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Finite-Time Blow-Up and Local Existence for Chemotaxis System with a General Memory Term

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Abstract. In this paper, we discuss the local existence of weak solutions for a parabolic system modelling chemotaxis with memory term, and we show the finite-time blow-up and chemotactic collapse for this system. The main methods we used are the fixed point theorem and the semigroup theory.

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Key Words: parabolic system; chemotaxis; memory term; fixed point theorem; blow-up.

1 Introduction

In this paper, we consider the following model:

$$\begin{cases} u_t=\nabla(\nabla u-\chi u\nabla v)+\int_0^t f[u(\cdot,\tau)]\mathrm{d}\tau & \text{in } \Omega\times(0,T), \\ v_t=\Delta v-v+u & \text{in } \Omega\times(0,T), \\ u(\cdot,0)=u_0 & \text{in } \Omega\times\{0\}, \\ v(\cdot,0)=v_0 & \text{in } \Omega\times\{0\}, \\ \dfrac{\partial u}{\partial\vec{n}}=\dfrac{\partial v}{\partial\vec{n}}=0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

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where $\Omega\subset\mathbb{R}^N$, a bounded open domain with smooth boundary $\partial\Omega$, $\vec{n}$ is the unit outer normal on $\partial\Omega$ and χ is a nonnegative constant. And f is a continuous linear function and it satisfies the condition: $\|f(x_1)-f(x_2)\|\le L\|x_1-x_2\|$, where L is a positive constant.

Our model is initiated by the PKS model which is a mathematical model of biological phenomena. And this model for chemosensitive movement has been developed by Patlak, Keller and Segel [1].

$$\begin{cases} u_t=\nabla(\nabla u-\chi(v)u\nabla v), \\ \varepsilon v_t=\Delta v+g(u,v), \end{cases} \tag{1.2}$$

where u represents the population density and v denotes the density of the external stimulus, χ is the sensitive coefficient, the time constant ε $(0\le\varepsilon\le 1)$ indicates that the spatial spread of the organisms u and the signal v are on different time scales. The case $\varepsilon=0$ corresponds to a quasi-steady state assumption for the signal distribution.

Since the PKS model is designed to describe the behavior of bacteria and bacteria aggregates, the question arises whether or not this model is able to show aggregation. Plenty of theoretical research uncovered exact conditions for aggregations and for blow up (see, e.g., Childress and Percus [2, 3], Jäger and Luckhaus [4], Nagai [5], Gajewski et al. [6], Senba [7], Rasde and Ziti [8], Herrera and Velasquez [9], Othmer and Stevens [10] or Levine and Sleeman [11]).

Global existence below these thresholds has been proven using a Lyapunov functional in Gajewski, et al. [6], Nagai, et al. [12] and Biler [13]. Besides, a number of theoretical research found exact conditions for aggregations and other properties [14–16]. Free boundary problems for the chemotaxis model are considered [17–20].

Our study of (1.1) is also motivated by the following problem for the heat equation with a general time integral boundary condition [21]:

$$\begin{cases} u_t=\Delta u & x\in\Omega,\ t>0, \\ \dfrac{\partial u}{\partial\vec{n}}=\displaystyle\int_0^t f[u(x,s)]\mathrm{d}s & x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x) & x\in\overline{\Omega}, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega\subset C^{1+\mu}(0<\mu<1)$, $\vec{n}$ is the outward normal, and $u_0(x)$ is a nonnegative function such that

$$\frac{\partial u_0}{\partial\vec{n}}=0 \quad \text{for } x\in\partial\Omega,$$

f is a nondecreasing function with $f\in C^1(0,\infty)$ and $f(0)>0$.

Considering the nonlinear time integral condition governing flux through the boundary, the model (1.1) involves a continuous time delay which is often referred to as a memory condition in the literature. This memory term can perfectly describe the movement of population density or the movement of single particles. Especially, the movement behavior of most species is guided by external signals: insects orient towards light sources, the smell of a sexual partner makes it favorable to choose a certain direction.

Models with memory terms present in the boundary flux have been formulated in many applied sciences. For example, in [22], a linear memory boundary condition is introduced for the study of thermodynamics. It takes into account the hereditary effects on the boundary as those studied in [23, 24]. Similar hereditary boundary conditions have been employed in models of time-dependent electromagnetic fields at dissipative boundaries [25].

From the mathematical point of view, it is significant to study the local existence of weak solution for chemotaxis system with memory terms and the finite-time blow-up for this system. In our previous work, we have done something for this [26].

