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Blow-up of Classical Solutions to the Isentropic Compressible Barotropic Navier-Stokes-Langevin-Korteweg Equations

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Abstract. In this paper, we study the barotropic Navier-Stokes-Langevin-Korteweg system in $\mathbb{R}^3$. Assuming the derivatives of the square root of the density and the velocity field decay to zero at infinity, we can prove the classical solutions blow up in finite time when the initial energy has a certain upper bound. We obtain this blow up result by a contradiction argument based on the conservation of the total mass and the total quasi momentum.

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1 Introduction

We are concerned with the Cauchy problem for the isentropic compressible barotropic Navier-Stokes-Langevin-Korteweg system in $\mathbb{R}^3$:

$$\begin{cases} \partial_t\rho+\mathrm{div}(\rho u)=0, \\ \partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)+\nabla P+\mu\rho u=\dfrac{\hbar^2}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)+\nu\mathrm{div}(\rho\mathbb{D}u), \\ \rho(0,x)=\rho_0(x), \quad u(0,x)=u_0(x), \end{cases} \tag{1.1}$$

where $\mathbb{D}u=(\nabla u+\nabla u^\top)/2$, the unknown functions $\rho:\mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}_+$ and $u:\mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}^3$ denote the density and the velocity field respectively, and $P:\mathbb{R}_+\times\mathbb{R}^3\to\mathbb{R}_+$ is the barotropic pressure of the form $P(\rho)=\rho^\gamma$ where $\gamma>1$ is the adiabatic constant. $\mu>0$,

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$\hbar > 0$ and $\nu > 0$ are the dissipation coefficient, the renormalized Plauck constant and the viscosity coefficient respectively.

The Navier-Stokes-Langevin-Korteweg equations include many classical equations, such as the compressible Navier-Stokes equations if $\mu = \hbar = 0$ which describe the law of mechanics of viscous fluids and are significant in fluid mechanics; the Navier-Stokes-Korteweg equations if $\mu = 0$ which are firstly considered by Van der Waal and Korteweg to model fluid capillarity effect and then developed by Dunn and Serrin [1] to reflect the variation of density at the interface of two phases; the Euler-Langevin-Korteweg equations if ν=0 which are firstly used as a stochastic interpretation of quantum mechanics [2], then applied to quantum semiconductor [3] and quantum trajectories of Bohmia mechanics [4], and recently have a renewed interest in statistical mechanics and cosmology [5].

There are many theoretical studies in blow up of smooth solutions of compressible Navier-Stokes equations. In 1998, Xin [6] showed a sufficient condition which leads to blow up of smooth solutions with initial density of compact support, and the key idea of proof is the total pressure decays faster in time in the presence of the vacuum. Then these results were improved by Xin and Wei [7] by showing the finite time blow up of smooth solutions with initial density containing vacuum and without the assumption of finite total energy. For the same questions, Lai [8] applied a contradiction argument to prove the classical solutions blow up in finite time, when the gradient of the velocity satisfies some decay constraint and the initial total momentum does not vanish. The blow up results of the full compressible case and the isentropic compressible case with constant and degenerate viscosities are obtained by Jiu [9], and a more precise blow up time can be computed out when the results are applied to Euler equations. Huang [10] presented a blow-up criterion for classical solutions in $\mathbb{R}^3$ in terms of the gradient of the velocity, which is the necessary condition of blow up. For the Navier-Stokes-Korteweg system, Tang [11] established a blow up result for the smooth solutions to the Cauchy problem of the symmetric barotropic case with initial density of compact support. The one-dimensional initial boundary value problem in a bounded domain was studied by Tang [12] and the blow up results of smooth solutions were obtained. Li [13] showed a Serrins type blow-up criterion for the density-dependent Navier-Stokes-Korteweg equations with vacuum in $\mathbb{R}^3$.

To our best knowledge, there isn't any blow up result for the Navier-Stokes-Langevin-Korteweg system yet and the dissipative term $\mu\rho u$ on the left side of $(1.1)_2$ brings some difficulties when we come to prove the total momentum is conserved. To overcome these difficulties, we define the total quasi momentum as

$$\int_{\mathbb{R}^3} e^{\mu t}\rho(t,x)u(t,x)\mathrm{d}x,$$

and we prove the classic solutions to the three-dimensional Navier-Stokes-Langevin-Korteweg system blow up in finite time by a contradiction argument based on the conservation of the total mass and the total quasi momentum in this paper.

Remark 1.1. Meanwhile, there are a lot of results on existence of weak solutions to these equations. Readers can refer to [14-20] if they have interest.

