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Global Well-Posedness and Asymptotic Behavior for the 2D Subcritical Dissipative Quasi-Geostrophic Equation in Critical Fourier-Besov-Morrey Spaces

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Abstract. In this paper, we study the subcritical dissipative quasi-geostrophic equation. By using the Littlewood Paley theory, Fourier analysis and standard techniques we prove that there exists v a unique global-in-time solution for small initial data belonging to the critical Fourier-Besov-Morrey spaces $\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}$. Moreover, we show the asymptotic behavior of the global solution v. i.e., $\|v(t)\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}$ decays to zero as time goes to infinity.

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1 Introduction

In this article, we consider the following Cauchy problem for the two-dimensional quasi-geostrophic equation (2DGQ) with subcritical dissipation $\alpha > 1/2$.

$$\begin{cases} \partial_t v + k\Lambda^{2\alpha} v + u_v \cdot \nabla v = 0, \quad x \in \mathbb{R}^2,\ t > 0, \\ u_v = (-\Re_2 v, \Re_1 v), \\ v(0,x) = v_0(x), \end{cases} \tag{1.1}$$

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where $\Re_j=\partial_{x_j}(-\Delta)^{-1/2}$, $j=1,2$, are the Riesz transforms, $\alpha>1/2$ is a real number, $k>0$ is a dissipative coefficient $\big($when $k=0$ Eq. (1.1) becomes the two-dimensional non-dissipative quasi-geostrophic equation$\big)$. Notice that (1.1) is called **subcritical** when $\alpha>1/2$, **critical** when $\alpha=1/2$ and **supercritical** when $\alpha<1/2$. Λ is the operator defined by the fractional power of $-\Delta$:

$$\Lambda v=(-\Delta)^{\frac{1}{2}}v, \quad \mathcal{F}(\Lambda v)=\mathcal{F}((-\Delta)^{\frac{1}{2}}v)=|\xi|\mathcal{F}(v),$$

and more generally

$$\mathcal{F}(\Lambda^{2\alpha}v)=\mathcal{F}((-\Delta)^{\alpha}v)=|\xi|^{2\alpha}\mathcal{F}(v),$$

where $\mathcal{F}$ is the Fourier transform. The scalar function $v(x,t)$ represents the potential temperature, and u_v is the divergence free velocity which is determined by the Riesz transformation of v. Since we are concerned with the dissipative case, we assume $k=1$ for the sake of simplicity.

The 2D quasi-geostrophic equation is an important model in geophysical fluid dynamics, which represents the potential temperature dynamics of atmospheric and ocean flow. For further information on the physical background of this equation, see [1, 2] and the references therein. It is well known that Eq. (1.1) is comparable to the three-dimensional Navier-Stokes equations (see [3–5]).

There is a rich literature about global-in-time well-posedness for fluid dynamics PDEs in different spaces, where the smallness conditions are taken in norms of critical spaces (i.e., the norm is invariant under the scaling of the equation/system). For instance, for Navier-Stokes equations, 2D quasi-geostrophic equations, and related models, we have well-posedness results in the critical case of the following spaces: Lebesgue space L^p [6,7], Marcinkiewicz space $L^{p,\infty}$ [8, 9], Morrey spaces $\mathcal{M}_{p,\mu}$ [10], Besov-Morrey spaces $\mathcal{N}^s_{p,\mu,q}$ [11], Fourier-Besov spaces $\mathcal{FB}^s_{p,q}$ [5, 12], Fourier-Herz spaces $\mathcal{FB}^s_{1,q}=\mathcal{B}^s_{1,q}$ [13], Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\mu,q}$ [14–19], and BMO^{-1} [20], among others. Moreover, in some of the above references, one can find results on decay and/or asymptotic behavior of solutions, such as the works [6–11, 19].

Now, we recall the scaling property of the equations:
if v solves (1.1) with initial data v_0, then v_γ with $v_\gamma(x,t):=\gamma^{2\alpha-1}v\big(\gamma x,\gamma^{2\alpha}t\big)$ is also a solution to (1.1) with the initial data

$$v_{0,\gamma}(x):=\gamma^{2\alpha-1}v_0(\gamma x). \tag{1.2}$$

Definition 1.1. *A critical space for initial data of Eq.* (1.1) *is any Banach space* $E\subset\mathcal{S}'(\mathbb{R}^n)$ *whose norm is invariant under the scaling* (1.2) *for all* $\gamma>0$*, i.e.*

$$\|v_{0,\gamma}(x)\|_E\approx\|v_0(x)\|_E.$$

In accordance with these scales, we can show that the space $\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}$ is critical for (1.1). In this respect, there are several papers on global-in-time well-posedness for

(1.1) in various critical spaces. For instance, Chae and Lee [21] established the global well-posedness for any small initial data in the Besov space $\dot{\mathcal{B}}^{2-2\alpha}_{2,1}$ for $\alpha < \frac{1}{2}$, Bae [22] get the global existence with small initial data belongs to the critical Besov spaces $\dot{\mathcal{B}}^{1+\frac{2}{p}-2\alpha}_{p,q}$ with $p < \infty$, and Benameur & Benhamed [23] obtained the global existence of (1.1) with subcritical dissipation in the critical Lei-Lin spaces $\mathcal{X}^s$ which are defined as (see [3,24]): For $s \in \mathbb{R}$

$$\mathcal{X}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n); \int_{\mathbb{R}^n} |\xi|^s |\widehat{f}(\xi)| \mathrm{d}\xi < \infty \right\}, \tag{1.3}$$

with the norm

$$\|f\|_{\mathcal{X}^s} = \int_{\mathbb{R}^n} |\xi|^s |\widehat{f}(\xi)| \mathrm{d}\xi. \tag{1.4}$$

Recently, Benhamed and Abusalim [25] studied the asymptotic behavior of (1.1) in Lei-Lin space $\mathcal{X}^{1-2\alpha}$ with subcritical dissipation. Other related results can be found in [6,26–28].

This paper plans to analyze the existence of the global solution of the $2D$ quasi-geostrophic equation in the framework of critical Fourier-Besov-Morrey spaces. Furthermore, we obtain the long time decay property of a given global solution for (1.1) (asymptotic behavior). Our results extend and complement some previous works such as [5,23,25]. Before stating our results, let us first define our setting.

Denote the set of all polynomials by $\mathcal{P}$ and the Morrey spaces by M^{λ}_p with norm

$$\|f\|_{M^{\lambda}_p} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0,r))} < \infty.$$

We define

$$\mathcal{FN}^s_{p,\lambda,q} = \left\{ f \in \mathcal{S}' \backslash \mathcal{P} \mid \|f\|_{\mathcal{FN}^s_{p,\lambda,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \varphi_j \hat{f} \right\|^q_{\mathrm{M}^{\lambda}_p} \right)^{\frac{1}{q}} < \infty \right\}, \tag{1.5}$$

where $\{\varphi_j\}$ is the Littlewood-Paley decomposition (see Section 2 for details).

For the convenience of description, we use $\mathcal{FN}^s_{p,\lambda,q}$ to denote the homogenous Fourier-Besov-Morrey spaces, $\dot{H}^s$ to denote the usual homogeneous Sobolev space, and χ_A to denote the indicator function of a set A. For f, we denote $u_f := (-\Re_2 f, \Re_1 f)$. Let X, Y be Banach spaces, we denote

$$\|v\|_{X \cap Y} := \|v\|_X + \|v\|_Y \quad \text{and} \quad \|(v,w)\|_X := \|v\|_X + \|w\|_X,$$

C will denote constants which can be different at different places, $A \sim B$ means that there are two constants $C_1, C_2 > 0$ such that

$$C_1 B \leq A \leq C_2 B,$$

$V \lesssim W$ denotes the estimate $V \leq CW$ for some constant $C \geq 1$, and p' is the conjugate of p satisfying $\frac{1}{p}+\frac{1}{p'}=1$ for $1\leq p\leq\infty$.

