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On Free Boundary Problem for the Non-Newtonian Shear Thickening Fluids

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Abstract. The aim of this paper is to explore the free boundary problem for the Non-Newtonian shear thickening fluids. These fluids not only have vacuum, but also have strong nonlinear properties. In this paper, a class of approximate solutions is first constructed, and some uniform estimates are obtained for these approximate solutions. Finally, the existence of free boundary problem solutions is proved by these uniform estimates.

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1 Introduction

It is well known that the non-newtonian shear thickening flows can be described by the following equations (for example, see [1-6])

$$\rho_t+(\rho u)_x=0, \tag{1.1}$$

$$(\rho u)_t+(\rho u^2)_x-((|u_x|^2+\mu)^{(p-2)/2}u_x)_x+(A\rho^\gamma)_x=0, \tag{1.2}$$

where $p>2$, $A>0$, $\mu>0$ and $\gamma>1$ are some given positive constants, and ρ,u,ρ^γ represent the density, velocity and pressure for the non-Newtonian fluids, respectively.

We assume that the initial density ρ_0 is some given nonnegative function satisfying $\text{supp}\rho_0=[a_0,b_0]$ for some constants a_0 and b_0, and $||\rho_0||_{L^1(a_0,b_0)}=1$. Let $x=a(t)$ and $x=b(t)$ represent the free boundary which is the interface between fluid and vacuum, and then

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have $\rho(a(t),t)=\rho(b(t),t)=0$, and $a'(t)=u(a(t),t)$ with $a(0)=a_0$, and $b'(t)=u(b(t),t)$ with $b(0)=b_0$.

We introduce the Lagrange coordinate transformation

$$s=t,\quad y=\int_{a(t)}^{x}\rho(z,t)\mathrm{d}z. \tag{1.3}$$

Clearly, the left dividing line $\Gamma_0: x=a(t)$ for the interface is a straight line $\Gamma_0: y=0$ in Lagrange coordinates. In addition, in the right dividing line $\Gamma_1: x=b(t)$ for the interface, we have

$$y=\int_{a(t)}^{b(t)}\rho(z,t)\mathrm{d}z=\int_{a_0}^{b_0}\rho_0(z)\mathrm{d}z=1. \tag{1.4}$$

Therefore, the right dividing line $\Gamma_1: x=b(t)$ for the interface is a straight line $\Gamma_1: y=1$ in Lagrange coordinates. In particular, in Lagrange coordinates, the original equations (1.1)-(1.2) are transformed into the following equations

$$\rho_s+\rho^2u_y=0, \tag{1.5}$$

$$u_s-(((\rho u_y)^2+\mu)^{(p-2)/2}\rho u_y)_y+(A\rho^\gamma)_y=0. \tag{1.6}$$

This paper is to solve the above equations (1.5)-(1.6) in $Q_S\equiv(0,1)\times(0,S)(S>0)$ with the following initial condition

$$(\rho(y,0),u(y,0))=(\rho_0(y),u_0(y)),\quad y\in[0,1], \tag{1.7}$$

and the following boundary condition

$$(\rho u_y)(0,s)=(\rho u_y)(1,s)=0,\quad s\geq 0, \tag{1.8}$$

where the initial density $\rho_0=\rho_0(y)$ and the initial velocity $u_0=u_0(y)$ have the following properties [A1]-[A3]:

[A1] The initial density $\rho_0\in C(-\infty,+\infty)\cap C^1(0,1)$ satisfies

$$\rho_0(y)>0\quad\forall y\in(0,1),\quad \rho_0(y)=0\quad \forall y\in(-\infty,0]\cup[1,+\infty). \tag{1.9}$$

[A2] The initial velocity $u_0\in C^3(-\infty,+\infty)$ satisfies $u_{0y}(0)=u_{0y}(1)=0$.

[A3] The initial value (ρ_0,u_0) also has the following property:

$$\begin{aligned}M_0\equiv 1&+||\rho_0(y)||_{L^\infty(-\infty,+\infty)}+||\rho_0^{-1}(y)||_{L^1(-\infty,+\infty)}+||\rho_0'(y)||_{L^2(-\infty,+\infty)}\\&+||u_0(y)||_{W^{3,\infty}(-\infty,+\infty)}<+\infty.\end{aligned}$$

Our main results are the following theorems.

Theorem 1.1. *Let $p>2$ and $\gamma>1$, and assume that* [A1]-[A3] *hold. Then there is a positive number $S_0\in(0,1)$ such that, the initial-boundary problem* (1.5)-(1.8) *has at least one solution $(\rho,u)=(\rho(y,s),u(y,s))$ for $(y,s)\in Q_{S_0}$. In particular, the solution (ρ,u) also has the following properties:*

(i) *There exist two positive constants μ_1 and μ_2 depending only on A,p,γ and M_0 such that*

$$\mu_1\rho_0(y)\le\rho(y,s)\le\mu_2\rho_0(y) \tag{1.10}$$

for almost all $(y,s)\in Q_{S_0}$.

(ii) *The solution (ρ,u) has the following regularity:*

$$\rho\in L^\infty(Q_{S_0}),\quad \rho_s\in L^\infty(Q_{S_0}),\quad \rho_y\in L^\infty(0,S_0;L^2(0,1)), \tag{1.11}$$
$$u\in L^\infty(Q_{S_0}),\quad u_s\in L^\infty(0,S_0;L^2(0,1)),\quad u_y\in L^\infty(0,S_0;L^1(0,1)), \tag{1.12}$$
$$\rho u_y\in L^\infty(Q_{S_0}),\quad ((|\rho u_y|^2+\mu)^{(p-2)/2}\rho u_y)_y\in L^\infty(0,S_0;L^2(0,1)). \tag{1.13}$$

(iii) *For almost all $(y,s)\in Q_{S_0}$, the solution $(\rho,u)=(\rho(y,s),u(y,s))$ satisfies Eqs.* (1.5)-(1.6).

(iv) *For almost all $s\in(0,S_0)$, the solution (ρ,u) satisfies initial conditions* (1.7) *in the following sense:*

$$||\rho(\cdot,s)-\rho_0(\cdot)||_{L^2(0,1)}+||u(\cdot,s)-u_0(\cdot)||_{L^2(0,1)}\le\mu_3 s, \tag{1.14}$$

where μ_3 is a positive constant depending only on A,p,μ,γ and M_0.

(v) *For almost all $(y,s)\in Q_{S_0}$, the solution(ρ,u) satisfies boundary condition* (1.8) *in the following sense:*

$$|(\rho u_y)(y,s)|\le\mu_4\min\{y^{1/2},(1-y)^{1/2}\}, \tag{1.15}$$

where μ_4 is a positive constant depending only on A,p,μ,γ and M_0.

We shall prove Theorem 1.1 in Section 4. In order to prove Theorem 1.1, we need some Lemmas in Sections 2-3.

2 Fundamental lemmas

In order to prove our results, we need following Lemmas.

Lemma 2.1. *Let $0<\epsilon<1$. We define*

$$G_\epsilon(r)=\left(\frac{r^2+\epsilon^2}{\epsilon^2r^2+1}\right)^{1/2} \tag{2.1}$$

for all $r\in(-\infty,+\infty)$. Then we have

$$\epsilon\le G_\epsilon(r)\le\min\{|r|+\epsilon,\ \epsilon^{-1}\},\quad |G'_\epsilon(r)|\le 1. \tag{2.2}$$

In addition, for any $q>0$, we also have

$$G_\epsilon^{-q}(r)\le C_1\epsilon^q+C_1(|r|+\epsilon)^{-q},2.3) \tag{()}$$

$$|\{G_\epsilon^q(r)-|r|^q\}|\le\begin{cases}C_1\epsilon^q(1+r^4), & \text{if } 0<q<1,\\ C_1\epsilon(1+|r|^{4q}), & \text{if } q\ge 1,\end{cases} \tag{2.4}$$

where C_1 is a positive constant depending only on q.

Proof. The conclusions (2.2)-(2.3) of Lemma 2.1 can be obtained by direct calculation. For (2.4), we have two cases: (i) $0<q<1$; (ii) $q\ge 1$.

To prove (2.4) in the case: (i) $0<q<1$. First, we have

$$|G_\epsilon(r)-|r||\le\epsilon(1+r^4). \tag{2.5}$$

By (2.2), we have

$$|G_\epsilon^q(r)-|r|^q|\le(|r|+\epsilon)^q+|r|^q\le C\epsilon^q, \tag{2.6}$$

for $|r|\le\epsilon$. For $|r|\ge\epsilon$, by (2.5), we compute

$$\begin{aligned}|G_\epsilon^q(r)-|r|^q|=&|G_\epsilon(r)-|r||\cdot\int_0^1 q(\theta|r|+(1-\theta)G_\epsilon(r))^{q-1}\mathrm{d}\theta\\ \le&\epsilon(1+r^4)\int_0^1 q(\theta\epsilon)^{q-1}\mathrm{d}\theta\le\epsilon^q(1+r^4).\end{aligned}$$

Combining the above inequality with (2.6) we have (2.4) in the case $0<q<1$. For $q\ge 1$, the proof is similarly, and then the details are omitted. Therefore (2.4) is proved and then the proof of Lemma 2.1 is completed. □

Lemma 2.2. *We define*

$$j_\eta(r)=\eta^{-1}j(\eta^{-1}r) \tag{2.7}$$

for all $r\in(-\infty,+\infty)$ and all $\eta\in(0,1)$, where $j\in C_0^\infty(-\infty,+\infty)$ is a nonnegative function satisfying

$$\mathrm{supp} j\subset(-1,1),\qquad j(-r)=j(r),\qquad \int_{-\infty}^{+\infty}j(r)\mathrm{d}r=1. \tag{2.8}$$

In addition, we also define

$$\rho_0^\epsilon(y)=(j_{\epsilon^2}*\rho_{0\epsilon})(y)+\epsilon^m, \tag{2.9}$$

where $m=1/4$, and

$$\rho_{0\epsilon}(y)=\rho_0\left(\frac{1}{2}+(1+4\epsilon^2)\left(y-\frac{1}{2}\right)\right). \tag{2.10}$$

Then we have

$$\mathrm{supp}\rho_{0\epsilon}\subset\left(\frac{2\epsilon^2}{1+4\epsilon^2},\frac{1+2\epsilon^2}{1+4\epsilon^2}\right). \tag{2.11}$$

In addition, for any $\epsilon\in(0,\epsilon_0)$, we have

$$\text{supp}(j_{\epsilon^2}*\rho_{0\epsilon})\subset(0,1),\quad \rho_0^\epsilon(0)=\rho_0^\epsilon(1)=\epsilon^m, \tag{2.12}$$

$$||\rho_0^\epsilon(y)||_{L^\infty(-\infty,+\infty)}+||\rho_{0y}^\epsilon(y)||_{L^2(-\infty,+\infty)}\le C_2, \tag{2.13}$$

$$||\rho_0^\epsilon-\rho_0||_{L^\infty(-\infty,+\infty)}\le C_2\epsilon^m, \tag{2.14}$$

$$\frac{1}{2}(\rho_0(y)+\epsilon^m)\le\rho_0^\epsilon(y)\le C_2(\rho_0(y)+\epsilon^m)\quad \forall y\in[0,1], \tag{2.15}$$

where $\epsilon_0\in(0,1/2)$ and $C_2\in(1,+\infty)$ are some positive constants depending only on p and M_0.

Proof. From (2.7)-(2.10), the conclusion (2.11)-(2.13) can be obtained by direct calculation. Therefore, the details of the proofs for (2.11)-(2.13) are omitted.