2 Some basic lemmas

Choose a constant σ which satisfies

$$1<\sigma<2, \tag{2.1}$$

and

$$N<2\sigma<N+2. \tag{2.2}$$

It is easy to check that (2.1) and (2.2) can be simultaneously satisfied in the case of $1\le N\le 3$. We define

$$\begin{aligned} X_u &= C\left([0,t_0],H^\sigma(\Omega)\cap\left\{\frac{\partial u}{\partial\vec{n}}=0 \text{ on } \partial\Omega\right\}\right),\\ X_v &= C\left([0,t_0],H^2(\Omega)\cap\left\{\frac{\partial v}{\partial\vec{n}}=0 \text{ on } \partial\Omega\right\}\right),\\ X &= H^\sigma(\Omega)\cap\left\{\frac{\partial u}{\partial\vec{n}}=0 \text{ on } \partial\Omega\right\}. \end{aligned}$$

Here, $u(x,t)\in C([0,t_0],H^\sigma(\Omega)\cap\{\frac{\partial u}{\partial\vec{n}}=0 \text{ on } \partial\Omega\})$ means that $u(x,t)\in H^\sigma(\Omega)\cap\{\frac{\partial u}{\partial\vec{n}}=0 \text{ on } \partial\Omega\}$ for each $t\in[0,t_0]$ and $\|u(\cdot,t)\|_{H^\sigma}\in C([0,t_0])$. In the Sections 2 and 3, inessential constants will be denoted by the same letter c, even if they may vary from line to line.

Lemma 2.1. *Let $p(z)$ be a holomorphic semigroup on a Banach space Y, with generator A. Then*

$$t>0, f\in Y \Rightarrow p(t)f\in D(A),$$

and

$$\|Ap(t)f\|_Y\le\frac{c}{t}\|f\|_Y, \quad for\ 0<t\le 1.$$

Proof. The proof can be found in [27, Proposition 7.2]. □

If Ω is a bounded open domain with smooth boundary, on which the Neumann boundary condition is placed, then we know that $e^{t\Delta}$ defines a holomorphic semigroup on the Hilbert space $L^2(\Omega)$. So by Lemma 2.1, we have that

$$f \in L^2(\Omega) \Rightarrow \|e^{t\Delta} f\|_{H^2(\Omega)} \le \frac{c}{t}\|f\|_{L^2(\Omega)}, \tag{2.3}$$

where

$$D(\Delta) = \left\{ u \in H^2(\Omega), x\, \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial\Omega \right\}.$$

Applying interpolation to (2.3) yields

$$\|e^{t\Delta} f\|_{H^\sigma(\Omega)} \le c t^{-\frac{\sigma}{2}} \|f\|_{L^2(\Omega)} \quad \text{for } 1<\sigma<2,\ 0<t\le 1. \tag{2.4}$$

Lemma 2.2. *We assume that $u \in X$, a Banach space of functions, and that there is another Banach space Y such that the following four conditions hold:*

$e^{t\Delta}: X \to X$ *is a strongly continuous semigroup, for* $t \ge 0$,

$\Psi: X \to Y$ *is Lipschitz, uniformly on bounded sets,*

$e^{t\Delta}: Y \to X$, *for* $t>0$,

and, for some $\gamma<1$,

$$\left\| e^{t\Delta} \right\|_{\mathcal{L}(Y,X)} \le C t^{-\gamma}, \quad \text{for } t \in (0,1].$$

Then we have a bound $\|\Psi(u(s))\|_Y \le K_1$ *and*

$$\left\| \int_0^t e^{(t-s)\Delta} \Psi(u(s))\mathrm{d}s \right\|_X \le C_\gamma t^{1-\gamma} K_1.$$

Proof. The proof can be found in [27]. □

Dividing system (1.1) into two parts:

$$\begin{cases} u_t = \nabla(\nabla u - \chi u \nabla v) + \int_0^t f[u(\cdot,\tau)]\mathrm{d}\tau & \text{in } \Omega\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega\times\{0\}, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{2.5}$$

and

$$\begin{cases} v_t = \Delta v - v + u & \text{in } \Omega\times(0,T), \\ v(\cdot,0) = v_0 & \text{in } \Omega\times\{0\}, \\ \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{2.6}$$

then we have the following lemmas.

Lemma 2.3. *For $u\in X_u$, $v_0\in H^2(\Omega)\cap\{\frac{\partial v}{\partial \vec{n}}=0 \text{ on } \partial\Omega\}$, $t_0>0$ small enough, problem* (2.6) *has a unique solution $v\in X_v$, and v satisfies*

$$\|v(\cdot,t)\|_{X_v} \le c\|v_0\|_{H^2} + ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|u(\cdot,t)\|_{L^2}, \quad 0\le t\le t_0, \tag{2.7}$$

where c is a constant which is independent of T.