In order to solve Eq. (1.1), we consider the following equivalent integral equation coming from Duhamels principle

$$v(t)=\mathcal{H}_\alpha(t)v_0+\mathcal{B}(v,v)(t), \tag{1.6}$$

where $\mathcal{H}_\alpha:=e^{-t(-\Delta)^\alpha}$ denotes the fractional heat semigroup operator, which can be regarded as the convolution operator with the kernel $K_t(x)=\mathcal{F}^{-1}(e^{-t|\xi|^{2\alpha}})$, and

$$\mathcal{B}(v,\psi)(t)=-\int_0^t \mathcal{H}_\alpha(t-\tau)(u_v\cdot\nabla\psi))(\tau)\mathrm{d}\tau. \tag{1.7}$$

First, we show the existence of global solutions for (1.1).

Theorem 1.1. (Well-posedness) *Let* $1\leq p<\infty$, $1\leq q\leq\infty$, $0\leq\lambda<2$ *and* $\frac{1}{2}<\alpha<2+\frac{\lambda-2}{2p}$. *There exists a constant* $\beta>0$ *such that for any* $v_0\in\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}$ *satisfying* $\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}<\beta$, *Eq.* (1.1) *admits a unique global solution*

$$v\in\mathcal{C}\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}\right)\cap\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3+\frac{\lambda-2}{p}}\right),$$

such that

$$\|v\|_{\mathcal{L}^\infty\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}\right)\cap\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3+\frac{\lambda-2}{p}}\right)}\lesssim\|v_0\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}.$$

Remark 1.1. Taking $n=2, a_{21}=1=-a_{12}, a_{11}=a_{22}=0, |P_j(\xi)|=|\xi| (j=1,2,\quad \beta=1)$ in the expressions (1.2) to (1.5) in [19], we obtain the 2DQG (1.1). Then, using Theorem 3.1 in [19], we obtain a well-posedness result in Fourier-Besov-Morrey spaces $\mathcal{FN}_{p,\lambda,\infty}^{3-2\alpha-\frac{2-\lambda}{p}}$ for (1) in the subcritical case, which is related to Theorem 1.1 but is different. It is worth noticing that in [19] it is used the persistence norm $BC\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,\infty}^{3-\frac{2-\lambda}{p}}\right)$ while Chemin-Lerner type norms are employed in our Theorem 1.1, namely the norms of $\mathcal{L}^\infty\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3-2\alpha-\frac{2-\lambda}{p}}\right)$ and $\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3-\frac{2-\lambda}{p}}\right)$.

Next, we present an asymptotic behavior result for solutions in the context of Fourier-Besov-Morrey spaces.

Theorem 1.2. (**Asymptotic behavior**) *Let* $\frac{2}{3}<\alpha<1$, $1\leq p$, $q\leq 2$, $0\leq\lambda\leq 2-p$, *and* $v\in\mathcal{C}\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}\right)$ *be a global solution of* (1.1) *given by Theorem* 1.1. *Then,*

$$\lim_{t\longrightarrow\infty}\|v(t)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}=0.$$

2 Preliminaries

In this section, we give some notations and recall basic properties of Fourier-Besov-Morrey spaces, which will be used throughout the article.

The Fourier-Besov-Morrey spaces, presented in [19], are built by using a type of localization on Morrey spaces. The function spaces M_p^λ are defined as follows.

Definition 2.1. ([10]) *Let $1\le p\le\infty$ and $0\le\lambda<n$.*

- *The homogeneous Morrey space M_p^λ is the set of all functions $f\in L^p(B(x_0,r))$ such that*

$$\|f\|_{M_p^\lambda}=\sup_{x_0\in\mathbb{R}^n}\sup_{r>0} r^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x_0,r))}<\infty, \tag{2.1}$$

  where $B(x_0,r)$ is the open ball in $\mathbb{R}^n$ centered at x_0 and with radius $r>0$. The space M_p^λ endowed with the norm $\|f\|_{M_p^\lambda}$ is a Banach space.

  When $p=1$, the L^1 -norm in (2.1) *is understood as the total variation of the measure f on $B(x_0,r)$ and M_p^λ as a subspace of Radon measures. When $\lambda=0$, we have $M_p^0=L^p$.*

- *The mixed Morrey-sequence space $l^q(M_p^\lambda)$ consists of all sequences $\{f_i\}_{i\in\mathbb{Z}}$ of measurable functions in $\mathbb{R}^n$ such that $\|\{f_i\}_{i\in\mathbb{Z}}\|_{l^q(M_p^\lambda)}<\infty$. For $\{f_i\}_{i\in\mathbb{Z}}\in l^q(M_p^\lambda)$ we define*

$$\|\{f_i\}_{i\in\mathbb{Z}}\|_{l^q(M_p^\lambda)}:=\left(\sum_{j\in\mathbb{Z}}\|f_i\|_{M_p^\lambda}^q\right)^{\frac{1}{q}}.$$

The proofs of the results discussed in this work are based on a dyadic partition of unity in the Fourier variables, known as the homogeneous Littlewood-Paley decomposition. We present briefly this construction below. For more detail, we refer the reader to [29].

Let $f\in S'(\mathbb{R}^n)$. Define the Fourier transform as

$$\hat{f}(\xi)=\mathcal{F}f(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\mathrm{d}x$$

and its inverse Fourier transform as

$$\check{f}(x)=\mathcal{F}^{-1}f(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{ix\cdot\xi}f(\xi)\mathrm{d}\xi.$$

Let $\varphi\in S(\mathbb{R}^d)$ be such that $0\le\varphi\le1$ and $\mathrm{supp}(\varphi)\subset\{\xi\in\mathbb{R}^d:\frac{3}{4}\le|\xi|\le\frac{8}{3}\}$ and

$$\sum_{j\in\mathbb{Z}}\varphi\left(2^{-j}\xi\right)=1,\quad \text{for all } \xi\neq0.$$

We denote

$$\varphi_j(\xi)=\varphi\left(2^{-j}\xi\right),\quad \delta_j(\xi)=\sum_{k\le j-1}\varphi_k(\xi)$$

and

$$h(x)=\mathcal{F}^{-1}\varphi(x), \quad g(x)=\mathcal{F}^{-1}\delta(x).$$

We now present some frequency localization operators:

$$\begin{aligned} \dot{\Delta}_j f &= \mathcal{F}^{-1}\varphi_j * f = 2^{dj}\int_{\mathbb{R}^d} h\left(2^j y\right) f(x-y)\mathrm{d}y, \\ \dot{S}_j f &= \sum_{k\le j-1}\dot{\Delta}_k f = \mathcal{F}^{-1}\delta_j * f = 2^{dj}\int_{\mathbb{R}^d} g\left(2^j y\right) f(x-y)\mathrm{d}y. \end{aligned}$$

where $\dot{\Delta}_j=\dot{S}_j-\dot{S}_{j-1}$ is a frequency projection to the annulus $\{|\xi|\sim 2^j\}$ and $\dot{S}_j$ is a frequency to the ball $\{|\xi|\lesssim 2^j\}$.