To prove (2.14). In fact, by (2.7)-(2.10), for $y\in[0,1]$, we compute

$$\begin{aligned}
|\rho_0^\epsilon(y)-(\rho_0(y)+\epsilon^m)|\le& C|(j_{\epsilon^2}*\rho_{0\epsilon})(y)-\rho_0(y)|\\
=&C\left|\int_{-\infty}^{+\infty}\left\{\rho_0\left(\frac{1}{2}+(1+4\epsilon^2)\left(y-\epsilon^2 z-\frac{1}{2}\right)\right)-\rho_0(y)\right\}j(z)\mathrm{d}z\right|\\
=&C\left|\int_{-1}^{1}j(z)\left\{\int_y^{1/2+(1+4\epsilon^2)(y-\epsilon^2z-1/2)}\left[\frac{\mathrm{d}(\rho_0(\lambda))}{\mathrm{d}\lambda}\right]\mathrm{d}\lambda\right\}\mathrm{d}z\right|\\
\le& C\epsilon\int_{-1}^{1}j(z)\left\{\int_{-\infty}^{+\infty}\left[\frac{\mathrm{d}(\rho_0(\lambda))}{\mathrm{d}\lambda}\right]^2\mathrm{d}\lambda\right\}^{1/2}\mathrm{d}z\le C\epsilon.
\end{aligned}$$

By the above inequality, we can obtain (2.14) and (2.15). Thus the proof of Lemma 2.2 is completed. □

Lemma 2.3. *Let $\epsilon\in(0,1)$. We denote*

$$u_0^\epsilon(y)=(j_{\epsilon^2}*u_0)(y)-(j_{\epsilon^2}*u_0)(0)\left(y-\frac{y^2}{2}\right)-\frac{1}{2}(j_{\epsilon^2}*u_0)(1)y^2 \tag{2.16}$$

for all $y\in(-\infty,+\infty)$, and then have

$$u_{0y}^\epsilon(0)=u_{1y}^\epsilon(1)=0. \tag{2.17}$$

In addition, we also have

$$||u_0^\epsilon||_{W^{2,\infty}[0,1]}\le C_3,\qquad ||u_0^\epsilon-u_0||_{W^{2,\infty}[0,1]}\le C_3\epsilon^2, \tag{2.18}$$

where C_3 is a positive constant depending only on M_0.

Proof. From (2.16), the conclusions (2.17)-(2.18) can be obtained by direct calculation. Therefore, the details of the proof for Lemma 2.3 are omitted. Thus the proof of Lemma 2.3 is completed. □

Lemma 2.4. *Assume that* $\{h_n(s):n=1,2,\cdots\}$ *is a sequence of nonnegative continuous functions satisfying the following inequalities*

$$h_{n+1}(s)\le C_4\int_0^s h_n(\tau)\mathrm{d}\tau+C_5$$

for all $s\in(0,1)$ *and all* $n=1,2,\cdots$, *where* C_4 *and* C_5 *are some given nonnegative real numbers. Then we have*

$$h_{n+1}(s)\le\frac{C_6(C_4s)^n}{n!}+C_5e^{C_4s} \tag{2.19}$$

for all $s\in(0,1)$ *and all* $n=1,2,\cdots$, *where* $C_6=\sup_{s\in(0,1)}h_1(s)$.

Proof. Applying the mathematical induction method, we immediately get

$$h_{n+1}(s)\le\frac{C_6(C_4s)^n}{n!}+C_5\sum_{k=0}^{n-1}\frac{(C_4s)^k}{k!}$$

for all $s\in(0,1)$ and all $n=1,2,\cdots$. This implies (2.19). Thus the proof of Lemma 2.4 is completed. □

Lemma 2.5. *We define a function*

$$H(r)=(r^2+\mu)^{(p-2)/2}r \tag{2.20}$$

for all $r\in(-\infty,+\infty)$. *Then we have*

$$\nu_1|r|\le|H(r)|\le\nu_2(1+|r|^{p-1}), \tag{2.21}$$

$$\nu_1(1+|r|^{p-2})\le H'(r)\le\nu_2(1+|r|^{p-2}), \tag{2.22}$$

where ν_1 *and* ν_2 *are some positive constants depending only on* μ *and* p.

Proof. From (2.20), the conclusions (2.21)-(2.22) can be obtained by direct calculation. Therefore, the details of the proof of Lemma 2.5 are omitted. Thus the proof of Lemma 2.5 is completed. □

3 The constructions and uniform estimates of the approximate solutions

By ρ_0^ϵ defined by Lemma 2.2 and u_0^ϵ defined by Lemma 2.3, we construct a sequence of the approximate solutions as follows.

Step 1. We define $\rho^0=\rho_0^\epsilon$ and then consider the following initial-boundary problem

$$\begin{cases}u_s^1-[H(G_\epsilon(\rho^0)u_y^1)]_y+(AG_\epsilon^\gamma(\rho^0))_y=0,\\ u^1|_{s=0}=u_0^\epsilon,\qquad u_y^1|_{y=0,1}=0.\end{cases} \tag{3.1}$$

By [7], the initial-boundary problem (3.1) has a unique smooth solution $u^1=u^1(y,s)$.

Step 2. We consider the following initial value problem

$$\begin{cases} \rho^1_s+\rho^1 G_\epsilon(\rho^0)u^1_y=0, \\ \rho^1|_{s=0}=\rho^\epsilon_0. \end{cases} \tag{3.2}$$

Clearly, the initial value problem (3.2) has a smooth solution $\rho^1=\rho^1(y,s)$.

Step 3. We consider the following initial-boundary problem

$$\begin{cases} u^2_s-[H(G_\epsilon(\rho^1)u^2_y)]_y+(AG^\gamma_\epsilon(\rho^2))_y=0, \\ u^2|_{s=0}=u^\epsilon_0, \qquad u^2_y|_{y=0,1}=0. \end{cases} \tag{3.3}$$

By [7], the initial boundary problem (3.3) has a unique smooth solution $u^2=u^2(y,s)$. In addition, we also consider the following initial value problem

$$\begin{cases} \rho^2_s+\rho^2 G_\epsilon(\rho^1)u^2_y=0, \\ \rho^2|_{s=0}=\rho^\epsilon_0. \end{cases} \tag{3.4}$$

Clearly, the initial value problem (3.4) also has a smooth solution $\rho^2=\rho^2(y,s)$.

Repeating the above process we can find a sequence $\{(\rho^n,u^n)\}_{n=1}^\infty$ of the approximate solutions, which are smooth and satisfy the following equations

$$\rho^n_s+\rho^n\Gamma^n=0, \tag{3.5}$$

$$u^n_s-F^n_y=0, \tag{3.6}$$

with initial conditions

$$(\rho^n,u^n)|_{s=0}=(\rho^\epsilon_0,u^\epsilon_0), \tag{3.7}$$

and boundary conditions

$$u^n_y|_{y=0,1}=0, \tag{3.8}$$

where

$$F^n=F^n(y,s)=H(\Gamma^n)-R^n, \tag{3.9}$$

$$R^n=R^n(y,s)=AG^\gamma_\epsilon(\rho^{n-1})(y,s), \tag{3.10}$$

$$\Gamma^n=\Gamma^n(y,s)=[G_\epsilon(\rho^{n-1})u^n_y](y,s). \tag{3.11}$$

Using [A1] and (2.14), by (3.5), we have

$$\rho^n=\rho^\epsilon_0(y)\exp\left(-\int_0^s\Gamma^n(y,\tau)\mathrm{d}\tau\right)>0. \tag{3.12}$$

Next, we shall find some uniform estimates of approximate solutions $\{(\rho^n,u^n)\}_{n=1}^\infty$. We have the following lemmas.

Lemma 3.1. *Let $p>2$. For any positive integer k, we define*

$$\Phi_k(s)=\sup_{1\le n\le k}\sup_{0\le\tau\le s}\Big\{1+||\rho^n(\cdot,\tau)||_{L^\infty(0,1)}+||\Gamma^n(\cdot,\tau)||_{L^\infty(0,1)}\Big\}, \tag{3.13}$$

$$S_k=\sup\{s\in(0,\epsilon_0):s\Phi_k^\alpha(s)\le 1\}, \tag{3.14}$$

where

$$\alpha=16p+8\gamma. \tag{3.15}$$

Then, for all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k]$, and $\epsilon\in(0,S_k]$, we have

$$\mu_1(\rho_0(y)+\epsilon^m)\le\rho^n(y,s)\le\mu_2(\rho_0(y)+\epsilon^m), \tag{3.16}$$

$$\rho^n(y,s)\le C_7, \tag{3.17}$$

where μ_1, μ_2 and C_7 are some positive constants depending only on A,p,γ and M_0.

Proof. By (3.13)-(3.14), using Lemma 2.2, we compute

$$\begin{aligned}\rho^n=&\rho_0^\epsilon(y)\exp\left(-\int_0^s\Gamma^n(y,\tau)\mathrm{d}\tau\right)\\ \le&C_2(\rho_0(y)+\epsilon^m)\exp\left(\int_0^s\Phi_k(s)\mathrm{d}\tau\right)\\ \le&C(\rho_0(y)+\epsilon^m)\exp(s\Phi_k(s))\le C(\rho_0(y)+\epsilon^m).\end{aligned}$$

Similarly to the above inequality, we also have $\rho^n\ge C^{-1}(\rho_0(y)+\epsilon^m)$. Therefore, we have (3.16). From (3.16), by [A3], we get (3.17). Thus the proof of Lemma 3.1 is completed. □

Lemma 3.2. *Let $p>2$ and denote*

$$S_k^1=\min\{\epsilon_1,\,S_k\},\qquad \epsilon_1=\min\left\{\epsilon_0,\,\left(\frac{\mu_1}{4C_1}\right)^4\right\}, \tag{3.18}$$

where μ_1 and C_1 are defined by Lemma 3.1 and Lemma 2.1, respectively. Then, for all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k^1]$, and $\epsilon\in(0,S_k^1)$, we have

$$|[(\rho^{n-1}+\epsilon^m)u_y^n](y,s)|\le C_8\Phi_k(s), \tag{3.19}$$

$$|\Gamma^n(y,s)-(\rho^{n-1}u_y^n)(y,s)|\le\epsilon^{1/8}, \tag{3.20}$$

where C_8 is a positive constant depending only on A,p,μ,γ and M_0.