Proof. It is obvious that Eq. (2.6) has a solution and the solution is unique. So what we need to proof is (2.7). Let $T(t)=e^{t\Delta}$, where $D(\Delta)=H^2(\Omega)\cap\{\frac{\partial v}{\partial \vec{n}}=0 \text{ on } \partial\Omega\}$, then

$$v(t)=T(t)v_0-\int_0^t T(t-s)v(s)\mathrm{d}s+\int_0^t T(t-s)u(s)\mathrm{d}s.$$

By (2.3), Lemma 2.2, we calculate

$$\begin{aligned} \|v(\cdot,t)\|_{H^2} \le & c\|v_0\|_{H^2} + ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|v(\cdot,t)\|_{L^2} + ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|u(\cdot,t)\|_{L^2} \\ \le & c\|v_0\|_{H^2} + ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|v(\cdot,t)\|_{H^2} + ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|u(\cdot,t)\|_{L^2}, \quad 0\le t\le t_0. \end{aligned}$$

Thus for small enough t_0, (2.7) holds. □

Lemma 2.4. *For each $u_0\in H^\sigma(\Omega)$ and $v\in X_v$, σ and N satisfy* (2.1) *and* (2.2)*. The problem* (2.5) *has a unique solution $u\in c([0,t_0],H^\sigma(\Omega))$, and the solution can be written as*

$$\begin{aligned} u(t)= & T(t)u_0 - \chi\int_0^t T(t-s)\nabla u\nabla v \mathrm{d}s - \chi\int_0^t T(t-s)u\Delta v\mathrm{d}s \\ & + \int_0^t T(t-s)\int_0^s f(u)\mathrm{d}\tau\mathrm{d}s. \end{aligned} \tag{2.8}$$

Proof. We consider the following problem first

$$\begin{cases} u_t = \Delta u - \chi\nabla(\tilde{u}\nabla v) + \displaystyle\int_0^t f[\tilde{u}(\cdot,\tau)]\mathrm{d}\tau & \text{in } \Omega\times(0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega\times\{0\}, \\ \dfrac{\partial u}{\partial \vec{n}} = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{2.9}$$

where $\tilde{u}\in X_u$ is fixed. Using the semigroup notation $T(t)=e^{t\Delta}$, we can write the solution of (2.9) as

$$u(t)=T(t)u_0-\chi\int_0^t T(t-s)\nabla v\nabla\tilde{u}\mathrm{d}s-\chi\int_0^t T(t-s)\tilde{u}\Delta v\mathrm{d}s \\ +\int_0^t T(t-s)\int_0^s f[\tilde{u}(\cdot,\tau)]\mathrm{d}\tau\mathrm{d}s.$$

Define a mapping

$$G_1:\ X_u\to X_u,$$
$$G_1(\tilde{u})=u,$$

where u is the corresponding solution of (2.9).

Then we claim that for t_0 small enough, G_1 is a contract mapping. In fact, let $\tilde{u}_1,\tilde{u}_2\in X_u$, we have

$$G_1(\tilde{u}_1)-G_1(\tilde{u}_2)=-\chi\int_0^t T(t-s)\nabla((\tilde{u}_1-\tilde{u}_2)\nabla v)\mathrm{d}s \\ +\int_0^t T(t-s)\int_0^s (f[\tilde{u}_1(\cdot,\tau)]-f[\tilde{u}_2(\cdot,\tau)])\mathrm{d}\tau\mathrm{d}s. \tag{2.10}$$

By Sobolev imbedding theorems, we have

$$H^1(\Omega)\hookrightarrow L^\infty(\Omega) \qquad \text{for } N=1,$$
$$H^1(\Omega)\hookrightarrow L^q(\Omega),1<q<\infty \qquad \text{for } N=2,$$
$$H^1(\Omega)\hookrightarrow L^{\frac{2N}{N-2}}(\Omega) \qquad \text{for } N=3.$$

If $N=1$,

$$\|\chi\nabla(\tilde{u}_1-\tilde{u}_2)\nabla v\|_{L^2}\le\chi\|\nabla(\tilde{u}_1-\tilde{u}_2)\|_{L^2}\|\nabla v\|_{L^\infty} \\ \le c\|\tilde{u}_1-\tilde{u}_2\|_{H^1}\|\nabla v\|_{H^1}\le c\|\tilde{u}_1-\tilde{u}_2\|_{H^\sigma}\|v\|_{H^2}.$$

If $N=2,3$, according to (2.1) and (2.2), we obtain that

$$H^1(\Omega)\hookrightarrow L^{\frac{N}{\sigma-1}}(\Omega), \qquad H^{\sigma-1}(\Omega)\hookrightarrow L^{\frac{2N}{N-2(\sigma-1)}}(\Omega).$$

We know that $\nabla(\tilde{u}_1-\tilde{u}_2)\in H^{\sigma-1}$ and $\nabla v\in H^1$, then $|\nabla(\tilde{u}_1-\tilde{u}_2)|^2\in L^{\frac{N}{N-2(\sigma-1)}}(\Omega)$ and $|\nabla v|^2\in L^{\frac{N}{2(\sigma-1)}}(\Omega)$. Hence, Hölder's inequality yields

$$\||\nabla(\tilde{u}_1-\tilde{u}_2)|^2\cdot|\nabla v|^2\|_{L^1}\le\||\nabla(\tilde{u}_1-\tilde{u}_2)|^2\|_{L^{\frac{N}{N-2(\sigma-1)}}}\||\nabla v|^2\|_{L^{\frac{N}{2(\sigma-1)}}},$$

which implies

$$\|\chi\nabla(\tilde{u}_1-\tilde{u}_2)\nabla v\|_{L^2}\leq\chi\|\nabla(\tilde{u}_1-\tilde{u}_2)\|_{L^{\frac{2N}{N-2(\sigma-1)}}}\|\nabla v\|_{L^{\frac{N}{\sigma-1}}}\leq c\|\tilde{u}_1-\tilde{u}_2\|_{H^\sigma}\|v\|_{H^2}.$$