From the definition of $\dot{\Delta}_j$ and $\dot{S}_j$, one easily derives that

$$\begin{aligned} &\dot{\Delta}_j\dot{\Delta}_k f = 0, && \text{if } |j-k|\ge 2, \\ &\dot{\Delta}_j\left(\dot{S}_{k-1} f\dot{\Delta}_k f\right) = 0, && \text{if } |j-k|\ge 5, \\ &\widehat{\dot{\Delta}_j f} = \varphi_j \widehat{f}. \end{aligned}$$

The following Bony paraproduct decomposition will be applied throughout the paper.

$$uv=\dot{T}_u v+\dot{T}_v u+R(u,v),$$

where

$$\dot{T}_u v=\sum_{j\in\mathbb{Z}}\dot{S}_{j-1}u\dot{\Delta}_j v, \quad \dot{R}(u,v)=\sum_{j\in\mathbb{Z}}\dot{\Delta}_j u\tilde{\Delta}_j v, \quad \tilde{\Delta}_j v=\sum_{|j'-j|\le 1}\dot{\Delta}_{j'} v.$$

Lemma 2.1. ([19])

Let $1\le p_1,p_2,p_3<\infty$ and $0\le\lambda_1,\lambda_2,\lambda_3<n$.
(i) (Hölder's inequality) Let $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{\lambda_3}{p_3}=\frac{\lambda_1}{p_1}+\frac{\lambda_2}{p_2}$, then we have

$$\|fg\|_{\mathrm{M}_{p_3}^{\lambda_3}}\le\|f\|_{\mathrm{M}_{p_1}^{\lambda_1}}\|g\|_{\mathrm{M}_{p_2}^{\lambda_2}}. \tag{2.2}$$

(ii) (Young's inequality) If $\varphi\in L^1$ and $g\in\mathrm{M}_{p_1}^{\lambda_1}$, then

$$\|\varphi*g\|_{\mathrm{M}_{p_1}^{\lambda_1}}\le\|\varphi\|_{L^1}\|g\|_{\mathrm{M}_{p_1}^{\lambda_1}}, \tag{2.3}$$

where $*$ denotes the standard convolution operator.

Now, we recall the Bernstein-type lemma in Fourier variables in Morrey spaces.

Lemma 2.2. ([19]) *Let $1\le q\le p<\infty, 0\le\lambda_1,\lambda_2<n$, $\frac{n-\lambda_1}{p}\le\frac{n-\lambda_2}{q}$ and let γ be a multi-index. If $supp(\widehat{f})\subset\{|\xi|\le A2^j\}$, then there is a constant $C>0$ independent of f and j such that*

$$\|(i\xi)^\gamma\widehat{f}\|_{\mathrm{M}_q^{\lambda_2}}\le C2^{j|\gamma|+j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})}\|\widehat{f}\|_{\mathrm{M}_p^{\lambda_1}}. \tag{2.4}$$

We give the definition of the homogeneous Fourier-Besov spaces $\dot{FB}^s_{p,q}$ [5].

Definition 2.2. *(Homogeneous Fourier-Besov spaces) Let $1\leq p,q\leq\infty$ and $s\in\mathbb{R}$. The homogeneous Fourier-Besov space $\dot{FB}^s_{p,q}$ is defined as the set of all distributions $f\in\mathcal{S}'\backslash\mathcal{P}$, $\mathcal{P}$ is the set of all polynomials, such that the norm $\|f\|_{\dot{FB}^s_{p,q}}$ is finite, where*

$$\|f\|_{\dot{FB}^s_{p,q}} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{j\in\mathbb{Z}}2^{jsq}\left\|\varphi_j\hat{f}\right\|^q_{L^p}\right)^{\frac{1}{q}} & \text{for } q<\infty \\ \sup_{j\in\mathbb{Z}}2^{js}\left\|\varphi_j\hat{f}\right\|_{L^p} & \text{for } q=\infty. \end{cases} \tag{2.5}$$

Remark 2.1. ([30]) Notice that in the case $p=q$ we have an equivalent norm on $\dot{FB}^s_{p,p}$, that is

$$\|f\|_{\dot{FB}^s_{p,p}}\sim\left(\int_{\mathbb{R}^n}|\xi|^{sp}|\hat{f}(\xi)|^p\mathrm{d}\xi\right)^{\frac{1}{p}}.$$

Let us now recall the definition of Fourier-Besov-Morrey spaces $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$, see [19].

Definition 2.3. (Homogeneous Fourier-Besov-Morrey spaces) *Let $1\leq p,q\leq\infty$, $0\leq\lambda<n$ and $s\in\mathbb{R}$. The homogeneous Fourier-Besov-Morrey space $\mathcal{FN}^s_{p,\lambda,q}$ is defined as the set of all distributions $f\in\mathcal{S}'\backslash\mathcal{P}$, $\mathcal{P}$ is the set of all polynomials, such that the norm $\|f\|_{\mathcal{FN}^s_{p,\lambda,q}}$ is finite, where*

$$\|f\|_{\mathcal{FN}^s_{p,\lambda,q}} \stackrel{\text{def}}{=} \begin{cases} \left(\sum_{j\in\mathbb{Z}}2^{jsq}\left\|\varphi_j\hat{f}\right\|^q_{\mathrm{M}^\lambda_p}\right)^{\frac{1}{q}} & \text{for } <\infty \\ \sup_{j\in\mathbb{Z}}2^{js}\left\|\varphi_j\hat{f}\right\|_{\mathrm{M}^\lambda_p} & \text{for } q=\infty. \end{cases} \tag{2.6}$$

Note that the space $\mathcal{FN}^s_{p,\lambda,q}(\mathbb{R}^n)$ equipped with the norm (2.6) *is a Banach space. Since $\mathrm{M}^0_p=L^p$, we have $\mathcal{FN}^s_{p,0,q}=\dot{FB}^s_{p,q}$, and $\mathcal{FN}^s_{1,0,1}=\chi^s$ where χ^s is the Lei-Lin space* [23].

Lemma 2.3. *The derivation $\partial^\alpha_\xi:\mathcal{FN}^{s+|\alpha|}_{p,\lambda,q}\to\mathcal{FN}^s_{p,\lambda,q}$ is a bounded operator.*

Proof. We have

$$\begin{aligned}\|\partial^\alpha_\xi v\|_{\mathcal{FN}^s_{p,\lambda,q}}&=\|\{2^{js}\varphi_j\widehat{\partial^\alpha_\xi v}\}_{j\in\mathbb{Z}}\|_{l^q(\mathcal{M}^\lambda_p)}=\|\{2^{js}\varphi_j|\xi|^\alpha\widehat{v}\}_{j\in\mathbb{Z}}\|_{l^q(\mathcal{M}^\lambda_p)}\\ &\lesssim\|\{2^{js}2^{j\alpha}\varphi_j\widehat{v}\}_{j\in\mathbb{Z}}\|_{l^q(\mathcal{M}^\lambda_p)}\lesssim\|v\|_{\mathcal{FN}^{s+|\alpha|}_{p,\lambda,q}},\end{aligned} \tag{2.7}$$

where in (2.7) we used the fact that $|\xi|\sim 2^j$ for all $j\in\mathbb{Z}$. □

Remark 2.2. As a consequence of Lemma 2.3, we have the following estimates:

$$\|\mathrm{div}(v)\|_{\mathcal{FN}^s_{p,\lambda,q}}\lesssim\|v\|_{\mathcal{FN}^{s+1}_{p,\lambda,q}},\qquad\|\Delta v\|_{\mathcal{FN}^s_{p,\lambda,q}}\lesssim\|v\|_{\mathcal{FN}^{s+2}_{p,\lambda,q}}.$$

Proposition 2.1. ([18]) *(Sobolev-type embedding) For $p_2 \leq p_1$ and $s_2 \leq s_1$ satisfying*

$$s_2+\frac{n-\lambda_2}{p_2}=s_1+\frac{n-\lambda_1}{p_1},$$

we have the continuous inclusion

$$\mathcal{FN}^{s_1}_{p_1,\lambda_1,r_1} \hookrightarrow \mathcal{FN}^{s_2}_{p_2,\lambda_2,r_2}$$

for all $1 \leq r_1 \leq r_2 \leq \infty$.

The definition of mixed space-time spaces is given below.

Definition 2.4. ([18]) *Let $s \in \mathbb{R}, 1 \leq p < \infty, 1 \leq q,\rho \leq \infty, 0 \leq \lambda < n$, and $I=[0,T), T \in (0,\infty]$. The space-time norm is defined on $u(t,x)$ by*

$$\|u(t,x)\|_{\mathcal{L}^{\rho}(I,\mathcal{FN}^s_{p,\lambda,q})}=\left\{\sum_{j\in\mathbb{Z}} 2^{jqs}\|\widehat{\dot{\Delta}_j u}\|^q_{L^{\rho}(I,\mathrm{M}^{\lambda}_p)}\right\}^{\frac{1}{q}},$$

and denote by $\mathcal{L}^{\rho}(I,\mathcal{FN}^s_{p,\lambda,q})$ the set of distributions in $S'(\mathbb{R}\times\mathbb{R}^n)/\mathcal{P}$ with finite $\|.\|_{\mathcal{L}^{\rho}(I,\mathcal{FN}^s_{p,\lambda,q})}$ norm.