Proof. By Lemmas 2.1 and 3.1, we compute

$$\begin{aligned}|\Gamma^n(y,s)-(\rho^{n-1}u_y^n)(y,s)|=&|(G_\epsilon(\rho^{n-1})u_y^n)(y,s)-(\rho^{n-1}u_y^n)(y,s)|\\ \le&C_1\epsilon(1+((\rho^{n-1})^4)|u_y^n(y,s)|\end{aligned}$$

$$\begin{aligned}
&\le C_1\epsilon(\rho^{n-1})^{-1}(1+\Phi_k^4(s))|(\rho^{n-1}u_y^n)(y,s)|\\
&\le C_1\epsilon[\mu_1(\rho_0(y)+\epsilon^m)]^{-1}(1+\Phi_k^4(s))|(\rho^{n-1}u_y^n)(y,s)|\\
&\le 2C_1\mu_1^{-1}\epsilon^{1-m}\Phi_k^4(s)|(\rho^{n-1}u_y^n)(y,s)|\\
&\le 2C_1\mu_1^{-1}\epsilon^{1/2}[\epsilon\Phi_k^{16}(s)]^{1/4}|(\rho^{n-1}u_y^n)(y,s)|\\
&\le 2C_1\mu_1^{-1}\epsilon^{1/2}|(\rho^{n-1}u_y^n)(y,s)|\\
&\le \frac{1}{2}\epsilon^{1/4}|(\rho^{n-1}u_y^n)(y,s)|,
\end{aligned}$$

which implies that

$$|\Gamma^n(y,s)-(\rho^{n-1}u_y^n)(y,s)|\le\frac{1}{2}\epsilon^{1/4}|(\rho^{n-1}u_y^n)(y,s)|. \tag{3.21}$$

Then, for $n=1,\cdots,k$ and $\epsilon\in(0,S_k^1)$, by (3.21), we have

$$|(\rho^{n-1}u_y^n)(y,s)|\le 2|\Gamma^n(y,s)|\le 2\Phi_k(s). \tag{3.22}$$

By (3.21) and (3.22), using Lemma 3.1 we get

$$|\Gamma^n(y,s)-(\rho^{n-1}u_y^n)(y,s)|\le\epsilon^{1/4}\Phi_k(s)\le\epsilon^{1/8}(\epsilon\Phi_k^8(s))^{1/8}\le\epsilon^{1/8}. \tag{3.23}$$

In addition, by Lemma 3.1, we compute

$$\begin{aligned}
\epsilon^m|u_y^n(y,s)|&=\epsilon^m(\rho^{n-1})^{-1}|(\rho^{n-1}u_y^n)(y,s)|\\
&\le\epsilon^m[\mu_1(\rho_0(y)+\epsilon^m)]^{-1}|(\rho^{n-1}u_y^n)(y,s)|\le 2\mu_1^{-1}\Phi_k(s).
\end{aligned}$$

Combining the above inequality with (3.22)-(3.23) we have (3.19)-(3.20). Thus the proof of Lemma 3.2 is completed. □

Lemma 3.3. *Let $p>2$ and $\gamma>1$. For all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k]$, and $\epsilon\in(0,S_k^1)$, we have*

$$|R^n(y,s)-A(\rho^{n-1})^\gamma|\le C_9\epsilon^{1/2}, \tag{3.24}$$

$$|H(\Gamma^n)-H(\rho^{n-1}u_y^n)|\le C_9\epsilon^{1/16}, \tag{3.25}$$

where C_9 is a positive constant depending only on A,p,μ and γ.

Proof. By Lemmas 2.1 and 3.1, we compute

$$\begin{aligned}
|R^n(y,s)-A(\rho^{n-1})^\gamma|&=|AG_\epsilon^\gamma(\rho^{n-1})-A(\rho^{n-1})^\gamma|\le C\epsilon(1+(\rho^{n-1})^{4\gamma})\\
&\le C\epsilon\Phi_k^{4\gamma}=C\epsilon^{1/2}(\epsilon\Phi_k^{8\gamma})^{1/2}\le C\epsilon^{1/2},
\end{aligned}$$

which implies (3.25). By Lemma 2.5 and Lemma 3.1, we compute

$$
\begin{aligned}
&|H(\Gamma^n(y,s))-H((\rho^{n-1}u_y^n)(y,s))|\\
=&\left|(\Gamma^n-\rho^{n-1}u_y^n)\int_0^1 H'(\theta\Gamma^n(y,s)+(1-\theta)(\rho^{n-1}u_y^n)(y,s))\mathrm{d}\theta\right|\\
\le&\epsilon^{1/8}\int_0^1 |H'(\theta\Gamma^n(y,s)+(1-\theta)(\rho^{n-1}u_y^n)(y,s))|\mathrm{d}\theta\\
\le&\epsilon^{1/8}\int_0^1 \nu_2(1+|(\theta\Gamma^n(y,s)+(1-\theta)(\rho^{n-1}u_y^n)(y,s))|^{p-2})\mathrm{d}\theta\\
\le&C\epsilon^{1/8}[1+\Phi_k^{p-2}(s)+(2\Phi_k(s))^{p-2}]\\
\le&C\epsilon^{1/16}[\epsilon\Phi_k^{16(p-2)}(s)]^{1/16}\le C\epsilon^{1/16},
\end{aligned}
$$

which implies (3.25). Thus the proof of Lemma 3.3 is completed. □

Lemma 3.4. *Let $1<p<2$ and $\gamma>1$. For all $n=1,2,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k^1]$, and $\epsilon\in(0,S_k^1]$, we have*

$$\int_0^1 |F_y^n(y,s)|^2\mathrm{d}y\le C_{10}, \tag{3.26}$$

where C_{10} is a positive constant depending only on A,p,μ,γ and M_0.

Proof. By (3.5)-(3.6) and (3.9)-(3.11), we compute

$$
\begin{aligned}
F_s^n=&\frac{\partial}{\partial s}\{H(\Gamma^n)-R^n\}\\
=&\frac{\partial}{\partial s}\left\{H(G_\epsilon(\rho^{n-1})u_y^n)-AG_\epsilon^\gamma(\rho^{n-1})\right\}\\
=&H'(G_\epsilon(\rho^{n-1})u_y^n)G_\epsilon(\rho^{n-1})u_{ys}^n+H'(G_\epsilon(\rho^{n-1})u_y^n)G'_\epsilon(\rho^{n-1})\rho_s^{n-1}u_y^n\\
&\quad-A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1})G'_\epsilon(\rho^{n-1})\rho_s^{n-1}\\
=&H'(G_\epsilon(\rho^{n-1})u_y^n)G_\epsilon(\rho^{n-1})F_{yy}^n-H'(G_\epsilon(\rho^{n-1})u_y^n)G'_\epsilon(\rho^{n-1})(\rho^{n-1}\Gamma^{n-1})u_y^n\\
&\quad+A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1})G'_\epsilon(\rho^{n-1})(\rho^{n-1}\Gamma^{n-1}),
\end{aligned}
$$

which implies

$$F_s^n-f_1^nF_{yy}^n=f_2^n+f_3^n, \tag{3.27}$$

where

$$f_1^n=H'(G_\epsilon(\rho^{n-1})u_y^n)G_\epsilon(\rho^{n-1}), \tag{3.28}$$
$$f_2^n=-H'(G_\epsilon(\rho^{n-1})u_y^n)G'_\epsilon(\rho^{n-1})(\rho^{n-1}\Gamma^{n-1})u_y^n, \tag{3.29}$$
$$f_3^n=A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1})G'_\epsilon(\rho^{n-1})(\rho^{n-1}\Gamma^{n-1}). \tag{3.30}$$

By (3.25), for all $s\in(0,1)$, we have

$$\int_0^s\int_0^1 (F_s^n-f_1^nF_{yy}^n)(-F_{yy}^n)(y,\tau))\mathrm{d}y\mathrm{d}\tau=\int_0^s\int_0^1 (f_2^n+f_3^n)(-F_{yy}^n)(y,\tau))\mathrm{d}y\mathrm{d}\tau. \tag{3.31}$$

We now calculate the items on both sides of (3.31). By (3.8) and (3.11), we have

$$\Gamma^n(0,s)=G_\epsilon(\rho^{n-1}(0,s))u_y^n(0,s)=0,\qquad \forall s\in[0,1]. \tag{3.32}$$

By (3.12) and (3.32), using Lemma 2.2 we get

$$\rho^n(0,s)=\rho_0^\epsilon(y)\exp\left(-\int_0^s\Gamma^n(0,\tau)\mathrm{d}\tau\right)=\epsilon^m,\qquad \forall s\in[0,1]. \tag{3.33}$$

By (3.9)-(3.11) and (3.32)-(3.33), we obtain

$$F^n(0,s)=H(\Gamma^n(0,s))-AG_\epsilon^\gamma(\rho^{n-1}(0,s))=-AG_\epsilon^\gamma(\epsilon^m),\qquad \forall s\in[0,1]. \tag{3.34}$$

This implies

$$F_s^n(0,s)=0,\qquad \forall s\in[0,1]. \tag{3.35}$$

Similarly to (3.35), we also have

$$F_s^n(1,s)=0,\qquad \forall s\in[0,1]. \tag{3.36}$$

Applying (3.35)-(3.36), we compute

$$\begin{aligned}&\int_0^s\int_0^1 (F_s^n-f_1^nF_{yy}^n)(-F_{yy}^n)(y,\tau))\mathrm{d}y\mathrm{d}\tau\\ =&\frac{1}{2}\int_0^1|F_y^n(y,s)|^2\mathrm{d}y+\int_0^s\int_0^1 f_1^n(F_{yy}^n)^2\mathrm{d}y\mathrm{d}\tau-\frac{1}{2}\int_0^1|F_y^n(y,0)|^2\mathrm{d}y.\end{aligned} \tag{3.37}$$

Using Young's inequality we compute

$$\begin{aligned}&\left|\int_0^s\int_0^1 (f_2^n+f_3^n)(-F_{yy}^n)(y,\tau))\mathrm{d}y\mathrm{d}\tau\right|\\ \le&\frac{1}{2}\int_0^s\int_0^1 f_1^n(F_{yy}^n)^2\mathrm{d}y\mathrm{d}\tau+\frac{1}{2}\int_0^s\int_0^1 (f_1^n)^{-1}(f_2^n+f_3^n)^2\mathrm{d}y\mathrm{d}\tau.\end{aligned} \tag{3.38}$$

Combining (3.37)-(3.38) with (3.31) we conclude that

$$\begin{aligned}&\int_0^1|F_y^n(y,s)|^2\mathrm{d}y+\int_0^s\int_0^1 f_1^n(F_{yy}^n)^2\mathrm{d}y\mathrm{d}\tau\\ \le&\int_0^s\int_0^1 (f_1^n)^{-1}(f_2^n+f_3^n)^2\mathrm{d}y\mathrm{d}\tau+\int_0^1|F_y^n(y,0)|^2\mathrm{d}y.\end{aligned} \tag{3.39}$$

We now calculate the two items on the right side of (3.39). First, applying Lemma 2.1, Lemma 2.5 and Lemma 3.1, by (3.28)-(3.30), we compute

$$
\begin{aligned}
&(f_1^n)^{-1}(f_2^n+f_3^n)^2\\
=&[H'(G_\epsilon(\rho^{n-1})u_y^n)G_\epsilon(\rho^{n-1})]^{-1}\{-H'(G_\epsilon(\rho^{n-1})u_y^n)G'_\epsilon(\rho^{n-1})(\rho^{n-1}\Gamma^{n-1})u_y^n\\
&\quad+A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1})G'_\epsilon(\rho^{n-1})(\rho^{n-1}\Gamma^{n-1})\}^2\\
\le& C[\nu_2(1+|G_\epsilon(\rho^{n-1})u_y^n|^{p-2})]^{-1}G_\epsilon^{-1}(\rho^{n-1})(\Gamma^{n-1})^2\{[H'(G_\epsilon(\rho^{n-1})u_y^n)]^2(\rho^{n-1}u_y^n)^2\\
&\quad+G_\epsilon^{2\gamma-2}(\rho^{n-1})(\rho^{n-1})^2\}\\
\le& C\left(\frac{\epsilon^2(\rho^{n-1})^2+1}{(\rho^{n-1})^2+\epsilon^2}\right)^{1/2}(\Gamma^{n-1})^2\{[1+|G_\epsilon(\rho^{n-1})u_y^n)|^{p-2}]^2(\rho^{n-1}u_y^n)^2\\
&\quad+G_\epsilon^{2\gamma}(\rho^{n-1})(\rho^{n-1})^2\left(\frac{\epsilon^2(\rho^{n-1})^2+1}{(\rho^{n-1})^2+\epsilon^2}\right)\Bigg\}\\
\le& C(\rho^{n-1})^{-1}(\Gamma^{n-1})^2\{[1+|(\rho^{n-1}+\epsilon)u_y^n)|^{p-2}]^2(\rho^{n-1}u_y^n)^2+(\rho^{n-1}+\epsilon)^{2\gamma}\}\\
\le& C[\mu_1(\rho_0(y)+\epsilon^m)]^{-1}\Phi_k^2(s)\{[1+\Phi_k^{p-2}(s)]^2\Phi_k^2(s)+1\}\\
\le& C(\rho_0(y)+\epsilon^m)^{-1}\Phi_k^{2p}(s),
\end{aligned}
$$

which implies

$$
(f_1^n)^{-1}(f_2^n+f_3^n)^2\le C(\rho_0(y)+\epsilon^m)^{-1}\Phi_k^{2p}(s). \tag{3.40}
$$

By (3.40) and [A3], using Lemma 3.1 we get

$$
\int_0^s\int_0^1(f_1^n)^{-1}(f_2^n+f_3^n)^2\mathrm{d}y\mathrm{d}\tau\le Cs\Phi_k^{2p}(s)\le C \tag{3.41}
$$

for $s\in[0,S_k^1]$, where C is a positive constant depending only on A,p,μ,γ and M_0.