We now introduce the definition of classic solutions and state the main result of this paper.

Definition 1.1. *Let $T>0$. We call $(\rho(t,x),u(t,x))$ are clssical solutions to the barotropic Navier-Stokes-Langevin-Korteweg system* (1.1) *on $[0,T)\times\mathbb{R}^3$ if*

$$\rho\in C^1([0,T),C^3(\mathbb{R}^3)),\quad u\in C^1([0,T),C^2(\mathbb{R}^3))$$

and satisfy (1.1) *on $[0,T)\times\mathbb{R}^3$ pointwisely.*

Theorem 1.1. *Support that (ρ,u) are classical solutions of the barotropic Navier-Stokes-Langevin-Korteweg system* (1.1) *with $\mu>0$, $\nu>0$. We assume that*

$$\sum_{|\alpha|\le 1}|\partial^\alpha u|\overset{|x|\to+\infty}{\longrightarrow}0,\qquad \sum_{|\alpha|\le 2}|\partial^\alpha\sqrt{\rho}|\overset{|x|\to+\infty}{\longrightarrow}0,\quad \forall t\ge 0. \tag{1.2}$$

Moreover, ρ satisfies

$$\int_{\mathbb{R}^3}\rho(t,x)\mathrm{d}x<+\infty,\qquad \int_{\mathbb{R}^3}|\nabla^2\sqrt{\rho}(t,x)|^2\mathrm{d}x<+\infty,\quad \forall t\ge 0. \tag{1.3}$$

If the initial data satisfies

$$0<\int_{\mathbb{R}^3}\rho_0(x)u_0(x)\mathrm{d}x<+\infty,\qquad 0<\int_{\mathbb{R}^3}\rho_0(x)\mathrm{d}x<+\infty,$$

and the initial energy satisfies

$$\begin{aligned} E(0)=&\frac{1}{2}\int_{\mathbb{R}^3}\rho_0(x)u_0^2(x)\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{R}^3}\rho_0^\gamma(x)\mathrm{d}x+\int_{\mathbb{R}^3}(\nabla\sqrt{\rho_0})^2(x)\mathrm{d}x\\ <&\frac{1}{2}\frac{\left(\int_{\mathbb{R}^3}\rho_0(x)u_0(x)\mathrm{d}x\right)^2}{\int_{\mathbb{R}^3}\rho_0(x)\mathrm{d}x}, \end{aligned} \tag{1.4}$$

then the classical solutions of system (1.1) *will blow up in a finite time T^*.*

Remark 1.2. Our method also works for other dimensions($n\ge 2$).

2 Preliminaries

We first give three lemmas showing the energy estimate and stating the conservation of the total mass and the total quasi momentum before demonstrating the proof of the main theorem.

Lemma 2.1. *Support that* (ρ,u) *are classical solutions of system* (1.1) *and satisfy* (1.2), *then we have the energy estimate for* $t\in(0,T^*)$

$$E(t)+\mu\int_0^t\int_{\mathbb{R}^3}\rho|u|^2\mathrm{d}x\mathrm{d}\tau+\nu\int_0^t\int_{\mathbb{R}^3}\rho|\mathbb{D}u|^2\mathrm{d}x\mathrm{d}\tau=E(0), \tag{2.1}$$

where $E(t)$ *is the total energy*

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{R}^3}\rho^\gamma\mathrm{d}x+\int_{\mathbb{R}^3}(\nabla\sqrt{\rho})^2\mathrm{d}x, \tag{2.2}$$

and T^* *denotes the lifespan of classical solutions.*

Proof. Multiplying $(1.1)_2$ with u and using $(1.1)_1$ twice, we have

$$\partial_t(\rho u)\cdot u=-\mathrm{div}(\rho u)|u|^2+\frac{1}{2}(\rho u^2)_t+\frac{1}{2}\mathrm{div}(\rho u)u^2. \tag{2.3}$$

And it is easy to obtain

$$\begin{aligned}\mathrm{div}(\rho u\otimes u)\cdot u&=\mathrm{div}(\rho u)|u|^2+\frac{1}{2}\rho u\cdot\nabla u^2\\&=\mathrm{div}(\rho u)|u|^2+\frac{1}{2}\mathrm{div}(\rho u|u|^2)-\frac{1}{2}\mathrm{div}(\rho u)u^2.\end{aligned} \tag{2.4}$$