By virtue of the Minkowski inequality, we have

$$L^{\rho}\left(I;\mathcal{FN}^s_{p,\lambda,q}\right) \hookrightarrow \mathcal{L}^{\rho}\left(I,\mathcal{FN}^s_{p,\lambda,q}\right), \quad \text{if } \rho \leq q, \tag{2.8}$$

$$\mathcal{L}^{\rho}\left(I,\mathcal{FN}^s_{p,\lambda,q}\right) \hookrightarrow L^{\rho}\left(I;\mathcal{FN}^s_{p,\lambda,q}\right), \quad \text{if } \rho \geq q, \tag{2.9}$$

where

$$\|u(t,x)\|_{L^{\rho}\left(I;\mathcal{FN}^s_{p,\lambda,q}\right)} := \left(\int_I \|u(\tau,\cdot)\|^{\rho}_{\mathcal{FN}^s_{p,\lambda,q}} \mathrm{d}\tau\right)^{\frac{1}{\rho}}.$$

At the end of this section, we will recall an existence and uniqueness result for an abstract operator equation in a Banach space that will be used to show Theorem 1.1 in the sequel. For the proof, we refer the reader to see [31,32].

Lemma 2.4. *Let X be a Banach space with norm $\|\cdot\|_X$ and $B: X \times X \longmapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X \leq \eta \|u\|_X \|v\|_X$$

for all $u,v \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $y \in X$ such that $\|y\|_X \leq \varepsilon$, the equation $x := y + B(x,x)$ has a solution $\overline{x}$ in X such that $\|\overline{x}\|_X \leq 2\varepsilon$. This solution is the only one in the ball $\overline{B}(0,2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \varepsilon, x' = y' + B(x',x')$, and $\|x'\|_X \leq 2\varepsilon$, then

$$\|\overline{x}-x'\|_X \leq \frac{1}{1-4\varepsilon\eta}\|y-y'\|_X.$$

3 Linear estimates in Fourier-Besov-Morrey spaces

We present some important lemmas from [33].

Lemma 3.1. *Let* $T>0$, $0\le\lambda<3, 1\le p<\infty, 1\le q,\rho\le\infty$, $s\in\mathbb{R}$ *and* $u_0\in\mathcal{FN}^{s}_{p,\lambda,q}$. *Then there exists a constant* $C>0$ *such that*

$$\left\| e^{-t(-\triangle)^{\alpha}(\cdot)}u_0 \right\|_{\mathcal{L}^{\rho}\left([0,T),\mathcal{FN}^{s+\frac{2\alpha}{\rho}}_{p,\lambda,q}\right)} \le C\|u_0\|_{\mathcal{FN}^{s}_{p,\lambda,q}}. \tag{3.1}$$

Lemma 3.2. *Let* $0<T\le\infty, s\in\mathbb{R}, 0\le\lambda<3, 1\le p<\infty, 1\le q,\rho,r\le\infty$ *and* $1\le r\le\rho$. *There exists a constant* $C>0$ *such that*

$$\left\| \int_0^t e^{-(-\triangle)^{\alpha}(t-\tau)}f(\tau)\mathrm{d}\tau \right\|_{\mathcal{L}^{\rho}([0,T),\mathcal{FN}^{s}_{p,\lambda,q})} \le C\|f\|_{\mathcal{L}^{r}\left([0,T),\mathcal{FN}^{s-2\alpha-\frac{2\alpha}{\rho}+\frac{2\alpha}{r}}_{p,\lambda,q}\right)}$$

for all $f\in\mathcal{L}^r([0,T),\mathcal{F}\dot{\mathcal{N}}^{s}_{p,\lambda,q})$.

4 Bilinear estimates in Fourier-Besov-Morrey spaces

In this section, we will establish the bilinear estimate which will be crucial in the proof of Theorem 1.1.

Proposition 4.1. *Set*

$$Y=\mathcal{L}^{\infty}\left(\mathbb{R}^+,\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)\cap\mathcal{L}^{1}\left(\mathbb{R}^+,\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right).$$

Under the hypothesis of Theorem 1.1, *there exists a constant* $\eta>0$ *such that*

$$\|\mathcal{B}(v,\psi)\|_Y\le\eta\|v\|_Y\|\psi\|_Y,\quad for\ all\ v,\psi\in Y.$$

Proof. First, using Lemma 3.2 we have

$$\begin{aligned}\|\mathcal{B}(v,\psi)\|_Y &\lesssim \|u_v\cdot\nabla\psi\|_{\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \lesssim \|\mathrm{div}(u_v\psi)\|_{\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)}\\ &\lesssim \|u_v\psi\|_{\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{4-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)}.\end{aligned} \tag{4.1}$$

Then, the remainder of the proof is to show that

$$\|u_v\psi\|_{\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{4-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \lesssim \|v\|_Y\|\psi\|_Y. \tag{4.2}$$

Recalling that $u_v=(-\Re_2 v,\Re_1 v)$. Since $\widehat{\Re_j v(\xi)}=-i\frac{\xi_j}{|\xi|}\widehat{v}(\xi), j=1,2$, then

$$\|\widehat{u_v}\|\lesssim\|\widehat{v}\|. \tag{4.3}$$

Applying Bony para-product decomposition and quasi-orthogonality property for Littlewood-Paley decomposition, for fixed j, we obtain

$$\begin{aligned}\dot{\Delta}_j(u_v\psi)=&\sum_{|k-j|\le 4}\dot{\Delta}_j(\dot{S}_{k-1}u_v\dot{\Delta}_k\psi)+\sum_{|k-j|\le 4}\dot{\Delta}_j(\dot{S}_{k-1}\psi\dot{\Delta}_k u_v)+\sum_{k\ge j-3}\dot{\Delta}_j(\dot{\Delta}_k u_v\widetilde{\dot{\Delta}}_k\psi)\\ =:&I_j^1+I_j^2+I_j^3.\end{aligned}$$

Then, by the triangle inequalities in M_p^λ and in $l^q(\mathbb{Z})$, we have

$$\begin{aligned}&\|u_v\psi\|_{\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{4-2\alpha+\frac{\lambda-2}{p}}\right)}=\left\{\sum_{j\in\mathbb{Z}}2^{j(4-2\alpha+\frac{\lambda-2}{p})q}\|\widehat{\dot{\Delta}_j(u_v\psi)}\|^q_{L^1(\mathbb{R}^+,M_p^\lambda)}\right\}^{\frac{1}{q}}\\ \le&\left\{\sum_{j\in\mathbb{Z}}2^{j(4-2\alpha+\frac{\lambda-2}{p})q}\|\widehat{I_j^1}\|^q_{L^1(\mathbb{R}^+,M_p^\lambda)}\right\}^{\frac{1}{q}}+\left\{\sum_{j\in\mathbb{Z}}2^{j(4-2\alpha+\frac{\lambda-2}{p})q}\|\widehat{I_j^2}\|^q_{L^1(\mathbb{R}^+,M_p^\lambda)}\right\}^{\frac{1}{q}}\\ &+\left\{\sum_{j\in\mathbb{Z}}2^{j(4-2\alpha+\frac{\lambda-2}{p})q}\|\widehat{I_j^3}\|^q_{L^1(\mathbb{R}^+,M_p^\lambda)}\right\}^{\frac{1}{q}}\\ =:&\mathrm{E}_1+\mathrm{E}_2+\mathrm{E}_3.\end{aligned}$$