Finally, let us calculate the second item on the right side of (3.39). Using (3.9)-(3.11), we compute we compute

$$
\begin{aligned}
F_y^n(y,s)=&\frac{\partial}{\partial y}\{H(\Gamma^n)-R^n\}(y,s)\\
=&\frac{\partial}{\partial y}\left\{H(G_\epsilon(\rho^{n-1})u_y^n)-AG_\epsilon^\gamma(\rho^{n-1})\right\}\\
=&H'(G_\epsilon(\rho^{n-1})u_y^n)G_\epsilon(\rho^{n-1})u_{yy}^n+H'(G_\epsilon(\rho^{n-1})u_y^n)G'_\epsilon(\rho^{n-1})\rho_y^{n-1}u_y^n\\
&\quad-A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1})G'_\epsilon(\rho^{n-1})\rho_y^{n-1}.
\end{aligned}
$$

By the above inequality and (3.7), using Lemma 2.2-2.3, we compute

$$
\begin{aligned}
|F_y^n(y,0)|=&\Big|H'(G_\epsilon(\rho^{n-1}(y,0))u_y^n(y,0))G_\epsilon(\rho^{n-1}(y,0))u_{yy}^n(y,0)\\
&\quad+H'(G_\epsilon(\rho^{n-1}(y,0))u_y^n(y,0))G'_\epsilon(\rho^{n-1}(y,0))\rho_y^{n-1}(y,0)u_y^n(y,0)
\end{aligned}
$$

$$\begin{aligned}&\quad -A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1}(y,0))G_\epsilon'(\rho^{n-1}(y,0))\rho_y^{n-1}(y,0)\Big|\\&=\Big|H'(G_\epsilon(\rho_0^\epsilon(y))u_{0y}^\epsilon(y))G_\epsilon(\rho_0^\epsilon(y))u_{0yy}^\epsilon(y)\\&\quad +H'(G_\epsilon(\rho_0^\epsilon(y))u_{0y}^\epsilon(y))G_\epsilon'(\rho_0^\epsilon(y))\rho_{0y}^\epsilon(y)u_{0y}^\epsilon(y)\\&\quad -A\gamma G_\epsilon^{\gamma-1}(\rho_0^\epsilon(y))G_\epsilon'(\rho_0^\epsilon(y))\rho_{0y}^\epsilon(y)\Big|\\&\le C(1+|\rho_{0y}^\epsilon(y)|).\end{aligned}$$

By the above inequality, using Lemma 2.2 we have

$$\int_0^1 |F_y^n(y,0)|^2 \mathrm{d}y \le C, \tag{3.42}$$

for all $s\in(0,S_k^1)$, where C is a positive constant depending only on A,p,μ,γ and M_0. Combining (3.41)-(3.42) with (3.39) we get (3.26). Thus the proof of Lemma 3.4 is completed. □

Lemma 3.5. *Let $p>2$ and $\gamma>1$. For all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k^1]$, and $\epsilon\in(0,S_k^1]$, we have*

$$|F^n(y,s)|+|R^n(y,s)|+|\Gamma^n(y,s)|+|H(\Gamma^n(y,s))|+|[(\rho^{n-1}(y)+\epsilon^m)u_y^n(y,s)|\le C_{11}, \tag{3.43}$$

where C_{11} is a positive constant depending only on A,p,γ,μ and M_0.

Proof. Applying Lemma 3.4, by (3.34), we compute

$$\begin{aligned}|F^n(y,s)|&\le|F^n(y,s)-F^n(0,s)|+|F^n(0,s)|\\&=\left|\int_0^y F_z^n(z,s)\mathrm{d}z\right|+|-AG_\epsilon^\gamma(\rho_0^\epsilon(0))|\\&\le\left(\int_0^1|F_z^n(z,s)|^2\mathrm{d}z\right)^{1/2}\left(\int_0^1\mathrm{d}z\right)^{1/2}+AG_\epsilon^\gamma(\epsilon^m)\le C,\end{aligned}$$

which implies

$$|F^n(y,s)|\le C. \tag{3.44}$$

On the other hand, using Lemmas 3.1 and 3.3, we have

$$|R^n(y,s)|\le|A(\rho^{n-1})^\gamma|+C_{10}\epsilon^{1/2}\le C. \tag{3.45}$$

In addition, by (3.9) and (3.44)-(3.45), we get

$$|H(\Gamma^n(y,s))|\le|F^n(y,s)|+|R^n(y,s)|\le C. \tag{3.46}$$

By (3.46), using Lemma 2.5 we get

$$|\Gamma^n(y,s)|\le C. \tag{3.47}$$

By (3.47), using Lemma 3.2 we get

$$|(\rho^{n-1}u_y^n)(y,s)|\le|\Gamma^n(y,s)|+\epsilon^{1/8}\le C. \tag{3.48}$$

By (3.48), using Lemma 3.1 we have

$$|\epsilon^m u_y^n(y,s)|=|\epsilon^m(\rho^{n-1})^{-1}(\rho^{n-1}u_y^n)(y,s)|\le C. \tag{3.49}$$

Using (3.44)-(3.49) we have (3.43). Thus the proof of Lemma 3.5 is completed. □

Lemma 3.6. *Let $p>2$ and $\gamma>1$. For all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k^1]$, and $\epsilon\in(0,S_k^1]$, we have*

$$|\rho_s^n(y,s)|+\int_0^1|\rho_y^n(y,s)|^2\mathrm{d}y\le C_{12}, \tag{3.50}$$

where C_{12} is a positive constant depending only on A,p,μ,γ and M_0.

Proof. By (3.5), using Lemmas 3.1 and 3.5, for all $n=1,\cdots,k$, and all $(y,s)\in(0,1)\times(0,S_k^1)$, and $\epsilon\in(0,S_k^1)$, we have

$$|\rho_s^n(y,s)|=|-\rho^n\Gamma^n|\le C. \tag{3.51}$$

In addition, by (3.5), for all $n=1,\cdots,k$, and all $s\in(0,S_k^1)$, and $\epsilon\in(0,S_k^1)$, we also have

$$\int_0^s\int_0^1\rho_y^n(y,\tau)\{\rho_{\tau y}^n(y,\tau)+(\rho^n\Gamma^n)_y(y,\tau)\}\mathrm{d}y\mathrm{d}\tau=0. \tag{3.52}$$

By [A3] and Lemma 2.2, we have

$$\int_0^s\int_0^1\rho_y^n\rho_{ys}^n\mathrm{d}y\mathrm{d}\tau\ge\frac{1}{2}\int_0^1|\rho_y^n(y,s)|^2\mathrm{d}y-C. \tag{3.53}$$

Using Schwarz's inequality and applying Lemmas 2.5, 3.1 and 3.5, by (3.9), we compute

$$\begin{aligned}
&\left|\int_0^s\int_0^1\rho_y^n\cdot(\rho^n\Gamma^n)_y\mathrm{d}y\mathrm{d}\tau\right|\\
=&\left|\int_0^s\int_0^1\rho_y^n\cdot(\rho_y^n\Gamma^n+\rho^n\Gamma_y^n)\mathrm{d}y\mathrm{d}\tau\right|\le C\int_0^s\int_0^1\{(\rho_y^n)^2+(\Gamma_y^n)^2\}\mathrm{d}y\mathrm{d}\tau\\
\le&C\int_0^s\int_0^1\{(\rho_y^n)^2+\{(H'(\Gamma^n))^{-1}(H(\Gamma^n))_y\}^2\}\mathrm{d}y\mathrm{d}\tau\\
\le&C\int_0^s\int_0^1\{(\rho_y^n)^2+\{(H(\Gamma^n))_y\}^2\}\mathrm{d}y\mathrm{d}\tau\le C\int_0^s\int_0^1\left\{(\rho_y^n)^2+(F_y^n)^2+(R_y^n)^2\right\}\mathrm{d}y\mathrm{d}\tau\\
\le&C\int_0^s\int_0^1\left\{(\rho_y^n)^2+(F_y^n)^2+(A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1})G_\epsilon'(\rho^{n-1})\rho_y^{n-1})^2\right\}\mathrm{d}y\mathrm{d}\tau\\
\le&C\int_0^s\int_0^1(\rho_y^n)^2\mathrm{d}y\mathrm{d}\tau+C\int_0^s\int_0^1(\rho_y^{n-1})^2\mathrm{d}y\mathrm{d}\tau+C.
\end{aligned}$$

Using the above inequality, by (3.52)-(3.53), we conclude that

$$\int_0^1 |\rho_y^n(y,s)|^2 \mathrm{d}y \le C_{13}\int_0^s\int_0^1 |\rho_y^n|^2 \mathrm{d}y\mathrm{d}\tau + C_{13}\int_0^s\int_0^1 |\rho_y^{n-1}|^2\mathrm{d}y\mathrm{d}\tau + C_{13} \tag{3.54}$$

for all $s\in(0,S_k^1)$, where C_{13} is a positive constant depending only on A,p,μ,γ and M_0. Applying Gronwall's inequality, by (3.54), we have

$$\begin{aligned}\int_0^s\int_0^1 |\rho_y^n|^2\mathrm{d}y\mathrm{d}\tau \le& \int_0^s e^{C_{13}(s-\tau)}\left\{C_{13}\int_0^\tau\int_0^1 |\rho_y^{n-1}|^2(y,\lambda)\mathrm{d}y\mathrm{d}\lambda + C_{13}\right\}\mathrm{d}\tau\\ \le& C\int_0^s\int_0^1 |\rho_y^{n-1}|^2(y,\lambda)\mathrm{d}y\mathrm{d}\lambda + C.\end{aligned}$$

By the above inequality and (3.54), we conclude that

$$\int_0^1 |\rho_y^n(y,s)|^2\mathrm{d}y \le C\int_0^s\int_0^1 |\rho_y^{n-1}|^2\mathrm{d}y\mathrm{d}\tau + C \tag{3.55}$$

for all $n=1,2,\cdots$, and all $s\in(0,S_k^1)$, where C is a positive constant depending only on A,p,μ,γ and M_0. Applying Lemma 2.4, by (3.51) and (3.55), we get (3.50). Thus the proof of Lemma 3.6 is completed. □

Lemma 3.7. *Let $p>2$ and $\gamma>1$. For all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times(0,S_k^1]$, and $\epsilon\in(0,S_k^1]$, we have*

$$|u^n(y,s)| + (\rho^{n-1}(y)+\epsilon^m)|u_y^n(y,s)| + \int_0^1 |u_s^n(y,s)|^2\mathrm{d}y + \int_0^1 |u_y^n(y,s)|\mathrm{d}y \le C_{14}, \tag{3.56}$$

where C_{14} is a positive constant depending only on A,p,γ and M_0.