From $(1.1)_1$, we have

$$\begin{aligned}\nabla P(\rho)\cdot u&=\frac{\gamma}{\gamma-1}\nabla(\rho^{\gamma-1})\cdot(\rho u)=\frac{\gamma}{\gamma-1}\mathrm{div}(\rho^\gamma u)-\frac{\gamma}{\gamma-1}\rho^{\gamma-1}\mathrm{div}(\rho u)\\&=\frac{\gamma}{\gamma-1}\mathrm{div}(\rho^\gamma u)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho^\gamma}{\gamma-1}.\end{aligned} \tag{2.5}$$

Using $(1.1)_1$ again, we get

$$\begin{aligned}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\cdot u&=\mathrm{div}(\sqrt{\rho}\Delta\sqrt{\rho}u)-\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\mathrm{div}(\rho u)\\&=\mathrm{div}(\sqrt{\rho}\Delta\sqrt{\rho}u)+2\Delta\sqrt{\rho}(\sqrt{\rho})_t\\&=\mathrm{div}(\sqrt{\rho}\Delta\sqrt{\rho}u)+2\mathrm{div}[\nabla\sqrt{\rho}(\sqrt{\rho})_t]-\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\sqrt{\rho})^2.\end{aligned} \tag{2.6}$$

And it is obvious that

$$\nu\mathrm{div}(\rho\mathbb{D}u)\cdot u=\nu\mathrm{div}(\rho\mathbb{D}u\cdot u)-\nu\rho|\mathbb{D}u|^2. \tag{2.7}$$

Combing (2.3)-(2.7), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\rho u^2)+\frac{1}{2}\mathrm{div}(\rho u|u|^2)+\frac{\gamma}{\gamma-1}\mathrm{div}(\rho^\gamma u)+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho^\gamma}{\gamma-1}+\mu\rho u^2$$

$$=\mathrm{div}(\sqrt{\rho}\Delta\sqrt{\rho}u)+2\mathrm{div}[\nabla\sqrt{\rho}(\sqrt{\rho})_t]-\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\sqrt{\rho})^2+\nu\mathrm{div}(\rho\mathbb{D}u\cdot u)-\nu\rho|\mathbb{D}u|^2. \tag{2.8}$$

Integrating (2.8) over $\mathbb{R}^3$, and according to the assumption (1.2), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)+\mu\int_{\mathbb{R}^3}\rho|u|^2\mathrm{d}x+\nu\int_{\mathbb{R}^3}\rho|\mathbb{D}u|^2\mathrm{d}x=0, \tag{2.9}$$

where

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x+\frac{1}{\gamma-1}\int_{\mathbb{R}^3}\rho^\gamma\mathrm{d}x+\int_{\mathbb{R}^3}(\nabla\sqrt{\rho})^2\mathrm{d}x.$$

Integrating (2.9) over $[0,t]$, we come to

$$E(t)+\mu\int_0^t\int_{\mathbb{R}^3}\rho|u|^2\mathrm{d}x\mathrm{d}\tau+\nu\int_0^t\int_{\mathbb{R}^3}\rho|\mathbb{D}u|^2\mathrm{d}x\mathrm{d}\tau=E(0),$$

and the proof of Lemma 2.1 is finished. □

Since the initial energy is finite, we can assert that

$$\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^3)},\quad \|\sqrt{\rho}\mathbb{D}u\|_{L^2(\mathbb{R}^3)},\quad \|P\|_{L^1(\mathbb{R}^3)},\quad \|\nabla\sqrt{\rho}\|_{L^2(\mathbb{R}^3)}$$

are all finite.

Lemma 2.2. *Support that* (ρ,u) *are classical solutions of system* (1.1) *and satisfy* $\int_{\mathbb{R}^3}\rho(t,x)\mathrm{d}x<+\infty$, *then the total mass of this system is conserved in the sense that*

$$\int_{\mathbb{R}^3}\rho(t,x)\mathrm{d}x=\int_{\mathbb{R}^3}\rho_0(x)\mathrm{d}x,\ \ \forall t>0.$$

Proof. Integrating $(1.1)_1$ over $[0,t]\times B_R=\left\{(\tau,x)\in\mathbb{R}_+\times\mathbb{R}^3\,|\,0\le\tau\le t,|x|\le R\right\}$, we yield

$$\Big|\int_{|x|\le R}\rho(t,x)-\rho_0(x)\mathrm{d}x\Big|\le\int_0^t\Big|\int_{|x|\le R}\mathrm{div}(\rho u)\mathrm{d}x\Big|\mathrm{d}\tau.$$