By Bernstein-type inequality (2.4) with $|\gamma|=0$, we have

$$\|\varphi_l\widehat{u_v}\|_{L^1}\le C2^{l(2-\frac{2-\lambda}{p})}\|\varphi_l\widehat{u_v}\|_{M_p^\lambda}\lesssim 2^{l(2+\frac{\lambda-2}{p})}\|\varphi_l\widehat{v}\|_{M_p^\lambda}, \tag{4.4}$$

where we have used (4.3). Thus, using Young's inequality in Morrey spaces (2.3) and the estimate (4.4), we get

$$\begin{aligned}\|\widehat{I_j^1}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}\le&\sum_{|k-j|\le 4}\|\widehat{(\dot{S}_{k-1}u_v\dot{\Delta}_k\psi)}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}\\ \le&\sum_{|k-j|\le 4}\|\varphi_k\widehat{\psi}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}\sum_{l\le k-2}\|\varphi_l\widehat{u_v}\|_{L^\infty(\mathbb{R}^+,L^1)}\\ \lesssim&\sum_{|k-j|\le 4}\|\varphi_k\widehat{\psi}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}\sum_{l\le k-2}2^{(2+\frac{\lambda-2}{p})l}\|\varphi_l\widehat{v}\|_{L^\infty(\mathbb{R}^+,M_p^\lambda)}\\ \lesssim&\sum_{|k-j|\le 4}\|\varphi_k\widehat{\psi}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}\sum_{l\le k-2}2^{(3-2\alpha+\frac{\lambda-2}{p})l}2^{(2\alpha-1)l}\|\varphi_l\widehat{v}\|_{L^\infty(\mathbb{R}^+,M_p^\lambda)}\\ \lesssim&\|v\|_{\mathfrak{L}^\infty\left(\mathbb{R}^+,\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}\right)}\sum_{|k-j|\le 4}\Big(\sum_{l\le k-2}2^{l(2\alpha-1)q'}\Big)^{\frac{1}{q'}}\|\varphi_k\widehat{\psi}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}\end{aligned}$$

$$\lesssim \|v\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \sum_{|k-j|\leq 4} 2^{k(2\alpha-1)} \|\varphi_k \widehat{\psi}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}.$$

Therefore, by using the Young inequality for series, one has

$$\begin{aligned}
\mathrm{E}_1 &\lesssim \|v\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \left\{ \sum_{j\in\mathbb{Z}} 2^{j(4-2\alpha+\frac{\lambda-2}{p})q} \left(\sum_{|k-j|\leq 4} 2^{k(2\alpha-1)} \|\varphi_k \widehat{\psi}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \right)^q \right\}^{\frac{1}{q}} \\
&\lesssim \|v\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \left\{ \sum_{j\in\mathbb{Z}} \left(\sum_{|k-j|\leq 4} 2^{(j-k)(4-2\alpha+\frac{\lambda-2}{p})q} 2^{k(3+\frac{\lambda-2}{p})} \|\varphi_k \widehat{\psi}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \right)^q \right\}^{\frac{1}{q}} \\
&\lesssim \|v\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \|\psi\|_{\mathfrak{L}^{1}\left(\mathbb{R}^{+},\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)}.
\end{aligned}$$

Similary, we get

$$\mathrm{E}_2 \lesssim \|\psi\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \|v\|_{\mathfrak{L}^{1}\left(\mathbb{R}^{+},\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)}.$$

For E_3, first we use the Young inequality in Morrey spaces (2.3), the Bernstein-type inequality with $|\gamma|=0$ together with the Hölder inequality, to get

$$\begin{aligned}
\|\widehat{I_j^3}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} &\leq \sum_{k\geq j-3} \|\widehat{(\dot{\Delta}_k u_v \tilde{\dot{\Delta}}_k \psi)}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \\
&\leq \sum_{k\geq j-3} \|(\widehat{\dot{\Delta}_k u_v} * \widehat{\tilde{\dot{\Delta}}_k \psi})\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \\
&\leq \sum_{k\geq j-3} \|\varphi_k \widehat{u_v}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \sum_{|l-k|\leq 1} \|\varphi_l \widehat{\psi}\|_{L^\infty(\mathbb{R}^+,L^1)} \\
&\lesssim \sum_{k\geq j-3} \|\varphi_k \widehat{v}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \sum_{|l-k|\leq 1} 2^{(2+\frac{\lambda-2}{p})l} \|\varphi_l \widehat{\psi}\|_{L^\infty(\mathbb{R}^+,M_p^\lambda)} \\
&\lesssim \sum_{k\geq j-3} \|\varphi_k \widehat{v}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \Big(\sum_{|l-k|\leq 1} 2^{l(2\alpha-1)q'} \Big)^{\frac{1}{q'}} \|\psi\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \\
&\lesssim \|\psi\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \sum_{k\geq j-3} 2^{k(2\alpha-1)} \|\varphi_k \widehat{v}\|_{L^1(\mathbb{R}^+,M_p^\lambda)}.
\end{aligned}$$

Then, applying the Young inequality for series, we obtain

$$\begin{aligned}
\mathrm{E}_3 &\lesssim \|\psi\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \left\{ \sum_{j\in\mathbb{Z}} 2^{j(4-2\alpha+\frac{\lambda-2}{p})q} \left(\sum_{k\geq j-3} 2^{k(2\alpha-1)} \|\varphi_k \widehat{v}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \right)^q \right\}^{\frac{1}{q}} \\
&\lesssim \|\psi\|_{\mathfrak{L}^{\infty}\left(\mathbb{R}^{+},\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \left\{ \sum_{j\in\mathbb{Z}} \left(\sum_{k\geq j-3} 2^{(j-k)(4-2\alpha+\frac{\lambda-2}{p})} 2^{k(3+\frac{\lambda-2}{p})} \|\varphi_k \widehat{v}\|_{L^1(\mathbb{R}^+,M_p^\lambda)} \right)^q \right\}^{\frac{1}{q}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\psi\|_{\mathfrak{L}^\infty\left(\mathbb{R}^+,\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \|v\|_{\mathfrak{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \sum_{i\le 3} 2^{i(4-2\alpha+\frac{\lambda-2}{p})} \\
&\lesssim \|\psi\|_{\mathfrak{L}^\infty\left(\mathbb{R}^+,\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \|v\|_{\mathfrak{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)},
\end{aligned}
$$

where the condition $\alpha<2+\frac{\lambda-2}{2p}$ ensures that the series $\sum_{i\le 3}2^{i(4-2\alpha+\frac{\lambda-2}{p})}$ converges. This finishes the proof of Proposition 4.1. □

Now, we will establish some crucial lemmas in the proof of Theorem 1.2.

Lemma 4.1. *Let* $\frac{2}{3}<\alpha<1, 1\le p,q\le 2$ *and* $0\le\lambda\le 2-p$. *Then we have*

$$
\|v\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \lesssim \|v\|_{L^2}^{\frac{3\alpha-2}{\alpha}} \|v\|_{\dot{H}^\alpha}^{\frac{2-2\alpha}{\alpha}}. \tag{4.5}
$$