Proof. By (3.6) and [A3], using Lemmas 2.3 and 3.5, we compute

$$\begin{aligned}|u^n(y,s)| =& \left|\int_0^1 [u^n(y,s)-u^n(z,s)]\mathrm{d}z + \int_0^1 [u^n(z,s)-u^n(z,0)]\mathrm{d}z + \int_0^1 u^n(z,0)\mathrm{d}z\right|\\ \le& \left|\int_0^1\left(\int_z^y u_r^n(r,s)\mathrm{d}r\right)\mathrm{d}z\right| + \left|\int_0^1\left(\int_0^s u_\tau^n(z,\tau)\mathrm{d}\tau\right)\mathrm{d}z\right| + \left|\int_0^1 u_0^\epsilon(z)\mathrm{d}z\right|\\ \le& \int_0^1\left(\int_0^1 |u_r^n(r,s)|\mathrm{d}r\right)\mathrm{d}z + \left|\int_0^1\left(\int_0^s F_z^n(z,\tau)\mathrm{d}\tau\right)\mathrm{d}z\right| + \int_0^1 |u_0^\epsilon(z)|\mathrm{d}z\\ \le& \int_0^1 C_{14}(\rho^{n-1}(r)+\epsilon^m)^{-1}\mathrm{d}r + \left|\int_0^s (F^n(1,\tau)-F^n(0,\tau))\mathrm{d}\tau\right| + \int_0^1 |u_0^\epsilon(z)|\mathrm{d}z\\ \le& C\int_0^1 \rho_0^{-1}(r)\mathrm{d}r + \int_0^1 |u_0^\epsilon(z)|\mathrm{d}z \le C,\end{aligned}$$

which implies

$$|u^n(y,s)| \le C. \tag{3.57}$$

By Lemmas 3.1 and 3.5, we get

$$(\rho_0(y)+\epsilon^m)|u_y^n(y,s)|\le C. \tag{3.58}$$

By (3.6) and Lemma 3.4, we have

$$\int_0^1 |u_s^n(y,s)|^2 \mathrm{d}y=\int_0^1 |F_y^n(y,s)|^2 \mathrm{d}y\le C_{10}. \tag{3.59}$$

By (3.58) and [A3], we have

$$\int_0^1 |u_y^n(y,s)|\mathrm{d}y\le C\int_0^1 (\rho_0(y)+\epsilon^m)^{-1}\mathrm{d}y\le C. \tag{3.60}$$

Combining (3.57)-(3.60) we get (3.56). Thus the proof of Lemma 3.7 is completed. □

Lemma 3.8. *Let $p>2$ and $\gamma>1$. For all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k^1]$, and $\epsilon\in(0,S_k^1]$, we have*

$$||\rho^n(\cdot,s)-\rho_0(\cdot)||_{L^2(0,1)}+||u^n(\cdot,s)-u_0(\cdot)||_{L^2(0,1)}\le C_{15}(s^{1/2}+\epsilon^m), \tag{3.61}$$

where C_{15} is a positive constant depending only on A,p,μ,γ and M_0.

Proof. From Lemmas 2.2-2.3 and Lemmas 3.6-3.7, we compute

$$\begin{aligned}
&\int_0^1 (|\rho^n(y,s)-\rho_0(y)|^2+|u^n(y,s)-u_0(y)|^2)\mathrm{d}y\\
\le&2\int_0^1 (|\rho^n(y,s)-\rho_0^\epsilon(y)|^2+|u^n(y,s)-u_0^\epsilon(y)|^2)\mathrm{d}y\\
&\quad+2\int_0^1 (|\rho_0^\epsilon(y)-\rho_0(y)|^2+|u_0^\epsilon(y)-u_0(y)|^2)\mathrm{d}y\\
\le&2\int_0^1 \left(\left|\int_0^s \rho_\tau^n(y,\tau)\mathrm{d}\tau\right|^2+\left|\int_0^s u_\tau^n(y,\tau)\mathrm{d}\tau\right|^2\right)\mathrm{d}y+2\int_0^1 (|C_2\epsilon^m|^2+|C_3\epsilon^2|^2)\mathrm{d}y\\
\le&2\left(\int_0^s\int_0^1 ((\rho_\tau^n)^2+(u_\tau^n)^2)\mathrm{d}y\mathrm{d}\tau\right)\cdot\left(\int_0^s\int_0^1 \mathrm{d}y\mathrm{d}\tau\right)+C\epsilon^{2m}\\
\le&Cs+C\epsilon^{2m},
\end{aligned}$$

which implies (3.61). Thus the proof of Lemma 3.8 is completed. □

Lemma 3.9. *Let $p>2$ and $\gamma>1$. For all $n=1,\cdots,k$, and all $(y,s)\in[0,1]\times[0,S_k^1]$, and $\epsilon\in(0,S_k^1]$, we have*

$$|(\rho^{n-1}u_y^n)(y,s)|\le C_{16}\min\left\{y^{1/2}+\epsilon^{1/9},(1-y)^{1/2}+\epsilon^{1/9}\right\}, \tag{3.62}$$

where C_{16} is a positive constant depending only on A,p,μ,γ and M_0.

Proof. Applying Lemmas 2.5 and 3.5, we compute

$$\begin{aligned}|H(\Gamma^n)-H(\rho^{n-1}u_y^n)|=&\left|(\Gamma^n-\rho^{n-1}u_y^n)\int_0^1 H'(\theta\Gamma^n+(1-\theta)\rho^{n-1}u_y^n)\mathrm{d}\theta\right|\\ \geq&\nu_1\left|\Gamma^n-\rho^{n-1}u_y^n\right|.\end{aligned}$$

By the above inequality and Lemma 3.3, we get

$$|\Gamma^n-\rho^{n-1}u_y^n|\leq C\epsilon^{1/16}. \tag{3.63}$$

On the other hand, by (3.9)-(3.10), using Lemma 3.4 and Lemma 3.6 we have

$$\begin{aligned}|\Gamma^n(y,s)|=&|\Gamma^n(y,s)-\Gamma^n(0,s)|+|\Gamma^n(0,s)|=\left|\int_0^y \Gamma_z^n(z,s)\mathrm{d}z\right|\\ \leq&\left(\int_0^y|\Gamma_z^n(z,s)|^2\mathrm{d}z\right)^{1/2}\left(\int_0^y\mathrm{d}z\right)^{1/2}\leq y^{1/2}\left(\int_0^1|\Gamma_y^n(y,s)|^2\mathrm{d}y\right)^{1/2}\\ \leq&y^{1/2}\left(\int_0^1|F_y^n(y,s)+R_y^n(y,s)|^2\mathrm{d}y\right)^{1/2}\\ \leq&y^{1/2}\left(\int_0^1|F_y^n(y,s)+A\gamma G_\epsilon^{\gamma-1}(\rho^{n-1})G_\epsilon'(\rho^{n-1})\rho_y^{n-1}|^2\mathrm{d}y\right)^{1/2}\leq Cy^{1/2},\end{aligned}$$

which implies

$$|\Gamma^n(y,s)|\leq Cy^{1/2}. \tag{3.64}$$

By (3.63)-(3.64), we have

$$|\rho^{n-1}(y,s)u_y^n(y,s)|\leq C(y^{1/2}+\epsilon^{1/16}). \tag{3.65}$$

Similar to (3.65), we also have

$$|(\rho^{n-1}u_y^n)(y,s)|\leq C\Big((1-y)^{1/2}+\epsilon^{1/16}\Big). \tag{3.66}$$

Combining (3.65)-(3.66) and applying Lemma 3.1, we get (3.62).Thus the proof of Lemma 3.9 is completed. □

4 The proof of Theorem 1.1

In order to prove Theorem 1.1 we need the following lemmas.

Lemma 4.1. *Let $p>2$ and $\gamma>1$, and denote*

$$S_0=\min\left\{\epsilon_1,\frac{1}{(1+C_7+C_{11})^\alpha}\right\}, \tag{4.1}$$

where ϵ_1 is defined by Lemma 3.2, *α is defined by* (3.15), *C_7 is defined by Lemma* 3.1, *C_{11} is defined by Lemma* 3.5. *Then, for any positive integer k, we have*

$$S_k \geq S_0. \tag{4.2}$$

Proof. For any given positive integer k, by (3.13)-(3.15), we only have two cases: Case I: $S_k \geq \epsilon_1$, Case II: $S_k \in (0,\epsilon_1)$.

We now prove (4.2) in the Cases I and Case II, respectively.

Case I: $S_k \geq \epsilon_1$.

In this case, we have (4.2), and then Lemma 4.1 in the Case I is proved.

Case II: $S_k \in (0,\epsilon_1)$.

In this case, by (3.14) and (3.18), we have

$$S_k \Phi_k^\alpha(S_k) = 1, \qquad S_k = S_k^1 \in (0,\epsilon_1).$$

Applying Lemmas 3.1 and 3.5, by the above equation, we get

$$S_k = \frac{1}{\Phi_k^\alpha(S_k)} \geq \frac{1}{(1+C_7+C_{11})^\alpha},$$

which implies (4.2). Therefore, Lemma 4.1 in the Case II is also proved. Combining Case I with Case II, we have (4.2) and then the proof of Lemma 4.1 is completed. □

Lemma 4.2. *Let $p>2$ and $\gamma>1$. Then, for S_0 defined by Lemma* 4.1, *there exist*

$$u^\epsilon \in L^\infty(0,S_0;L^2(0,1)) \cap L^2(0,S_0;H^1(0,1))$$

and $\rho^\epsilon \in L^\infty(0,S_0;L^2(0,1))$ such that

$$u^n \to u^\epsilon \tag{4.3}$$

strongly in $L^\infty(0,S_0;L^2(0,1)) \cap L^2(0,S_0;H^1(0,1))$ as $n \to \infty$, and

$$\rho^n \to \rho^\epsilon \tag{4.4}$$

strongly in $L^\infty(0,S_0;L^2(0,1))$ as $n \to \infty$. In addition, we have

$$\rho_s^n \rightharpoonup \rho_s^\epsilon, \qquad \rho_y^n \rightharpoonup \rho_y^\epsilon, \qquad u_s^n \rightharpoonup u_s^\epsilon, \qquad u_y^n \rightharpoonup u_y^\epsilon, \tag{4.5}$$

weakly in $L^2(Q_{S_0})$ as $n \to \infty$. In particular, for almost all $(y,s) \in Q_{S_0}$, we also have

$$\mu_1(\rho_0(y)+\epsilon^m) \leq \rho^\epsilon(y,s) \leq \mu_2(\rho_0(y)+\epsilon^m), \tag{4.6}$$

$$\rho^\epsilon(y,s) + |\rho_s^\epsilon(y,s)| + \int_0^1 |\rho_y^\epsilon(y,s)|^2 \mathrm{d}y \leq C_7 + C_{12}, \tag{4.7}$$

$$|u^\epsilon(y,s)| + (\rho_0^\epsilon(y)+\epsilon^m)|u_y^\epsilon(y,s)| + \int_0^1 |u_s^\epsilon(y,s)|^2 \mathrm{d}y + \int_0^1 |u_y^\epsilon(y,s)| \mathrm{d}y \leq C_{14}, \tag{4.8}$$

where μ_1,μ_2,C_7 are defined by Lemma 3.1, *C_{12} and C_{14} are defined by Lemma* 3.6 *and Lemma* 3.7, *respectively.*

