(y)}{|x-y|^{n+2\alpha}}\mathrm{d}y, \tag{1.4}$$

where $0<\alpha<1$, $C_{n,\alpha}$ is a constant, $u$ belongs to the Schwartz space, the limit stands for the Cauchy principle value. Since the nonlocal property of the operator $(-\Delta)^{\alpha}$ brings new difficulties to investigate, Caffarelli and Silvestre in [1] put forward the extension method which can reduce the nonlocal problem relating to $(-\Delta)^{\alpha}$ to a local one in higher dimensions. This method has been applied to deal with equations involving the fractional Laplacian and fruitful results were obtained, see [2] and the references therein. But the method has tha limitation that $\frac{1}{2}\le\alpha<1$. Chen, Li and Li [4] developed a direct method of moving planes to handle the problem containing $(-\Delta)^{\alpha}$ for $0<\alpha<1$ and this direct method has been used successfully to deduce symmetry, monotonicity and nonexistence for many fractional Laplace equations, see [3,4] and references therein. Recently, Wang and Niu [5] considered (1.2) in $\mathbb{H}^n$, and showed properties of solutions.

To the elliptic equation

$$-\Delta u=f(u)\qquad \text{in } \mathbb{R}^n, \tag{1.5}$$

Gidas, Ni and Nirenberg [6] proved that the positive solutions are radial symmetric with the assumptions that the limitation of $u$ is zero at the infinity. In $\mathbb{H}^n$ we can also give a similar result of [6] to the equation (1.3).

There are many interesting results about subLaplace equations on the Heisenberg group (see [7–11]. It appears several different definitions of the fractional power subLaplacian in $\mathbb{H}^n$ (see [12–14] etc.). The definition of fractional power subLaplacian in Roncal and Thangavelu [14] is indeed a generalization of Cowling and Haagerup [15] about the Heat semigroup. The fractional power subLaplace equations can also be studied by generalizing the extension method in [1] to $\mathbb{H}^n$, for example, see [13, 16, 17]. Extending the method of moving planes in [3,4,18] to $\mathbb{H}^n$, we establish the Liouville type result of the solution to (1.2) on $\mathbb{H}^n_+=\{\xi\in\mathbb{H}^n|t>0\}$, and the symmetric and monotone of the solution to (1.3) on $\mathbb{H}^n$.

Our main results are the following:

**Theorem 1.1.** *Let $0<s<1$ and $1<p<\infty$. If $u\in L_{2s}(\mathbb{H}^n_+)\cap C^{1,1}_{loc}(\mathbb{H}^n_+)$ is a nonnegative cylindrical solution to*

$$\begin{cases}(-\Delta_{\mathbb{H}})^s u(\xi)=u^p & \xi\in\mathbb{H}^n_+,\\ u(\xi)=0 & \xi\notin\mathbb{H}^n_+,\end{cases} \tag{1.6}$$

*with*

$$\lim_{|\xi|_{\mathbb{H}}\to\infty}u(\xi)=0, \tag{1.7}$$

*and lower semi-continuous on $\bar{\mathbb{H}}^n_+$, then $u\equiv 0$.*