Proof. By using the definition of the Fourier-Besov-Morrey spaces, and Bernstein-type inequality (2.4) with $|\gamma|=0$, we have

$$
\begin{aligned}
&\|v\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \\
=&\left\{\sum_{j\in\mathbb{Z}} 2^{j\left(3-2\alpha+\frac{\lambda-2}{p}\right)q} \|\varphi_j\hat{u}\|^q_{M_p^\lambda}\right\}^{\frac{1}{q}} \\
\lesssim&\left\{\sum_{j\in\mathbb{Z}} 2^{j\left(3-2\alpha+\frac{\lambda-2}{p}\right)q} 2^{j\left(\frac{2-\lambda}{p}-1\right)q} \|\varphi_j\hat{u}\|^q_{L^2(B(x_0,r))}\right\}^{\frac{1}{q}} \\
\lesssim&\left\{\sum_{j\le N} 2^{j(2-2\alpha)q} \|\varphi_j\hat{u}\|^q_{L^2(\mathbb{R}^2)}\right\}^{\frac{1}{q}} + \left\{\sum_{j>N} 2^{j(2-3\alpha)q}2^{j\alpha q} \|\varphi_j\hat{u}\|^q_{L^2(\mathbb{R}^2)}\right\}^{\frac{1}{q}} \\
\lesssim& 2^{(2-2\alpha)N}\left\{\sum_{j\in\mathbb{Z}} \|\varphi_j\hat{u}\|^2_{L^2(\mathbb{R}^2)}\right\}^{\frac{1}{2}} + 2^{(2-3\alpha)N}\left\{\sum_{j\in\mathbb{Z}} 2^{2\alpha j}\|\varphi_j\hat{u}\|^2_{L^2(\mathbb{R}^2)}\right\}^{\frac{1}{2}} \\
\lesssim& 2^{(2-2\alpha)N}\|v\|_{F\dot{B}^0_{2,2}} + 2^{(2-3\alpha)N}\|v\|_{F\dot{B}^\alpha_{2,2}}.
\end{aligned}
$$

From Remark 2.1, one can see that $F\dot{B}^\alpha_{2,2}=\dot{H}^\alpha$ and $F\dot{B}^0_{2,2}=L^2$. Then, by Taking N such that $2^N=\left(\frac{\|v\|_{\dot{H}^\alpha}}{\|v\|_{L^2}}\right)^{1/\alpha}$, we obtain

$$
\|v\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \lesssim \left(\frac{\|v\|_{\dot{H}^\alpha}}{\|v\|_{L^2}}\right)^{\frac{2-\alpha}{\alpha}} \|v\|_{L^2} + \left(\frac{\|v\|_{\dot{H}^\alpha}}{\|v\|_{L^2}}\right)^{\frac{2-3\alpha}{\alpha}} \|v\|_{\dot{H}^\alpha} \lesssim \|v\|_{\dot{H}^\alpha}^{\frac{2-\alpha}{\alpha}} \|v\|_{L^2}^{\frac{3\alpha-2}{\alpha}}.
$$

□

Lemma 4.2. *Let* $\frac{1}{2}<\alpha<1$, $1\le p<\infty$ *and* $1\le q\le 2$. *Then we have*

$$\|fg\|_{\dot{H}^{1-\alpha}}\lesssim\|f\|_{L^2}\|g\|_{\mathcal{FN}^{3-\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}+\|f\|_{\dot{H}^{\alpha}}\|g\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}. \tag{4.6}$$

Proof. To prove this lemma, we will follow the method described in the proof of Proposition 4.1, for fixed j, we have

$$\begin{aligned}\dot{\Delta}_j(fg)&=\sum_{|k-j|\le 4}\dot{\Delta}_j(\dot{S}_{k-1}f\dot{\Delta}_k g)+\sum_{|k-j|\le 4}\dot{\Delta}_j(\dot{S}_{k-1}g\dot{\Delta}_k f)+\sum_{k\ge j-3}\dot{\Delta}_j(\dot{\Delta}_k f\widetilde{\dot{\Delta}}_k g)\\&=:II_j^1+II_j^2+II_j^3.\end{aligned}$$

One can write

$$\begin{aligned}\|fg\|_{\dot{H}^{1-\alpha}}=\|fg\|_{F\dot{B}^{1-\alpha}_{2,2}}&\le\left\{\sum_{j\in\mathbb{Z}}2^{2j(1-\alpha)}\left\|\widehat{II_j^1}\right\|^2_{L^2}\right\}^{\frac{1}{2}}+\left\{\sum_{j\in\mathbb{Z}}2^{2j(1-\alpha)}\left\|\widehat{II_j^2}\right\|^2_{L^2}\right\}^{\frac{1}{2}}\\&\quad+\left\{\sum_{j\in\mathbb{Z}}2^{2j(1-\alpha)}\left\|\widehat{II_j^3}\right\|^2_{L^2}\right\}^{\frac{1}{2}}=:\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_2.\end{aligned}$$

By using the Young inequality in Morrey spaces (2.3) and Bernstein-type inequality with $|\gamma|=0$, we have

$$\left\|\hat{f}_j\right\|_{L^1}\le C2^{j(2-\frac{2-\lambda}{p})}\left\|\hat{f}_j\right\|_{M_p^\lambda}.$$

Then

$$\begin{aligned}\left\|\widehat{II}_j^1\right\|_{L^2}&\le\sum_{|k-j|\le 4}\left\|\dot{S}_{k-1}f\widehat{\Delta}_k g\right\|_{L^2}\le\sum_{|k-j|\le 4}\|\widehat{g}_k\|_{L^2}\sum_{l\le k-2}\left\|\widehat{f}_l\right\|_{L^1}\\&\lesssim\sum_{|k-j|\le 4}\|\widehat{g}_k\|_{L^2}\sum_{l\le k-2}2^{l\left(2-\frac{2-\lambda}{p}\right)}\left\|\widehat{f}_l\right\|_{M_p^\lambda}\\&\lesssim\sum_{|k-j|\le 4}\|\widehat{g}_k\|_{L^2}\sum_{l\le k-2}2^{l\left(2-\frac{2-\lambda}{p}\right)}2^{-l(2\alpha-1)}2^{l(2\alpha-1)}\left\|\widehat{f}_l\right\|_{M_p^\lambda}\\&\lesssim\sum_{|k-j|\le 4}\|\widehat{g}_k\|_{L^2}\left(\sum_{l\le k-2}2^{l(2\alpha-1)q'}\right)^{\frac{1}{q'}}\|f\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}\\&\lesssim\sum_{|k-j|\le 4}2^{k(2\alpha-1)}\|\widehat{g}_k\|_{L^2}\|f\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}.\end{aligned}$$

Multiplying by $2^{2j(1-\alpha)}$, and taking l^2-norm of both sides in the above estimate, we obtain

$$\mathrm{J}_1\lesssim\|f\|_{\dot{H}^{\alpha}}\|g\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}. \tag{4.7}$$

Similary,

$$\begin{aligned}\left\|\widehat{II}_j^2\right\|_{L^2} &\le \sum_{|k-j|\le 4}\left\|\dot{S}_{k-1}g\widehat{\Delta}_k f\right\|_{L^2} \le \sum_{|k-j|\le 4}\left\|\widehat{f}_k\right\|_{L^1}\sum_{l\le k-2}\|\widehat{g}_l\|_{L^2}\\ &\lesssim \|g\|_{L^2}\sum_{|k-j|\le 4}2^{k\left(2-\frac{2-\lambda}{p}\right)}\left\|\widehat{f}_k\right\|_{M_p^\lambda}.\end{aligned}$$

Therefore, by the Young inequality for series, we get

$$\begin{aligned}J_2 &\lesssim \|g\|_{L^2}\left\{\sum_{j\in\mathbb{Z}}2^{2j(1-\alpha)}\left(\sum_{|k-j|\le 4}2^{k\left(2-\frac{2-\lambda}{p}\right)}\left\|\widehat{f}_k\right\|_{M_p^\lambda}\right)^2\right\}^{\frac{1}{2}}\\ &\lesssim \|g\|_{L^2}\left\{\sum_{j\in\mathbb{Z}}\left(\sum_{|k-j|\le 4}2^{(j-k)(1-\alpha)}2^{k(3-\alpha+\frac{\lambda-2}{p})}\left\|\widehat{f}_k\right\|_{M_p^\lambda}\right)^2\right\}^{\frac{1}{2}}\\ &\lesssim \|g\|_{L^2}\|f\|_{\mathcal{FN}_{p,\lambda,2}^{3-\alpha+\frac{\lambda-2}{p}}}\sum_{i\le 3}2^{i(1-\alpha)}\\ &\lesssim \|g\|_{L^2}\|f\|_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}},\end{aligned}$$

where we have used Proposition 2.1 and the fact that $\alpha<1$.