Proof. Denote

$$\bar{\rho}^{n+1}=\rho^{n+1}-\rho^{n}, \qquad \bar{u}^{n+1}=u^{n+1}-u^{n}. \tag{4.9}$$

By (3.5)-(3.6), we have

$$\bar{\rho}_s^{n+1}+(\rho^{n+1}\Gamma^{n+1}-\rho^n\Gamma^n)=0, \qquad \bar{u}_s^{n+1}-(F^{n+1}-F^n)_y=0,$$

which implies

$$\int_0^1 \Big\{[\bar{\rho}_s^{n+1}+(\rho^{n+1}\Gamma^{n+1}-\rho^n\Gamma^n)]\bar{\rho}^{n+1}+[\bar{u}_s^{n+1}-(F^{n+1}-F^n)_y]\bar{u}^{n+1}\Big\}(y,s)\mathrm{d}y=0.$$

By the above equation, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_0^1(|\bar{\rho}^{n+1}|^2+|\bar{u}^{n+1}|^2)(y,s)\mathrm{d}y\\ &\quad+\int_0^1\Big\{(F^{n+1}-F^n)\bar{u}_y^{n+1}+(\rho^{n+1}\Gamma^{n+1}-\rho^n\Gamma^n)\bar{\rho}^{n+1}\Big\}(y,s)\mathrm{d}y=0.\end{aligned} \tag{4.10}$$

Applying Young's inequality and using Lemmas 2.1, 2.5 and 3.5, by (3.9)-(3.11), we compute

$$\begin{aligned}&(F^{n+1}-F^n)\bar{u}_y^{n+1}+(\rho^{n+1}\Gamma^{n+1}-\rho^n\Gamma^n)\bar{\rho}^{n+1}\\
=&\Big(H(\Gamma^{n+1})-H(\Gamma^n)\Big)\bar{u}_y^{n+1}-(R^{n+1}-R^n)\bar{u}_y^{n+1}+(\rho^{n+1}-\rho^n)\Gamma^{n+1}\bar{u}_y^{n+1}+\rho^n(\Gamma^{n+1}-\Gamma^n)\bar{\rho}^{n+1}\\
=&\bar{u}_y^{n+1}(\Gamma^{n+1}-\Gamma^n)\int_0^1 H'(\theta\Gamma^{n+1}+(1-\theta)\Gamma^n)\mathrm{d}\theta\\
&\quad-\Big(AG^\gamma(\rho^n)-AG^\gamma(\rho^{n-1})\Big)\bar{u}_y^{n+1}+(\rho^{n+1}-\rho^n)\Gamma^{n+1}\bar{u}_y^{n+1}+\rho^n(\Gamma^{n+1}-\Gamma^n)\bar{\rho}^{n+1}\\
=&\bar{u}_y^{n+1}(G_\epsilon(\rho^n)u_y^{n+1}-G_\epsilon(\rho^{n-1})u_y^n)\int_0^1 H'(\theta\Gamma^{n+1}+(1-\theta)\Gamma^n)\mathrm{d}\theta\\
&\quad-\int_0^1 A\gamma G^{\gamma-1}\Big(\theta\rho^n+(1-\theta)\rho^{n-1}\Big)G'\Big(\theta\rho^n+(1-\theta)\rho^{n-1}\Big)\mathrm{d}\theta(\rho^n-\rho^{n-1})\bar{u}_y^{n+1}\\
&\quad+(\rho^{n+1}-\rho^n)\Gamma^{n+1}\bar{u}_y^{n+1}+\rho^n\Big(G_\epsilon(\rho^n)u_y^{n+1}-G_\epsilon(\rho^{n-1})u_y^n\Big)\bar{\rho}^{n+1}\\
=&\bar{u}_y^{n+1}G_\epsilon(\rho^n)(u_y^{n+1}-u_y^n)\int_0^1 H'\Big(\theta\Gamma^{n+1}+(1-\theta)\Gamma^n\Big)\mathrm{d}\theta\\
&\quad+\bar{u}_y^{n+1}\Big(G_\epsilon(\rho^n)-G_\epsilon(\rho^{n-1})\Big)u_y^n\int_0^1 H'\Big(\theta\Gamma^{n+1}+(1-\theta)\Gamma^n\Big)\mathrm{d}\theta\\
&\quad-\int_0^1 A\gamma G^{\gamma-1}\Big(\theta\rho^n+(1-\theta)\rho^{n-1}\Big)G'\Big(\theta\rho^n+(1-\theta)\rho^{n-1}\Big)\mathrm{d}\theta(\rho^n-\rho^{n-1})\bar{u}_y^{n+1}\\
&\quad+(\rho^{n+1}-\rho^n)\Gamma^{n+1}\bar{u}_y^{n+1}+\rho^n\Big(G_\epsilon(\rho^n)u_y^{n+1}-G_\epsilon(\rho^{n-1})u_y^n\Big)\bar{\rho}^{n+1}\\
\geq&\nu_1\epsilon|\bar{u}_y^{n+1}|^2-C|\bar{\rho}^n\bar{u}_y^{n+1}|-C|\bar{\rho}^{n+1}\bar{u}_y^{n+1}|\end{aligned}$$

$$
\begin{aligned}
&+\rho^n G_\epsilon(\rho^n)(u_y^{n+1}-u_y^n)\bar{\rho}^{n+1}+\rho^n\Big(G_\epsilon(\rho^n)-G_\epsilon(\rho^{n-1})\Big)u_y^n\bar{\rho}^{n+1}\\
\ge&\nu_1\epsilon|\bar{u}_y^{n+1}|^2-C|\bar{\rho}^n\bar{u}_y^{n+1}|-C|\bar{\rho}^{n+1}\bar{u}_y^{n+1}|-C|\bar{u}_y^{n+1}\bar{\rho}^{n+1}|-C|\bar{\rho}^n\bar{\rho}^{n+1}|\\
\ge&\frac{\nu_1\epsilon}{2}|\bar{u}_y^{n+1}|^2-C(|\bar{\rho}^n|^2+|\bar{\rho}^{n+1}|^2),
\end{aligned}
$$

which implies

$$
(F^{n+1}-F^n)\bar{u}_y^{n+1}+(\rho^{n+1}\Gamma^{n+1}-\rho^n\Gamma^n)\bar{\rho}^{n+1}\ge\frac{\nu_1\epsilon}{2}|\bar{u}_y^{n+1}|^2-C(|\bar{\rho}^n|^2+|\bar{\rho}^{n+1}|^2). \tag{4.11}
$$

Combining (4.10)-(4.11) with (3.7) we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^1(|\bar{\rho}^{n+1}|^2+|\bar{u}^{n+1}|^2)(y,s)\mathrm{d}y+\frac{\nu_1\epsilon}{2}\int_0^s\int_0^1|\bar{u}_y^{n+1}(y,\tau)|^2\mathrm{d}y\mathrm{d}\tau\\
\le&C\int_0^s\int_0^1(|\bar{\rho}^n|^2+|\bar{\rho}^{n+1}|^2)(y,\tau)\mathrm{d}y\mathrm{d}\tau.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\int_0^1(|\bar{\rho}^{n+1}|^2+|\bar{u}^{n+1}|^2)(y,s)\mathrm{d}y+\int_0^s\int_0^1|\bar{u}_y^{n+1}(y,\tau)|^2\mathrm{d}y\mathrm{d}\tau\\
\le&C_{17}\int_0^s\int_0^1(|\bar{\rho}^n|^2+|\bar{\rho}^{n+1}|^2)(y,\tau)\mathrm{d}y\mathrm{d}\tau
\end{aligned} \tag{4.12}
$$

for all $s\in[0,S_0]$, where C_{17} is a positive constant independent of n. Applying Gronwall's inequality, by (4.12), we obtain

$$
\int_0^s\int_0^1|\bar{\rho}^{n+1}|^2\mathrm{d}y\mathrm{d}\tau\le\int_0^s\left\{e^{C_{17}(s-\lambda)}\left[C_{17}\int_0^\lambda\int_0^1|\bar{\rho}^n(y,\tau)|^2\mathrm{d}y\mathrm{d}\tau\right]\right\}\mathrm{d}\lambda. \tag{4.13}
$$

By (4.12)-(4.13) we have

$$
\begin{aligned}
&\int_0^1(|\bar{\rho}^{n+1}|^2+|\bar{u}^{n+1}|^2)(y,s)\mathrm{d}y+\int_0^s\int_0^1|\bar{u}_y^{n+1}(y,\tau)|^2\mathrm{d}y\mathrm{d}\tau\\
\le&C_{18}\int_0^s\int_0^1|\bar{\rho}^n(y,\tau)|^2\mathrm{d}y\mathrm{d}\tau
\end{aligned} \tag{4.14}
$$

for all $s\in(0,S_0)$, where C_{18} is a positive constant independent of n. Applying Lemma 2.5, by (4.14), we get

$$
\int_0^1|\bar{\rho}^{n+1}(y,s)|^2\mathrm{d}y\le\frac{C_{19}(C_{18}s)^n}{n!}, \tag{4.15}
$$

where

$$
C_{19}=\sup_{s\in(0,1)}\int_0^1|\bar{\rho}^1(y,s)|^2\mathrm{d}y. \tag{4.16}
$$

Applying Lemma 3.1, by (4.9) and (4.15)-(4.16), we conclude that

$$\int_0^1 |\bar{\rho}^{n+1}(y,s)|^2 \mathrm{d}y \le \frac{(Cs)^n}{n!} \tag{4.17}$$

for all positive integer n, where C is a positive constant independent of n. Combining (4.14) with (4.17) we get

$$\int_0^1 (|\bar{\rho}^{n+1}|^2 + |\bar{u}^{n+1}|^2)(y,s)\mathrm{d}y + \int_0^s \int_0^1 |\bar{u}_y^{n+1}(y,\tau)|^2 \mathrm{d}y\mathrm{d}\tau \le \frac{(Cs)^n}{n!}$$

for all $s \in [0,S_0]$, where C is a positive constant independent of n. This implies that, the sequence $\{\rho^n\}_{n=1}^{\infty}$ is a Cauchy's sequence in $L^\infty(0,S_0;L^2(0,1))$. Therefore, we have (4.4). Similarly, we also have (4.3). In addition, applying Lemma 3.4, Lemmas 3.7-3.8, by (4.3)-(4.4), we have (4.5)-(4.8). Thus the proof of Lemma 4.2 is completed. ☐

Lemma 4.3. *Let $p>2$ and $\gamma>1$. Then, for S_0 defined by Lemma 4.1, we have*

$$R^n \to R^\epsilon, \qquad F^n \to F^\epsilon, \qquad \Gamma^n \to \Gamma^\epsilon, \qquad H(\Gamma^n) \to H(\Gamma^\epsilon) \tag{4.18}$$

strongly in $L^2(Q_{S_0})$ as $n\to\infty$, where $R^\epsilon = AG_\epsilon^\gamma(\rho^\epsilon)$, $F^\epsilon = \Psi^\epsilon - R^\epsilon$ and $\Gamma^\epsilon = G_\epsilon(\rho^\epsilon)u_y^\epsilon$. In addition, we also have

$$R_y^n \rightharpoonup R_y^\epsilon, \qquad F_y^n \rightharpoonup F_y^\epsilon, \qquad (H(\Gamma^n))_y \rightharpoonup (H(\Gamma^\epsilon))_y \tag{4.19}$$

weakly in $L^2(Q_{S_0})$ as $n\to\infty$. In particular, we also have

$$|F^\epsilon(y,s)| + |\Gamma^\epsilon(y,s)| + |H(\Gamma^\epsilon(y,s))| + |R^\epsilon(y,s)| + |[(\rho^\epsilon(y)+\epsilon^m)u_y^\epsilon(y,s)| \le C_{20}, \tag{4.20}$$

$$||F_y^\epsilon||_{L^\infty(0,S_0;L^2(0,1))} + ||(H(\Gamma^\epsilon))_y||_{L^\infty(0,S_0;L^2(0,1))} + ||R_y^\epsilon||_{L^\infty(0,S_0;L^2(0,1))} \le C_{20}, \tag{4.21}$$

for almost all $(y,s)\in Q_{S_0}$, where C_{20} is a positive constant depending only on A,p,μ,γ and M_0.