Hence for $N=1,2,3$, we have

$$\|\chi\nabla(\tilde{u}_1-\tilde{u}_2)\nabla v\|_{L^2}\leq c\|\tilde{u}_1-\tilde{u}_2\|_{H^\sigma}\|v\|_{H^2}. \tag{2.11}$$

Similarly, we have

$$\|\chi(\tilde{u}_1-\tilde{u}_2)\Delta v\|_{L^2}\leq c\|\tilde{u}_1-\tilde{u}_2\|_{H^\sigma}\|v\|_{H^2}. \tag{2.12}$$

So for the first term on the right side of (2.10), $N=1,2,3$, by (2.11) and (2.12), we have

$$\begin{aligned}&\left\|\int_0^t T(t-s)\nabla((\tilde{u}_1-\tilde{u}_2)\nabla v)\mathrm{d}s\right\|_{H^\sigma}\\ \leq& ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq s\leq t}\|\nabla((\tilde{u}_1-\tilde{u}_2)\nabla v)\|_{L^2}\\ \leq& ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq s\leq t}\left\{\|\nabla(\tilde{u}_1-\tilde{u}_2)\nabla v\|_{L^2}+\|(\tilde{u}_1-\tilde{u}_2)\Delta v\|_{L^2}\right\}\\ \leq& ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq s\leq t}\|\tilde{u}_1-\tilde{u}_2\|_{H^\sigma}\|v\|_{H^2}.\end{aligned} \tag{2.11}$$

For the second term on the right side of (2.10), we have

$$\begin{aligned}&\left\|\int_0^t T(t-s)\int_0^s(f[\tilde{u}_1(\cdot,\tau)]-f[\tilde{u}_2(\cdot,\tau)])\mathrm{d}\tau\mathrm{d}s\right\|_{H^\sigma}\\ \leq&\int_0^t\|T(t-s)\int_0^s(f[\tilde{u}_1(\cdot,\tau)]-f[\tilde{u}_2(\cdot,\tau)])\mathrm{d}\tau\|_{H^\sigma}\mathrm{d}s\\ \leq&\int_0^t c(t-s)^{-\frac{\sigma}{2}}\|\int_0^s(f[\tilde{u}_1(\cdot,\tau)]-f[\tilde{u}_2(\cdot,\tau)])\mathrm{d}\tau\|_{L^2}\mathrm{d}s\\ \leq& ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq s\leq t}\left\|\int_0^s(\tilde{u}_1-\tilde{u}_2)\mathrm{d}\tau\right\|_{L^2}\leq ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq s\leq t}\int_0^s\|(\tilde{u}_1-\tilde{u}_2)\|_{H^\sigma}\mathrm{d}\tau\\ \leq& ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq s\leq t}\sup_{0\leq\tau\leq s}\|\tilde{u}_1-\tilde{u}_2\|_{H^\sigma}\leq ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq\tau\leq t}\|\tilde{u}_1-\tilde{u}_2\|_{H^\sigma}.\end{aligned} \tag{2.12}$$

So we have

$$\|G_1(\tilde{u}_1)-G_1(\tilde{u}_2)\|_{X_u}\leq ct_0^{1-\frac{\sigma}{2}}\|\tilde{u}_1-\tilde{u}_2\|_{X_u}\|v\|_{X_v}+ct_0^{1-\frac{\sigma}{2}}\|\tilde{u}_1-\tilde{u}_2\|_{X_u},$$

which implies for $t_0>0$ small enough, G_1 is contract. By Banach fixed point theorem, there exists a unique fixed point $\tilde{u}$ such that $G_1(\tilde{u})=\tilde{u}$. Then we have the local solution of the problem (2.5):

$$\begin{aligned} u(t)=&T(t)u_0-\chi\int_0^t T(t-s)\nabla u\nabla v\mathrm{d}s-\chi\int_0^t T(t-s)u\Delta v\mathrm{d}s\\ &+\int_0^t T(t-s)\int_0^s f(u)\mathrm{d}\tau\mathrm{d}s. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.5. *Assume σ,N as given by* (2.1) *and* (2.2). *For solution $u\in X_u$ of* (2.5), *we have*

$$\|u\|_{X_u}\le c\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma}{2}}\|v\|_{X_v}\|u\|_{X_u},\ 0\le t\le t_0. \tag{2.13}$$