Now we deal with the third term J_3, again employing Young's inequality in Morrey spaces (2.3), the Bernstein-type inequality with $|\gamma|=0$, Hölder's inequality and the fact that $\alpha<1$, we obtain

$$\begin{aligned}\|\widehat{II_j^3}\|_{L^2} &\le \sum_{k\ge j-3}\|\widehat{(\dot{\Delta}_k f\tilde{\dot{\Delta}}_k g)}\|_{L^2} \le \sum_{k\ge j-3}\|\widehat{(\dot{\Delta}_k f}*\widehat{\tilde{\dot{\Delta}}_k g}\|_{L^2}\\ &\le \sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^2}\sum_{|l-k|\le 1}\|\varphi_l\widehat{g}\|_{L^1}\\ &\lesssim \sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^2}\sum_{|l-k|\le 1}2^{(2+\frac{\lambda-2}{p})l}\|\varphi_l\widehat{g}\|_{M_p^\lambda}\\ &\lesssim \sum_{k\ge j-3}\|\varphi_k\widehat{f}\|_{L^2}\Big(\sum_{|l-k|\le 1}2^{l(2\alpha-1)q'}\Big)^{\frac{1}{q'}}\|g\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}\\ &\lesssim \|g\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}\sum_{k\ge j-3}2^{k(2\alpha-1)}\|\varphi_k\widehat{f}\|_{L^2}.\end{aligned}$$

Thus

$$J_3 \lesssim \|g\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}\left\{\sum_{j\in\mathbb{Z}}2^{2j(1-\alpha)}\left(\sum_{k\ge j-3}2^{k(2\alpha-1)}\|\varphi_k\widehat{f}\|_{L^2}\right)^2\right\}^{\frac{1}{2}}$$

$$
\begin{aligned}
&\lesssim \|g\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \left\{ \sum_{j\in\mathbb{Z}} \left(\sum_{k\geq j-3} 2^{(j-k)(1-\alpha)} 2^{k\alpha} \|\varphi_k \widehat{f}\|_{L^2} \right)^2 \right\}^{\frac{1}{2}} \\
&\lesssim \|g\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \|f\|_{F\dot{B}^{\alpha}_{2,2}} \sum_{i\leq 3} 2^{i(1-\alpha)} \\
&\lesssim \|f\|_{\dot{H}^{\alpha}} \|g\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}.
\end{aligned}
$$

Combining J_1, J_2 and J_3, we conclude the desired result. □

5 Proof of Theorems

We are now ready to give the proof of Theorems 1.1 and 1.2.

5.1 Proof of Theorem 1.1

From Proposition 4.1, we have

$$
\|\mathcal{B}(v,\psi)\|_Y \leq \eta \|v\|_Y \|\psi\|_Y. \tag{5.1}
$$

By Lemma 3.1, we have the linear part in (1.6) satisfied

$$
\|\mathcal{H}_\alpha(t) v_0\|_Y \leq C_0 \|v_0\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}. \tag{5.2}
$$

So, if

$$
\|v_0\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} < \beta \text{ with } \beta = \frac{1}{4\eta C_0},
$$

then Lemma 2.4 together with the estimates (5.1) and (5.2) assures that Eq. (1.1) has a unique global solution $v\in Y$ such that

$$
\|v\|_Y \leq 2C_0 \|v_0\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}.
$$

The continuity with respect to time is standard. □

5.2 Proof of Theorem 1.2

To show the asymtotic stability for the global solution, we follow ideas from [34], in which the authors used standard interpolation in the Fourier space and energy estimates in L^2, (see [5,25,26]).

Let $\varepsilon > 0$, such that $\varepsilon \leq \beta$. For $m\in\mathbb{N}$, setting

$$
S_m = \left\{ \xi \in \mathbb{R}^2; |\xi| \leq m \text{ and } |\widehat{v}_0(\xi)| \leq m \right\}.
$$

Clearly

$$\mathcal{F}^{-1}(\chi_{S_m}\hat{v}_0)\to v_0 \text{ in } \mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}.$$

Then, there is $m_0\in\mathbb{N}$ such that

$$\left\|v_0-\mathcal{F}^{-1}(\chi_{S_m}\hat{v}_0)\right\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}\le\frac{\varepsilon}{2},\quad \forall m\ge m_0.$$

Let m be a fixed integer such that $m\ge m_0$.

Set $v_{0,m}=\mathcal{F}^{-1}(\chi_{S_m}\hat{v}_0)$, $b_{0,m}=v_0-\mathcal{F}^{-1}(\chi_{S_m}\hat{v}_0)$. Therefore, we have proved that

$$\|b_{0,m}\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}\le\frac{\varepsilon}{2},\qquad v_{0,m}\in\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\cap\mathrm{L}^2.$$

Next, we consider the system

$$\begin{cases}\partial_t b_m+\Lambda^{2\alpha}b_m+u_{b_m}\cdot\nabla b_m=0, & x\in\mathbb{R}^2, t>0,\\ b_m(0,x)=b_{0,m}(x).\end{cases}\tag{5.3}$$

For all $m\ge m_0$, we have $\|b_{0,m}\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}\le\frac{\varepsilon}{2}$. So we can conclude from Theorem 1.1 that there exists a unique global solution

$$b_m\in\mathcal{C}\left(\mathbb{R}^+,\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)\cap\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right).$$

In addition, $\forall t\ge 0$ we have

$$\|b_m(t)\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}+\|b_m\|_{\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)}\lesssim\|b_{0,m}\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}}.$$

Put

$$v=v-b_m+b_m=:v_m+b_m,$$

with v is the solution of Eq. (1.1). Then, v_m is a solution of the system:

$$\begin{cases}\partial_t v_m+\Lambda^{2\alpha}v_m+u_{v_m}\cdot\nabla v_m+u_{v_m}\cdot\nabla b_m+u_{b_m}\cdot\nabla v_m=0, & x\in\mathbb{R}^2,\ t>0,\\ v_m(0,x)=v_{0,m}(x)\in\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\cap\mathrm{L}^2.\end{cases}$$

Furthermore,

$$v_m\in\mathcal{C}\left(\mathbb{R}^+,\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)\cap\mathcal{L}^1\left(\mathbb{R}^+,\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right).$$

By taking the inner products in $L^2(\mathbb{R}^2)$ with v_m and integrating by parts, we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m\|_{L^2}^2+\|\Lambda^\alpha v_m\|_{L^2}^2\le|<u_{v_m}\cdot\nabla b_m,v_m>_{L^2}|.$$

Since

$$\|\Lambda^\alpha v_m\|_{L^2}=\|v_m\|_{\dot{H}^\alpha}.$$

Thus, Cauchy-Schwarz inequality, the estimate (4.3), Lemma 4.2, and Young's inequality give

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_m\|_{L^2}^2+\|v_m\|_{\dot{H}^\alpha}^2\\
&\le|<\mathrm{div}(u_{v_m}b_m),v_m>_{L^2}|\lesssim\left\|\Lambda^{1-\alpha}(u_{v_m}b_m)\right\|_{L^2}\|\Lambda^\alpha v_m\|_{L^2}\\
&\lesssim\|u_{v_m}b_m\|_{\dot{H}^{1-\alpha}}\|v_m\|_{\dot{H}^\alpha}\lesssim\|v_mb_m\|_{\dot{H}^{1-\alpha}}\|v_m\|_{\dot{H}^\alpha}\\
&\lesssim\|v_m\|_{L^2}\|b_m\|_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}}\|v_m\|_{\dot{H}^\alpha}+\|b_m\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}\|v_m\|_{\dot{H}^\alpha}^2\\
&\lesssim\|v_m\|_{L^2}^2\|b_m\|^2_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}}+\|v_m\|_{\dot{H}^\alpha}^2+\|b_m\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}\|v_m\|_{\dot{H}^\alpha}^2\\
&\lesssim\|v_m\|_{L^2}^2\|b_m\|^2_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}}+\|v_m\|_{\dot{H}^\alpha}^2,
\end{aligned}\tag{5.4}$$

where in (5.4) we used that $\|b_m\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}}\le\frac{\varepsilon}{2}$. Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_m\|_{L^2}^2+\|v_m\|_{\dot{H}^\alpha}^2\lesssim\|v_m\|_{L^2}^2\|b_m\|^2_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}}.$$