Proof. By Lemma 2.4, we compute

$$\begin{aligned}
&\iint_{Q_{S_0}} |R^n - R^\epsilon|^2 \mathrm{d}y\mathrm{d}s \\
=&\iint_{Q_{S_0}} \left| (\rho^{n-1}-\rho^\epsilon) \int_0^1 A\gamma G_\epsilon^{\gamma-1}\Big(\theta\rho^\epsilon + (1-\theta)\rho^{n-1}\Big) G_\epsilon'\Big(\theta\rho^\epsilon+(1-\theta)\rho^{n-1}\Big)\mathrm{d}\theta \right|^2 \mathrm{d}y\mathrm{d}s \\
\le&\iint_{Q_{S_0}} \left| (\rho^{n-1}-\rho^\epsilon) \int_0^1 A\gamma(\epsilon^{-1})^{\gamma-1}\mathrm{d}\theta\right|^2 \mathrm{d}y\mathrm{d}s \le C\iint_{Q_{S_0}} \left|\rho^{n-1}-\rho^\epsilon\right|^2 \mathrm{d}y\mathrm{d}s.
\end{aligned}$$

By the above inequality, applying Lemma 4.2, we get $\lim_{n\to\infty}||R^n - R^\epsilon||_{L^2(Q_{S_0})} = 0$. Similarly, we also have

$$\lim_{n\to\infty}\Big\{||F^n - F^\epsilon||_{L^2(Q_{S_0})} + ||\Gamma^n - \Gamma^\epsilon||_{L^2(Q_{S_0})} + ||H(\Gamma^n) - H(\Gamma^\epsilon)||_{L^2(Q_{S_0})}\Big\} = 0.$$

Therefore we have (4.18). In addition, from the lower half continuity of the norm, using Lemma 3.5, by (4.18), we get (4.20). Using Lemma 3.4 and Lemma 3.6-3.7, by (3.9), we have

$$||F_y^n||_{L^\infty(0,S_0;(0,1))}+||(H(\Gamma^n))_y||_{L^\infty(0,S_0;(0,1))}+||R_y^n||_{L^\infty(0,S_0;(0,1))}\leq C,$$

where C is a positive constant depending only on A,p,μ,γ and M_0. In addition, by the above inequality and (4.19), from the weak lower half continuity of the norm, we get (4.21). Thus the proof of Lemma 4.3 is completed. □

Lemma 4.4. *Let $p>2$ and $\gamma>1$. Then, for S_0 defined by Lemma 4.1, there exist $(\rho,u)\in L^\infty(Q_{S_0})$ and a subsequence $\{(\rho^{\epsilon_j},u^{\epsilon_j})\}_{j=1}^\infty$ of $\{(\rho^\epsilon,u^\epsilon)\}_{\epsilon\in(0,S_0)}$ such that*

$$(\rho^\epsilon,u^\epsilon)\to(\rho,u) \tag{4.22}$$

strongly in $L^2(Q_{S_0})$ as $\epsilon=\epsilon_j\to0^+$. In addition, we also have

$$\rho_s^\epsilon\rightharpoonup\rho_s,\quad \rho_y^\epsilon\rightharpoonup\rho_y,\quad u_s^\epsilon\rightharpoonup u_s, \tag{4.23}$$

weakly in $L^2(Q_{S_0})$ as $\epsilon=\epsilon_j\to0^+$, and

$$u_y^\epsilon\rightharpoonup u_y, \tag{4.24}$$

weakly in $L^2(0,S_0;L^2_{loc}(0,1))$ as $\epsilon=\epsilon_j\to0^+$. In particular, for almost all $(y,s)\in Q_{S_0}$, we also have

$$\mu_1\rho_0(y)\leq\rho(y,s)\leq\mu_2\rho_0(y), \tag{4.25}$$

$$\rho(y,s)+|\rho_s(y,s)|+\int_0^1|\rho_y(y,s)|^2\mathrm{d}y\leq C_7+C_{12}, \tag{4.26}$$

$$|u(y,s)|+\rho_0(y)|u_y(y,s)|+\int_0^1|u_s(y,s)|^2\mathrm{d}y+\int_0^1|u_y(y,s)|\mathrm{d}y\leq C_{14}, \tag{4.27}$$

where μ_1,μ_2 and C_7 are defined by Lemma 3.1; C_{12} and C_{14} are defined Lemma 3.6 and Lemma 3.7, respectively.

Proof. From the lower half continuity of the norm, applying Sobolev's imbedding theorem (see, e.g., [8]), by Lemma 4.2, we have (4.22)-(4.27). Thus the proof of Lemma 4.4 is completed. □

Lemma 4.5. *Let $p>2$ and $\gamma>1$. Then, for S_0 defined by Lemma 4.1, we have*

$$\rho^\epsilon u_y^\epsilon\rightharpoonup\rho u_y \tag{4.28}$$

weakly in $L^2(Q_{S_0})$ as $\epsilon=\epsilon_j\to0^+$. In particular, we also have

$$|\rho(y,s)u_y(y,s)|\leq C_{11} \tag{4.29}$$

for almost all $(y,s)\in Q_{S_0}$, where C_{11} is defined by Lemma 3.5.

Proof. We define

$$\omega(r)=1+\sup_{r\le y\le 1-r}\frac{1}{\rho_0(y)} \tag{4.30}$$

for all $r\in(0,1)$. By [A1] and (4.28), for any $r\in(0,1)$, we have

$$1<\omega(r)<+\infty. \tag{4.31}$$

For any given $\phi\in L^2(Q_{S_0})$ and for all $\nu\in(0,1)$, by (4.30)-(4.31), applying Lemmas 4.3-4.4, we compute

$$\begin{aligned}
&\left|\iint_{Q_{S_0}}\phi(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right|\\
=&\left|\int_0^{S_0}\int_\nu^{1-\nu}\phi(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s+\int_0^{S_0}\int_0^{\nu}\phi(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right.\\
&\quad\left.+\int_0^{S_0}\int_{1-\nu}^{1}\phi(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right|\\
\le&\left|\int_0^{S_0}\int_\nu^{1-\nu}\phi\rho(u_y^\epsilon-u_y)\mathrm{d}y\mathrm{d}s\right|+\left|\int_0^{S_0}\int_\nu^{1-\nu}\phi(\rho^\epsilon-\rho)u_y^\epsilon\mathrm{d}y\mathrm{d}s\right|\\
&\quad+C||\rho^\epsilon u_y^\epsilon-\rho u_y||_{L^\infty(Q_{S_0})}||\phi||_{L^2(Q_{S_0})}\left[\left(\int_0^{S_0}\int_0^{\nu}\mathrm{d}y\mathrm{d}s\right)^{1/2}+\left(\int_0^{S_0}\int_{1-\nu}^{1}\mathrm{d}y\mathrm{d}s\right)^{1/2}\right]\\
\le&\left|\int_0^{S_0}\int_\nu^{1-\nu}\phi\rho(u_y^\epsilon-u_y)\mathrm{d}y\mathrm{d}s\right|+C\sup_{\nu\le y\le 1-\nu,0\le s\le S_0}(\rho^\epsilon(y,s)+\epsilon^m)^{-1}\cdot||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\sqrt{\nu}^{1/2}\\
\le&\left|\int_0^{S_0}\int_\nu^{1-\nu}\phi\rho(u_y^\epsilon-u_y)\mathrm{d}y\mathrm{d}s\right|+C\omega(\nu)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\sqrt{\nu}^{1/2},
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left|\iint_{Q_{S_0}}\phi(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right|\\
\le&\left|\int_0^{S_0}\int_\nu^{1-\nu}\phi\rho(u_y^\epsilon-u_y)\mathrm{d}y\mathrm{d}s\right|+C\omega(\nu)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\nu^{1/2}
\end{aligned}$$

for all $\nu\in(0,1)$, where C is a positive constant independent of ϵ and ν. In the above inequality, letting $\epsilon=\epsilon_j\to0^+$ and $\nu\to0^+$ in turn, using Lemma 4.4, we get

$$\lim_{\epsilon=\epsilon_j\to0}\iint_{Q_{S_0}}\phi(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s=0$$

for all $\phi\in L^2(Q_{S_0})$. This implies (4.28). From the weak lower half continuity of the norm, by (4.20) and (4.28), we get (4.29). Thus the proof of Lemma 4.5 is completed. □

Lemma 4.6. *Let $p>2$ and $\gamma>1$. Then, for S_0 defined by Lemma 4.1, we have*

$$\rho^\epsilon u_y^\epsilon \to \rho u_y \tag{4.32}$$

strongly in $L^2(Q_{S_0})$ as $\epsilon=\epsilon_j\to 0^+$.

Proof. For any given $\delta\in(0,1/4)$, we define a cutoff function $\xi^\delta\in C_0^\infty(-\infty,+\infty)$ such that

$$\begin{cases} \xi^\delta(y)=1 \quad \forall y\in(2\delta,1-2\delta); \qquad \xi^\delta(y)=0 \quad \forall y\notin(\delta,1-\delta); \\ 0\le\xi^\delta(y)\le 1, \quad |\xi_y^\delta(y)|\le C_{21}\delta^{-1}, \quad \forall y\in(-\infty,+\infty), \end{cases} \tag{4.33}$$

where C_{21} is an absolute constant independent of δ. Applying Lemma 2.5, Lemmas 4.2-4.5, by (4.33), for any given $\delta\in(0,1/4)$, we compute