Proof. By Lemma 2.3, (2.5) has a unique solution, and the solution can be written as

$$\begin{aligned} u(t)=&T(t)u_0-\chi\int_0^t T(t-s)\nabla u\nabla v\mathrm{d}s-\chi\int_0^t T(t-s)u\Delta v\mathrm{d}s\\ &+\int_0^t T(t-s)\int_0^s f(u)\mathrm{d}\tau\mathrm{d}s. \end{aligned}$$

Next we prove estimate (2.13). By (2.4) we have

$$\left\|\int_0^t T(t-s)\nabla u\nabla v\mathrm{d}s\right\|_{H^\sigma}\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le s\le t}\|\nabla u(\cdot,s)\nabla v(\cdot,s)\|_{L^2}.$$

By Sobolev imbedding theorem, $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ for $N=1$, we have

$$\|\nabla u\nabla v\|_{L^2}\le\|\nabla u\|_{L^2}\|\nabla v\|_{L^\infty}\le c\|u\|_{H^1}\|v\|_{H^2}\le c\|u\|_{H^\sigma}\|v\|_{H^2}.$$

For $N=2,3$, we have

$$\begin{aligned} \|\nabla u\nabla v\|_{L^2}\le&\|\nabla u\|_{L^{\frac{2N}{N-2(\sigma-1)}}}\|\nabla v\|_{L^{\frac{N}{\sigma-1}}}\le c\|\nabla u\|_{L^{\frac{2N}{N-2(\sigma-1)}}}\|\nabla v\|_{H^1}\\ \le&c\|\nabla u\|_{H^{\sigma-1}}\|v\|_{H^2}\le c\|u\|_{H^\sigma}\|v\|_{H^2}. \end{aligned}$$

So we obtain that, for $0\le t\le t_0$,

$$\begin{aligned} \left\|\int_0^t T(t-s)\nabla u\nabla v\mathrm{d}s\right\|_{H^\sigma}\le& ct^{1-\frac{\sigma}{2}}\sup_{0\le s\le t}\|\nabla u\nabla v\|_{L^2}\\ \le&ct^{1-\frac{\sigma}{2}}\sup_{0\le s\le t}\|u\|_{H^\sigma}\|v\|_{H^2}\le ct_0^{1-\frac{\sigma}{2}}\|u\|_{X_u}\|v\|_{X_v}. \end{aligned}$$

Meanwhile, we deduce

$$\left\|\int_0^t T(t-s)u\Delta v\mathrm{d}s\right\|_{H^\sigma}\le ct^{1-\frac{\sigma}{2}}\sup_{0\le s\le t}\|u\Delta v\|_{L^2}$$

$$\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le s\le t_0}\|u\|_{L^\infty}\|\Delta v\|_{L^2}\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le s\le t_0}\|u\|_{H^\sigma}\sup_{0\le s\le t_0}\|v\|_{H^2}\le ct_0^{1-\frac{\sigma}{2}}\|u\|_{X_u}\|v\|_{X_v},$$

and

$$\left\|\int_0^t T(t-s)\int_0^s f[u(\cdot,\tau)]\mathrm{d}\tau\mathrm{d}s\right\|_{H^\sigma}\le\int_0^t\left\|T(t-s)\int_0^s f(u)\mathrm{d}\tau\right\|_{H^\sigma}\mathrm{d}s$$
$$\le\int_0^t c(t-s)^{-\frac{\sigma}{2}}\left\|\int_0^s f(u)\mathrm{d}\tau\right\|_{L^2}\mathrm{d}s\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le s\le t}\left\|\int_0^s f(u)\mathrm{d}\tau\right\|_{L^2}$$
$$\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le s\le t}\int_0^s\|u\|_{H^\sigma}\mathrm{d}\tau\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le s\le t}\sup_{0\le\tau\le s}\|u\|_{H^\sigma}\le ct_0^{1-\frac{\sigma}{2}}\sup_{0\le\tau\le t_0}\|u\|_{H^\sigma}.$$

Hence we declare that

$$\|u(t)\|_{H^\sigma}\le\|T(t)u_0\|_{H^\sigma}+\chi\left\|\int_0^t T(t-s)\nabla u\nabla v\mathrm{d}s\right\|_{H^\sigma}$$
$$+\chi\left\|\int_0^t T(t-s)u\Delta v\mathrm{d}s\right\|_{H^\sigma}+\left\|\int_0^t T(t-s)\int_0^s u\mathrm{d}\tau\mathrm{d}s\right\|_{H^\sigma}$$
$$\le c\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma}{2}}\|u\|_{X_u}\|v\|_{X_v}+ct_0^{1-\frac{\sigma}{2}}\|u\|_{X_u},\quad 0\le t\le t_0,$$

which implies for t_0 small enough

$$\|u(t)\|_{X_u}\le c\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma}{2}}\|u\|_{X_u}\|v\|_{X_v}.$$

Thus, Lemma 2.5 is proved. □

3 Local existence of solution

In this section, we establish the local solution of system (1.1).