Using Hölder's inequality and the assumption that $\int_{\mathbb{R}^3}\rho(t,x)\mathrm{d}x<+\infty$, we have

$$\begin{aligned}\int_0^\infty\Big|\int_{|x|\le R}\mathrm{div}(\rho u)\mathrm{d}x\Big|\mathrm{d}R&=\int_0^\infty\Big|\int_{|x|=R}\frac{x}{R}\cdot\rho u\mathrm{d}x\Big|\mathrm{d}R\le\int_{\mathbb{R}^3}\rho|u|\mathrm{d}x\\&\le\|\sqrt{\rho}\|_{L^2(\mathbb{R}^3)}\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^3)}<+\infty,\end{aligned}$$

which means that for any fixed $t>0$,

$$\int_0^\infty\Big|\int_{|x|\le R}\rho(t,x)-\rho_0(x)\mathrm{d}x\Big|\mathrm{d}R\le t\|\sqrt{\rho}\|_{L^2(\mathbb{R}^3)}\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^3)}<+\infty.$$

Hence there is a sequence $R_n \stackrel{n\to+\infty}{\longrightarrow} +\infty$ such that

$$\lim_{n\to+\infty}\Big|\int_{|x|\le R_n}\rho(t,x)-\rho_0(x)\mathrm{d}x\Big|\to 0,$$

which is

$$\int_{\mathbb{R}^3}\rho(t,x)\mathrm{d}x=\int_{\mathbb{R}^3}\rho_0(x)\mathrm{d}x,$$

and the proof of Lemma 2.2 is finished. □

Next we will state the conservation of the total quasi momentum $\int_{\mathbb{R}^3}e^{\mu t}\rho(t,x)u(t,x)\mathrm{d}x$.

Lemma 2.3. *Support that* (ρ,u) *are classical solutions of system* (1.1) *and satisfy* (1.3)*, then the total quasi momentum of this system is conserved in the sense that*

$$\int_{\mathbb{R}^3}e^{\mu t}\rho(t,x)u(t,x)\mathrm{d}x=\int_{\mathbb{R}^3}\rho_0(x)u_0(x)\mathrm{d}x.$$

Proof. Multiplying $(1.1)_2$ with $e^{\mu t}$, we yield

$$\partial_t(e^{\mu t}\rho u)=e^{\mu t}\left\{-\mathrm{div}(\rho u\otimes u)-\nabla P+\frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)+\nu\mathrm{div}(\rho\mathbb{D}u)\right\}. \tag{2.10}$$

Integrating (2.10) over $[0,t]\times B_R$, we have

$$\begin{aligned}&\Big|\int_{|x|\le R}e^{\mu t}\rho(t,x)u(t,x)-\rho_0(x)u_0(x)\mathrm{d}x\Big|\\ \le&\int_0^t e^{\mu\tau}\Big\{\Big|\int_{|x|\le R}\mathrm{div}(\rho u\otimes u)\mathrm{d}x\Big|+\Big|\int_{|x|\le R}\nabla P\mathrm{d}x\Big|\\ &+\Big|\int_{|x|\le R}\frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\mathrm{d}x\Big|+\Big|\int_{|x|\le R}\nu\mathrm{div}(\rho\mathbb{D}u)\mathrm{d}x\Big|\Big\}\mathrm{d}\tau.\end{aligned} \tag{2.11}$$

Integrating (2.11) over $(0,+\infty)$ with respect to R, we notice that

$$\int_0^\infty\Big|\int_{|x|\le R}\mathrm{div}(\rho u\otimes u)\mathrm{d}x\Big|\mathrm{d}R\le\sqrt{2}\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x=\sqrt{2}\|\sqrt{\rho}u\|^2_{L^2(\mathbb{R}^3)}<+\infty. \tag{2.12}$$

By using the Gauss's formula, we have

$$\int_0^\infty\Big|\int_{|x|\le R}\nabla P\mathrm{d}x\Big|\mathrm{d}R=\int_0^\infty\Big|\int_{|x|=R}\frac{x}{R}P\mathrm{d}x\Big|\mathrm{d}R\le\|P\|_{L^1(\mathbb{R}^3)}<+\infty. \tag{2.13}$$