Integrating with respect to time, we obtain

$$\|v_m\|_{L^2}^2+\int_0^t\|v_m\|_{\dot{H}^\alpha}^2\lesssim\|v_{0,m}\|_{L^2}^2+\int_0^t\|v_m\|_{L^2}^2\|b_m\|^2_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}}.\tag{5.5}$$

By Gronwalls lemma, we get

$$\|v_m\|_{L^2}^2\lesssim\|v_{0,m}\|_{L^2}^2\exp\int_0^t\|b_m\|^2_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}}\lesssim\|v_{0,m}\|_{L^2}^2,\tag{5.6}$$

where the following fact is used in the last inequality:

$$\int_0^t\|b_m\|^2_{\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}}\le C.$$

Indeed, note that $q\le2$, then by Minkowski's inequality (2.9) we have

$$\mathcal{L}^2\left(I,\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}\right)\hookrightarrow L^2\left(I;\mathcal{FN}_{p,\lambda,q}^{3-\alpha+\frac{\lambda-2}{p}}\right).\tag{5.7}$$

Thus, using (5.7) and Hölder's inequality for series, we get

$$
\begin{aligned}
\int_0^t \|b_m\|^2_{\mathcal{FN}^{3-\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} &\lesssim \left\| 2^{j\left(3-\alpha+\frac{\lambda-2}{p}\right)} \left\| \varphi_j \hat{b}_m \right\|_{L^2([0,t),\mathrm{M}^\lambda_p)} \right\|^2_{l^q} \\
&\lesssim \left\| 2^{\frac{j}{2}\left(3-2\alpha+\frac{\lambda-2}{p}\right)} 2^{\frac{j}{2}\left(3+\frac{\lambda-2}{p}\right)} \left\| \varphi_j \hat{b}_m \right\|^{\frac{1}{2}}_{L^\infty([0,t),\mathrm{M}^\lambda_p)} \left\| \varphi_j \hat{b}_m \right\|^{\frac{1}{2}}_{L^1([0,t),\mathrm{M}^\lambda_p)} \right\|^2_{l^q} \\
&\lesssim \left\| 2^{j\left(3-2\alpha+\frac{\lambda-2}{p}\right)} \left\| \varphi_j \hat{b}_m \right\|_{L^\infty([0,t),\mathrm{M}^\lambda_p)} \right\|_{l^q} \left\| 2^{j\left(3+\frac{\lambda-2}{p}\right)} \left\| \varphi_j \hat{b}_m \right\|_{L^1([0,t),\mathrm{M}^\lambda_p)} \right\|_{l^q} \\
&\lesssim \|b_m\|_{\mathcal{L}^\infty\left([0,t),\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \|b_m\|_{\mathcal{L}^1\left([0,t),\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \\
&\lesssim \left(\|b_m\|_{\mathcal{L}^\infty\left([0,t),\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} + \|b_m\|_{\mathcal{L}^1\left([0,t),\mathcal{FN}^{3+\frac{\lambda-2}{p}}_{p,\lambda,q}\right)} \right)^2 \\
&\lesssim \|b_{0,m}\|^2_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \leq C.
\end{aligned}
$$

Combining (5.5) and (5.6), we obtain

$$\|v_m\|^2_{L^2} + \int_0^t \|v_m\|^2_{\dot{H}^\alpha} \leq \|v_{0,m}\|^2_{L^2} + \|v_{0,m}\|^2_{L^2} \int_0^t \|b_m\|^2_{\mathcal{FN}^{3-\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \leq C.$$

Applying Lemma 4.1, we get

$$\|v_m\|^{\frac{\alpha}{1-\alpha}}_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \lesssim \|v_m\|^{\frac{3\alpha-2}{1-\alpha}}_{L^2} \|v_m\|^2_{\dot{H}^\alpha}. \tag{5.8}$$

Therefore,

$$\|v_m\|^{\frac{\alpha}{1-\alpha}}_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \lesssim \|v_m\|^2_{\dot{H}^\alpha}, \tag{5.9}$$

where we have used (5.6). Finally, by integrating in time between 0 and ∞, we obtain

$$\int_0^\infty \|v_m\|^{\frac{\alpha}{1-\alpha}}_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}} \leq C \int_0^\infty \|v_m\|^2_{\dot{H}^\alpha}. \tag{5.10}$$

Let us consider the following subset of $[0,\infty[$:

$$\mathcal{A}_\varepsilon = \left\{ t \geq 0;\ \|v_m(t)\|_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}(\mathbb{R}^2)} \geq \frac{\varepsilon}{2} \right\}.$$

Then, for all $\frac{1}{2} < \alpha < 1$,

$$\int_0^\infty \|v_m(t)\|^{\frac{\alpha}{1-\alpha}}_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}(\mathbb{R}^2)} \mathrm{d}t \geq \int_{\mathcal{A}_\varepsilon} \|v_m(t)\|^{\frac{\alpha}{1-\alpha}}_{\mathcal{FN}^{3-2\alpha+\frac{\lambda-2}{p}}_{p,\lambda,q}(\mathbb{R}^2)} \mathrm{d}t \geq \mu(\mathcal{A}_\varepsilon) \left(\frac{\varepsilon}{2}\right)^{\frac{\alpha}{1-\alpha}}, \tag{5.11}$$

where $\mu(\mathcal{A}_\varepsilon)$ is the Lebesgue measure of $\mathcal{A}_\varepsilon$.

Using the estimates (5.10) and (5.11), we get

$$\mu(\mathcal{A}_\varepsilon) < \infty \text{ and } \mu(\mathcal{A}_\varepsilon) \lesssim \left(\frac{2}{\varepsilon}\right)^{\frac{\alpha}{1-\alpha}} \int_0^\infty \|v_m\|_{\dot{H}^\alpha}^2.$$

For $\eta > 0$, there exists $t_0 \in [0, \mu(\mathcal{A}_\varepsilon) + \eta]$ such that $t_0 \notin \mathcal{A}_\varepsilon$. Then,

$$\|v_m(t_0)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} \leq \frac{\varepsilon}{2}.$$

Therefore we have

$$\begin{aligned}
\|v(t_0)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} &\leq \|v_m(t_0)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} + \|b_m(t_0)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} \\
&\leq \frac{\varepsilon}{2} + \|b_{0,m}\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

Consequently

$$\|v(t_0)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} \leq \varepsilon. \tag{5.12}$$

Now, we consider the quasi-geostrophic equations starting at $t = t_0$.

$$\begin{cases} \partial_t Z + u_Z \cdot \nabla Z + \Lambda^{2\alpha} Z = 0, & x \in \mathbb{R}^2, t > 0, \\ Z(0,x) = Z_0 = v(t_0). \end{cases} \tag{5.13}$$

Using inequality (5.12) and Theorem 1.1, we infer that there exists a unique solution

$$Z \in \mathcal{C}\left(\mathbb{R}^+, \mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}\right) \cap \mathcal{L}^1\left(\mathbb{R}^+, \mathcal{FN}_{p,\lambda,q}^{3+\frac{\lambda-2}{p}}\right)$$

of the problem (1.1).

The existence and uniqueness of a solution to the quasi-geostrophic equation gives $\forall t \geq 0$, $Z(t) = v(t_0 + t)$. Then,

$$\|v(t_0+t)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} = \|Z(t)\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} \leq \|Z_0\|_{\mathcal{FN}_{p,\lambda,q}^{3-2\alpha+\frac{\lambda-2}{p}}(\mathbb{R}^2)} \leq \varepsilon.$$

This completes the proof of Theorem 1.2. □

Remark 5.1. If $k \neq 1$, the results of Theorems 1.1 and 1.2 remain true, but with more complicated calculations.

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