Theorem 3.1. *Under conditions* (2.1) *and* (2.2), *for each initial data* $u_0\in X, v_0\in H^2(\Omega)\cap\{\frac{\partial v}{\partial\vec{n}}=0 \text{ on } \partial\Omega\}$, *problem* (1.1) *has a unique solution* $(u,v)\in X_u\times X_v$ *for some* $t_0>0$.

Proof. Consider $g\in X_u$ and $g(x,0)=u_0(x)$ and let $v=v(g)$ denotes the corresponding solution of the equation

$$\begin{cases} v_t=\Delta v-v+g & \text{in } \Omega\times(0,t_0),\\ v(\cdot,0)=v_0 & \text{in } \Omega\times\{0\},\\ \dfrac{\partial v}{\partial\vec{n}}=0 & \text{on } \partial\Omega\times(0,t_0). \end{cases}\tag{3.1}$$

By Lemma 2.2, we have $v\in X_v$ and

$$\|v(\cdot,t)\|_{X_v}\le c\|v_0\|_{H^2}+ct_0^{1-\frac{\sigma}{2}}\sup_{0\le t\le t_0}\|g(\cdot,t)\|_{L^2},\quad t\in(0,t_0). \tag{3.2}$$

For the solution v of (3.1), define $u=u(v(g))$ to be the corresponding solution of

$$\begin{cases} u_t=\nabla(\nabla u-\chi u\nabla v)+\int_0^t f[u(\cdot,\tau)]\mathrm{d}\tau & \text{in }\Omega\times(0,t_0),\\ u(x,0)=u_0(x) & \text{in }\Omega\times\{0\},\\ \dfrac{\partial u}{\partial\vec{n}}=0 & \text{on }\partial\Omega\times(0,t_0). \end{cases} \tag{3.3}$$

Define a mapping

$$G_2g=u(v(g)).$$

Then Lemma 2.3 shows that $G_2:X_u\to X_u$. Take $M=2c\|u_0\|_{H^\sigma}$ and a ball

$$B_M=\Big\{g\in X_u\Big|:\|g(\cdot,t)\|_{H^\sigma}\le M,\ g(x,0)=u_0(x),\ 0\le t\le t_0\Big\},$$

where the constant c is given by (2.13). Then we conclude from (2.13) and (3.2) that

$$\begin{aligned}\|G_2g\|_{X_u}\le& c\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma}{2}}\|v\|_{X_v}\|G_2g\|_{X_u}\\ \le& c\|u_0\|_{H^\sigma}+ct_0^{1-\frac{\sigma}{2}}\big(c\|v_0\|_{H^2}+ct_0^{1-\frac{\sigma}{2}}\sup_{0\le t\le t_0}\|g(\cdot,t)\|_{L^2}\big)\|G_2g\|_{X_u}.\end{aligned}$$

If $g\in B_M,\|g\|_{L^2}\le c\|g\|_{H^\sigma}\le cM$, then for $t_0>0$ small enough $\|G_2g\|_{X_u}\le 2c\|u_0\|_{H^\sigma}$. So for $t_0>0$ small enough, G_2 maps B_M into B_M.

Next we demonstrate that for t_0 small enough, G_2 is a contract mapping. In fact, let $g_1,g_2\in B_M\subset X_u$ and v_1,v_2 denote the corresponding solutions of (3.1). Then

$$\begin{aligned}&G_2g_1-G_2g_2=u_1-u_2\\ =&-\chi\int_0^t T(t-s)(u_1\Delta v_1-u_2\Delta v_2)\mathrm{d}s-\chi\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s\\ &+\int_0^t T(t-s)\int_0^s f(u_1)-f(u_2)\mathrm{d}\tau\mathrm{d}s.\end{aligned} \tag{3.4}$$

For the first term on the right side of (3.4),

$$\begin{aligned}&\Big\|\int_0^t T(t-s)(u_1\Delta v_1-u_2\Delta v_2)\mathrm{d}s\Big\|_{H^\sigma}\\ \le&\Big\|\int_0^t T(t-s)u_1(\Delta v_1-\Delta v_2)\mathrm{d}s\Big\|_{H^\sigma}+\Big\|\int_0^t T(t-s)(u_1-u_2)\Delta v_2\mathrm{d}s\Big\|_{H^\sigma},\end{aligned}$$

where

$$\left\| \int_0^t T(t-s)u_1(\Delta v_1-\Delta v_2)\mathrm{d}s \right\|_{H^\sigma} \le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le s\le t_0} \|u_1(\Delta v_1-\Delta v_2)\|_{H^\sigma}$$
$$\le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le s\le t_0} \|u_1\|_{L^\infty} \|\Delta(v_1-v_2)\|_{L^2} \le cMt_0^{1-\frac{\sigma}{2}} \sup_{0\le s\le t_0} \|v_1-v_2\|_{H^2},$$

and

$$\left\| \int_0^t T(t-s)(u_1-u_2)\Delta v_2\mathrm{d}s \right\|_{H^\sigma} \le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le s\le t_0} \|(u_1-u_2)\Delta v_2\|_{L^2}$$
$$\le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le s\le t_0} \|v_2\|_{H^2}\|u_1-u_2\|_{L^\infty} \le ct_0^{1-\frac{\sigma}{2}} \|v_2\|_{X_v}\|u_1-u_2\|_{X_u}.$$