$$\begin{aligned}
&\left|\iint_{Q_{S_0}} \xi^\delta H(\rho^\epsilon u_y^\epsilon)(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right| \\
\le&\left|\iint_{Q_{S_0}} \xi^\delta H(\Gamma^\epsilon)(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right|+\left|-\iint_{Q_{S_0}} \xi^\delta (H(\Gamma^\epsilon)-H(\rho^\epsilon u_y^\epsilon))(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right| \\
\le&\left|\iint_{Q_{S_0}} \xi^\delta H(\Gamma^\epsilon)\rho(u_y^\epsilon-u_y)\mathrm{d}y\mathrm{d}s\right|+\left|\iint_{Q_{S_0}} \xi^\delta H(\Gamma^\epsilon)(\rho^\epsilon-\rho)u_y^\epsilon\mathrm{d}y\mathrm{d}s\right| \\
&\quad +C\left|\iint_{Q_{S_0}} \xi^\delta |H(\Gamma^\epsilon)-H(\rho^\epsilon u_y^\epsilon)|\mathrm{d}y\mathrm{d}s\right| \\
\le& C\left|\iint_{Q_{S_0}} (-\xi^\delta H(\Gamma^\epsilon)\rho)_y(u^\epsilon-u)\mathrm{d}y\mathrm{d}s\right|+C\left|\int_0^{S_0}\int_\delta^{1-\delta} |\rho^\epsilon-\rho|(\rho^\epsilon+\epsilon^m)^{-1}\mathrm{d}y\mathrm{d}s\right| \\
&\quad +C\iint_{Q_{S_0}} \xi^\delta\left|(\Gamma^\epsilon-\rho^\epsilon u_y^\epsilon))\int_0^1 H'(\theta\Gamma^\epsilon+(1-\theta)\rho^\epsilon u_y^\epsilon))\mathrm{d}\theta\right|\mathrm{d}y\mathrm{d}s \\
\le& C\iint_{Q_{S_0}} |\xi_y^\delta H(\Gamma^\epsilon)\rho+\xi^\delta(H(\Gamma^\epsilon))_y\rho+\xi^\delta H(\Gamma^\epsilon)\rho_y||u^\epsilon-u|\mathrm{d}y\mathrm{d}s \\
&\quad +C\omega(\delta)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\iint_{Q_{S_0}} \xi^\delta|\Gamma^\epsilon-\rho^\epsilon u_y^\epsilon|\mathrm{d}y\mathrm{d}s \\
\le& C(1+\delta^{-1})||H(\Gamma^\epsilon)\rho+(H(\Gamma^\epsilon))_y\rho+H(\Gamma^\epsilon)\rho_y||_{L^2(Q_{S_0})}||u^\epsilon-u||_{L^2(Q_{S_0})} \\
&\quad +C\omega(\delta)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\iint_{Q_{S_0}} \xi^\delta|G_\epsilon(\rho^\epsilon)u_y^\epsilon-\rho^\epsilon u_y^\epsilon|\mathrm{d}y\mathrm{d}s \\
\le& C(1+\delta^{-1})||u^\epsilon-u||_{L^2(Q_{S_0})}+C\omega(\delta)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})} \\
&\quad +C\iint_{Q_{S_0}} \xi^\delta\rho_0^{-1}(y)C_1\epsilon(1+|\rho^\epsilon|^4)|\rho^\epsilon u_y^\epsilon|\mathrm{d}y\mathrm{d}s \\
\le& C(1+\delta^{-1})||u^\epsilon-u||_{L^2(Q_{S_0})}+C\omega(\delta)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\epsilon,
\end{aligned}$$

which implies that

$$\begin{aligned}&\iint_{Q_{S_0}}\xi^\delta H(\rho^\epsilon u_y^\epsilon)(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\\ \le& C(1+\delta^{-1})||u^\epsilon-u||_{L^2(Q_{S_0})}+C\omega(\delta)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\epsilon.\end{aligned}\tag{4.34}$$

By (4.34), using Lemma 2.5, we compute

$$\begin{aligned}&\iint_{Q_{S_0}}|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s\\ =&\iint_{Q_{S_0}}\xi^\delta|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+\iint_{Q_{S_0}}(1-\xi^\delta)|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s\\ \le&\iint_{Q_{S_0}}\xi^\delta\left\{\nu^{-1}\int_0^1 H'(\theta(\rho^\epsilon u_y^\epsilon)+(1-\theta)(\rho u_y))\mathrm{d}\theta\right\}|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+C\delta\\ =&\nu^{-1}\iint_{Q_{S_0}}\xi^\delta(H(\rho^\epsilon u_y^\epsilon)-H(\rho u_y))(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s+C\delta\\ \le&C\left|\iint_{Q_{S_0}}\xi^\delta H(\rho^\epsilon u_y^\epsilon)(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right|+C\left|\iint_{Q_{S_0}}\xi^\delta H(\rho u_y)(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right|+C\delta.\end{aligned}$$

Combining the above inequality with (4.34) we get

$$\begin{aligned}&\iint_{Q_{S_0}}|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s\\ \le&C(1+\delta^{-1})||u^\epsilon-u||_{L^2(Q_{S_0})}+C\omega(\delta)||\rho^\epsilon-\rho||_{L^2(Q_{S_0})}+C\epsilon\\ &+C\left|\iint_{Q_{S_0}}\xi^\delta(|\rho u_y|^{p-2}\rho u_y)(\rho^\epsilon u_y^\epsilon-\rho u_y)\mathrm{d}y\mathrm{d}s\right|+C\delta,\end{aligned}$$

where C is a positive constant independent of ϵ and δ. In the above inequality, letting $\epsilon=\epsilon_j\to 0^+$ and $\delta\to 0^+$ in turn, using Lemmas 4.4-4.5, we get

$$\lim_{\epsilon=\epsilon_j\to 0}\iint_{Q_{S_0}}|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s=0,$$

which implies (4.32). Thus the proof of Lemma 4.6 is completed. □

Lemma 4.7. *Let $p>2$ and $\gamma>1$. Then, for S_0 defined by Lemma 4.1, we have*

$$\Gamma^\epsilon\to\rho u_y,\quad H(\Gamma^\epsilon)\to H(\rho u_y),\quad \rho^\epsilon\Gamma^\epsilon\to\rho^2 u_y,\quad R^\epsilon\to A\rho^\gamma\tag{4.35}$$

strongly in $L^2(Q_{S_0})$ as $\epsilon=\epsilon_j\to 0^+$. In addition, we also have

$$(H(\Gamma^\epsilon))_y\to(H(\rho u_y))_y,\quad R_y^\epsilon\rightharpoonup(A\rho^\gamma)_y\tag{4.36}$$

weakly in $L^2(Q_{S_0})$ *as* $\epsilon=\epsilon_j\to 0^+$. *In particular, we also have*

$$||(H(\rho u_y))_y||_{L^\infty(0,S_0;L^2(0,1))}+||(A\rho^\gamma)_y||_{L^\infty(0,S_0;L^2(0,1))}\le C_{20} \tag{4.37}$$

for almost all $(y,s)\in Q_{S_0}$, *where* C_{20} *is defined by Lemma* 4.3.

Proof. Applying Lemma 2.1, Lemmas 4.2-4.3, for ξ^δ defined by (4.33), we compute

$$\begin{aligned}
&\iint_{Q_{S_0}}|\Gamma^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s\\
=&\iint_{Q_{S_0}}\xi^\delta|\Gamma^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+\iint_{Q_{S_0}}(1-\xi^\delta)|\Gamma^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s\\
=&\iint_{Q_{S_0}}\xi^\delta|G_\epsilon(\rho^\epsilon)u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+C\delta\\
\le& C\iint_{Q_{S_0}}\xi^\delta|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+C\iint_{Q_{S_0}}\xi^\delta|(G_\epsilon(\rho^\epsilon)-\rho^\epsilon)u_y^\epsilon|^2\mathrm{d}y\mathrm{d}s+C\delta\\
\le&\iint_{Q_{S_0}}\xi^\delta|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+C\iint_{Q_{S_0}}\xi^\delta(\rho^\epsilon)^{-2}C_1\epsilon(1+|\rho^\epsilon|^4)|\rho^\epsilon u_y^\epsilon|^2\mathrm{d}y\mathrm{d}s+C\delta\\
\le&\iint_{Q_{S_0}}\xi^\delta|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+C\epsilon\iint_{Q_{S_0}}\xi^\delta(y)\rho_0^{-2}(y)\mathrm{d}y\mathrm{d}s+C\delta\\
\le&\iint_{Q_{S_0}}\xi^\delta|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+C\epsilon\omega^2(\delta)+C\delta,
\end{aligned}$$

which implies

$$\iint_{Q_{S_0}}|\Gamma^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s\le\iint_{Q_{S_0}}\xi^\delta|\rho^\epsilon u_y^\epsilon-\rho u_y|^2\mathrm{d}y\mathrm{d}s+C\epsilon\omega^2(\delta)+C\delta, \tag{4.38}$$

where C is a positive constant independent of ϵ and δ. In (4.38), letting $\epsilon=\epsilon_j\to 0^+$ and $\delta\to 0^+$ in turn, by Lemma 4.6, we get

$$\lim_{\epsilon=\epsilon_j\to 0}||\Gamma^\epsilon-\rho u_y||_{L^2(Q_{S_0})}=0. \tag{4.39}$$

Using Lemmas 4.3-4.4, we compute

$$\begin{aligned}
&\iint_{Q_{S_0}}|H(\Gamma^\epsilon)-H(\rho u_y)|^2\mathrm{d}y\mathrm{d}s\\
=&\iint_{Q_{S_0}}\left|(\Gamma^\epsilon-\rho u_y)\int_0^1H'(\theta\rho u_y+(1-\theta)\Gamma^\epsilon)\mathrm{d}\theta\right|^2\mathrm{d}y\mathrm{d}s\\
\le& C\iint_{Q_{S_0}}\left|\Gamma^\epsilon-\rho u_y\right|^2\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

By the above inequality and (4.39), we have

$$\lim_{\epsilon=\epsilon_j\to 0}||H(\Gamma^\epsilon)-H(\rho u_y)||_{L^2(Q_{S_0})}=0. \tag{4.40}$$

Similar to (4.40), we also have

$$\lim_{\epsilon=\epsilon_j\to 0}||\rho^\epsilon\Gamma^\epsilon-\rho^2 u_y||_{L^2(Q_{S_0})}=\lim_{\epsilon=\epsilon_j\to 0}||R^\epsilon-A\rho^\gamma||_{L^2(Q_{S_0})}=0. \tag{4.41}$$

Combining (4.41) with (4.39)-(4.40) we have (4.35). From the weak lower half continuity of the norm, by Lemma 4.3 and (4.35), we have (4.36)-(4.37). Thus the proof of Lemma 4.7 is completed. □

Now, let us prove the Theorem 1.1. In fact, the conclusions (i)-(ii) of Theorem 1.1 can be obtained by using Lemma 4.4. To prove the conclusion (iii) of Theorem 1.1, we choose any $\phi\in L^2(Q_{S_0})$, by (3.6), we have

$$\iint_{Q_{S_0}}\phi(u_s^n-F_y^n)\mathrm{d}y\mathrm{d}s=0.$$

Letting $n\to\infty$ in the above equation, by Lemma 4.2-4.3, we get

$$\iint_{Q_{S_0}}\phi\{u_s^\epsilon-(H(\Gamma^\epsilon))_y+R_y^\epsilon\}\mathrm{d}y\mathrm{d}s=0.$$

In addition, letting $\epsilon=\epsilon_j\to 0^+$ in the above equation, by Lemma 4.4 and Lemma 4.7, we get

$$\iint_{Q_{S_0}}\phi(u_s-(H(\rho u_y))_y+(A\rho^\gamma)_y)\mathrm{d}y\mathrm{d}s=0$$

for all $\phi\in L^2(Q_{S_0})$. This implies

$$u_s-(H(\rho u_y))_y+(A\rho^\gamma)_y=0 \tag{4.42}$$

for almost all $(y,s)\in Q_{S_0}$. Similarly, we also have

$$\rho_s-\rho^2 u_y=0 \tag{4.43}$$

for almost all $(y,s)\in Q_{S_0}$. Combining (4.42)-(4.43) we get the conclusion (iii) of Theorem 1.1. The conclusions (iv)-(v) can be obtained by Lemmas 3.8-3.9 and Lemmas 4.2 and 4.4, and the details are omitted here. Thus the proof of Theorem 1.1 is completed. □

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