According to the assumption (1.3), we get

$$\int_0^\infty\Big|\int_{|x|\le R}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)\mathrm{d}x\Big|\mathrm{d}R$$

$$
\begin{aligned}
&=\int_0^\infty \Big|\int_{|x|\le R}\nabla(\sqrt{\rho}\Delta\sqrt{\rho})-2\nabla\sqrt{\rho}\Delta\sqrt{\rho}\mathrm{d}x\Big|\mathrm{d}R\\
&=\int_0^\infty \Big|\int_{|x|\le R}\nabla(\sqrt{\rho}\Delta\sqrt{\rho})-2\mathrm{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho})+\nabla(\nabla\sqrt{\rho})^2\mathrm{d}x\Big|\mathrm{d}R\\
&\le C\left(\|\sqrt{\rho}\|_{L^2(\mathbb{R}^3)}\|\nabla^2\sqrt{\rho}\|_{L^2(\mathbb{R}^3)}+\|\nabla\sqrt{\rho}\|^2_{L^2(\mathbb{R}^3)}\right)\\
&<+\infty. \qquad (2.14)
\end{aligned}
$$

Then using the assumption that $\int_{\mathbb{R}^3}\rho(x)\mathrm{d}x<+\infty$, we yield

$$
\begin{aligned}
\int_0^\infty \Big|\int_{|x|\le R}\mathrm{div}(\rho\mathbb{D}u)\mathrm{d}x\Big|\mathrm{d}R=&\int_0^\infty \Big|\int_{|x|=R}\frac{x}{R}\cdot(\rho\mathbb{D}u)\mathrm{d}x\Big|\mathrm{d}R\\
\le&\sqrt{2}\int_{\mathbb{R}^3}\rho|\mathbb{D}u|\mathrm{d}x\\
\le&\sqrt{2}\|\sqrt{\rho}\|_{L^2(\mathbb{R}^3)}\|\sqrt{\rho}|\mathbb{D}u|\|_{L^2(\mathbb{R}^3)}<+\infty. \qquad (2.15)
\end{aligned}
$$

Combining (2.11)-(2.15), we can obtain that for any fixed $t>0$,

$$
\begin{aligned}
&\int_0^\infty \Big|\int_{|x|\le R}e^{\mu t}\rho(t,x)u(t,x)-\rho_0(x)u_0(x)\mathrm{d}x\Big|\mathrm{d}R\\
\le& C\left(\int_0^t e^{\mu\tau}\mathrm{d}\tau\right)=\frac{C}{\mu}(e^{\mu t}-1)<+\infty,
\end{aligned}
$$

where

$$
C=c\cdot\left(E(0)+\int_{\mathbb{R}^3}\rho_0(x)\mathrm{d}x+\sup_{t\ge0}\int_{\mathbb{R}^3}|\nabla^2\sqrt{\rho}|^2\mathrm{d}x\right)
$$

and c is a positive constant. Hence, we come to

$$
\int_{\mathbb{R}^3}e^{\mu t}\rho(t,x)u(t,x)\mathrm{d}x=\int_{\mathbb{R}^3}\rho_0(x)u_0(x)\mathrm{d}x, \qquad (2.16)
$$

and the proof of Lemma 2.3 is finished. □

3 Proof of the main theorem

In this section, we are going to prove Theorem 1.1 by a contradiction argument. On account of Lemma 2.2 and Lemma 2.3, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3}\rho_0(x)u_0(x)\mathrm{d}x=&e^{\mu t}\int_{\mathbb{R}^3}\rho(t,x)u(t,x)\mathrm{d}x\\
\le& e^{\mu t}\left(\int_{\mathbb{R}^3}\rho(t,x)\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x\right)^{\frac12}
\end{aligned}
$$

$$=e^{\mu t}\left(\int_{\mathbb{R}^3}\rho_0(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x\right)^{\frac{1}{2}},$$

which means

$$\left(\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x\right)^{\frac{1}{2}}\geq e^{-\mu t}\frac{\int_{\mathbb{R}^3}\rho_0(x)u_0(x)\mathrm{d}x}{\left(\int_{\mathbb{R}^3}\rho_0(x)\mathrm{d}x\right)^{\frac{1}{2}}}:=Ae^{-\mu t}.$$

Hence we have

$$\mu\int_0^t\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x\mathrm{d}\tau\geq A^2\mu\int_0^t e^{-2\mu\tau}\mathrm{d}\tau=\frac{A^2}{2}\left(1-e^{-2\mu t}\right).$$

Using the assumption (1.4) that $E(0)<A^2/2$, we can see there is $0<T<\infty$ such that

$$\mu\int_0^T\int_{\mathbb{R}^3}\rho u^2\mathrm{d}x\mathrm{d}\tau=E(0).$$

Substituting this result into (2.1), we obtain there is $T^*\in(0,T)$ such that $E(T^*)=0$ which contradicts to the conservation of the total mass, and the proof of Theorem 1.1 is finished.

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