Therefore

$$\left\| \int_0^t T(t-s)(u_1\Delta v_1-u_2\Delta v_2)\mathrm{d}s \right\|_{H^\sigma}$$
$$\le ct_0^{1-\frac{\sigma}{2}} \|v_1-v_2\|_{X_v} + ct_0^{1-\frac{\sigma}{2}} \|v_2\|_{X_v}\|u_1-u_2\|_{X_u}, \tag{3.5}$$

where $0\le t\le t_0$. For the second term on the right side of (3.4), we have

$$\left\| \int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s \right\|_{H^\sigma}$$
$$\le \left\| \int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_1)\mathrm{d}s \right\|_{H^\sigma} + \left\| \int_0^t T(t-s)(\nabla u_2\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s \right\|_{H^\sigma},$$

where

$$\left\| \int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_1)\mathrm{d}s \right\|_{H^\sigma}$$
$$\le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|\nabla v_1\nabla(u_1-u_2)\|_{L^2}, \qquad 0\le t\le t_0.$$

As we have done in Lemma 2.3 and 2.4, we obtain that

$$\left\| \int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_1)\mathrm{d}s \right\|_{H^\sigma} \le ct_0^{1-\frac{\sigma}{2}} \|v_1\|_{X_v}\|u_1-u_2\|_{X_u},\ \ 0\le t\le t_0.$$

Similarly

$$\left\| \int_0^t T(t-s)(\nabla u_2\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s \right\|_{H^\sigma} \le ct_0^{1-\frac{\sigma}{2}} \sup_{0\le t\le t_0} \|\nabla u_2\nabla(v_1-v_2)\|_{L^2}$$

$$\leq ct_0^{1-\frac{\sigma}{2}}\|u_2\|_{X_u}\|v_1-v_2\|_{X_v}\leq cMt_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{X_v},\qquad 0\leq t\leq t_0.$$

Then

$$\begin{aligned}&\left\|\int_0^t T(t-s)(\nabla u_1\nabla v_1-\nabla u_2\nabla v_2)\mathrm{d}s\right\|_{H^\sigma}\\ \leq& ct_0^{1-\frac{\sigma}{2}}\|v_1\|_{X_v}\|u_1-u_2\|_{X_u}+ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{X_v},\quad 0\leq t\leq t_0.\end{aligned}\tag{3.6}$$

For the last term on the right side of (3.4), we have

$$\left\|\int_0^t T(t-s)\int_0^s f(u_1)-f(u_2)\mathrm{d}\tau\mathrm{d}s\right\|_{H^\sigma}\leq ct_0^{1-\frac{\sigma}{2}}\|u_1-u_2\|_{X_u}.\tag{3.7}$$

Combining the estimates (3.5), (3.6), and (3.7), it follows that

$$\begin{aligned}&\|G_2g_1-G_2g_2\|_{X_u}\\ \leq& ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{X_v}+ct_0^{1-\frac{\sigma}{2}}\|v_2\|_{X_v}\|u_1-u_2\|_{X_u}\\ &+ct_0^{1-\frac{\sigma}{2}}\|v_1\|_{X_v}\|u_1-u_2\|_{X_u}+ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{X_v}+ct_0^{1-\frac{\sigma}{2}}\|u_1-u_2\|_{X_u},\end{aligned}$$

which implies

$$\begin{aligned}&\|G_2g_1-G_2g_2\|_{X_u}\\ \leq& 2ct_0^{1-\frac{\sigma}{2}}\|v_1-v_2\|_{X_v}+ct_0^{1-\frac{\sigma}{2}}\Big(\|v_2\|_{X_v}+\|v_1\|_{X_v}+1\Big)\|G_2g_1-G_2g_2\|_{X_u}.\end{aligned}$$

Consider the following equation

$$\begin{cases}(v_1-v_2)_t=\Delta(v_1-v_2)-(v_1-v_2)+(g_1-g_2) & \text{in }\Omega\times(0,t_0),\\ (v_1-v_2)(\cdot,0)=0 & \text{in }\Omega\times\{0\},\\ \dfrac{\partial(v_1-v_2)}{\partial\vec{n}}=0 & \text{on }\partial\Omega\times(0,t_0).\end{cases}$$

By (2.7), we obtain

$$\|v_1-v_2\|_{X_v}\leq ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq t\leq t_0}\|g_1-g_2\|_{L^2}\leq ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq t\leq t_0}\|g_1-g_2\|_{H^\sigma}.$$

Moreover, we have

$$\|v_1\|_{X_v}\leq c\|v_0\|_{H^2}+ct_0^{1-\frac{\sigma}{2}}\sup_{0\leq t\leq t_0}\|g_1\|_{L^2}\leq c\|v_0\|_{H^2}+ct_0^{1-\frac{\sigma}{2}}M,\quad 0\leq t\leq t_0,$$

$$\|v_2\|_{X_v} \leq c\|v_0\|_{H^2} + ct_0^{1-\frac{\sigma}{2}} \sup_{0\leq t\leq t_0} \|g_2\|_{L^2} \leq c\|v_0\|_{H^2} + ct_0^{1-\frac{\sigma}{2}} M, \quad 0\leq t\leq t_0.$$

Thus for $t_0>0$ small enough, G_2 is contract.

From the process above, we have proved that problem (1.1) has a solution $(u,v)\in X_u\times X_v$ by Lemmas 2.2, 2.3 and 2.4. We derive the uniqueness by Banach fixed point theorem. □

4 Blow-up in finite time

We then introduce an auxiliary function $F(u)$ defined by

$$F(u)=\int_0^u f(\sigma)\mathrm{d}\sigma.$$

And we suppose that $N=1$, then we have the following result.

Theorem 4.1. *If f(u) is a convex function on* $[0,\infty)$, *and F(u) satisfies*

$$\int_0^\infty F^{-1/2}(u)\mathrm{d}u<\infty, \tag{4.1}$$

then all nonnegative solutions of (1.1) *blow up in finite time.*

Proof. In the section, without causing any confusion, we may use C_i $(i=0,1,2...)$ to denote various positive constants.

$$u_t=\nabla(\nabla u-\chi u\nabla v)+\int_0^\tau f[u(x,s)]\mathrm{d}s.$$

Integrate both sides of the equation on Ω,

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega u\mathrm{d}x=\int_\Omega\Big(\nabla(\nabla u-\chi u\nabla v)\Big)\mathrm{d}x+\int_\Omega\int_0^\tau f[u(x,s)]\mathrm{d}s\mathrm{d}x.$$

By using $\frac{\partial u}{\partial\vec{n}}=\frac{\partial v}{\partial\vec{n}}=0$, we can get the following equality:

$$\int_\Omega u\mathrm{d}x-\int_\Omega u_0\mathrm{d}x=\int_\Omega\int_0^\tau\int_0^t f[u(x,s)]\mathrm{d}s\mathrm{d}x\mathrm{d}t.$$

Set

$$K(t)=\int_\Omega u(x,t)\mathrm{d}x \quad \text{for} \quad t>0.$$

By using Jensen's inequality, we find

$$C_1\int_0^t\int_0^\tau f(K(s))\mathrm{d}s\mathrm{d}\tau\leq\int_\Omega\int_0^\tau\int_0^t f[u(x,s)]\mathrm{d}s\mathrm{d}x\mathrm{d}t.$$

Then we can get that $K(t)$ satisfies

$$K(t)\geq C_2+C_1\int_0^t\int_0^\tau f(K(s))\mathrm{d}s\mathrm{d}\tau \qquad t>0.$$

Assume to the contrary that (1.1) has a global solution u. Then for any positive number T, we have

$$K(t)\geq C_2+C_1\int_T^\tau\int_T^t f(K(s))\mathrm{d}s\mathrm{d}\tau \quad \text{for } T\leq t\leq 2T.$$

Thus, by comparision, $K(t)\geq k(t)$ on $[T,2T]$, where

$$k(t)\geq C_2+C_1\int_T^\tau\int_T^t f(k(s))\mathrm{d}s\mathrm{d}\tau \quad \text{for } \leq t\leq 2T.$$

Clearly, $k(t)$ satisfies

$$\begin{aligned} &k''(t)=C_1 f(k(t)), \qquad T<t<2T,\\ &k(T)=C_2,\ k'(T)=0. \end{aligned} \tag{4.2}$$

Multiplying the equation in (4.2) by $k^{'}(t)$ and integrating from T to t, we obtain

$$k'(t)=C_3|F(k(t))-F(k(T))|^{1/2}.$$

Integration of this relation over (T,2T) then leads to

$$\begin{aligned} C_3T=&\int_{k(T)}^{k(2T)}[F(z)-F(k(T))]^{-1/2}\mathrm{d}z\\ \leq&\int_{k(T)}^{c}[F(z)-F(k(T))]^{-1/2}\mathrm{d}z+\int_c^\infty[F(z)-F(k(T))]^{-1/2}\mathrm{d}z\\ \leq&[f(C_2)]^{-1/2}\int_{C_2}^{c}[z-C_2]^{-1/2}\mathrm{d}z+\int_c^\infty[F(z)-F(z)/2]^{-1/2}\mathrm{d}z\\ =&2(c-C_2)^{1/2}f^{-1/2}(C_2)+\sqrt{2}\int_c^\infty F^{-1/2}(z)\mathrm{d}z, \end{aligned} \tag{4.3}$$

where c is a positive constant chosen so that $F(c)=2F(k(T))=2F(C_2)$. For sufficiently large T, inequality (4.3) yields a contraction to condition (4.1), which completes the proof. □

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