

A De Giorgi Type Result to Divergence Degenerate Elliptic Equation with Bounded Coefficients Related to Hörmander's Vector Fields

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Abstract. In this paper, we consider the divergence degenerate elliptic equation with bounded coefficients constructed by Hörmander's vector fields. We prove a De Giorgi type result, i.e., the local Hölder continuity for the weak solutions to the equation by providing a De Giorgi type lemma and extending the Moser iteration to the setting here. As a consequence, the Harnack inequality of weak solutions is also given.

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1 Introduction

In 1957, De Giorgi has found the local Hölder continuity of weak solutions to the following divergence elliptic equation with bounded coefficients

$$Lu = - \sum_{i,j=1}^n D_i \left(a^{ij}(x) D_j u \right) = 0, \quad x \in \mathbb{R}^n$$

and a priori estimate of Hölder norm (see [1]). Nash in [2] used a different approach and derived the similar result to the parabolic equation with bounded coefficients. In [3] Hou and Niu have taken into account of Nash's approach to obtain the Hölder regularity and Harnack inequality to divergence parabolic equation related to Hörmander's vector fields. Moser [4] developed a new method (nowadays it has been called the Moser iteration) and applied it to prove forenamed results with respect to elliptic and parabolic equations. These important ideas opened a new prospect for the study of regularity to

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partial differential equations. A natural and interesting problem is whether De Giorgi's result is true to the divergence degenerate elliptic equation related to Hörmander's vector fields. We will affirmatively answer it.

Hörmander introduced the square sum operator constructed by smooth vector fields and proved that it is hypoelliptic if vector fields satisfy the finite rank condition (see [5]). Many authors continued his research ([6–11]) and obtained numerous insight, such as, fundamental solutions ([12]), the Poincaré inequality ([13]), potential estimates ([14]) and sub-elliptic estimates ([15, 16]). Nagel, Stein and Wainger ([17]) deduced the basic properties of balls and metrics defined by Hörmander's vector fields, which are the starting point for treating many problems on the Hörmander square sum operator and related sub-elliptic operators. Lu found the Harnack inequality and Hölder continuity for solutions to quasilinear degenerate elliptic equations formed by Hörmander's vector fields, see [18, 19].

Rothschild and Stein in [11] have proved regularity to the second order subelliptic equation. Xu and Zuily in [20] dealt with the interior regularity of weak solutions to the quasilinear degenerate elliptic system

$$\sum_{i,j=1}^q X_j^* \left(a^{ij}(x,u) X_i u^\alpha \right) = f^\alpha(x,u, Xu).$$

The Hölder regularity and Harnack inequality of the functions in the De Giorgi class related to Hörmander's vector fields are arrived at by Marchi in [21]. Bramanti and Brandolini in [22] gave regularity to the nondivergence degenerate elliptic equation of Hörmander's vector fields. The partial Hölder regularity for weak solutions to the quasilinear degenerate elliptic system was settled by Gao, Niu and Wang [23]. Dong and Niu in [24] obtained regularity of weak solutions to the nondiagonal quasilinear degenerate elliptic system

$$-X_\alpha^* \left(a_{ij}^{\alpha\beta}(x,u) X_\beta u^j \right) = g_i(x,u, Xu) - X_\alpha^* f_i^\alpha(x,u, Xu).$$

Schauder estimates to degenerate elliptic operators related to noncommutative vector fields have been derived in [25, 26] etc.

Throughout this paper we are concerned with the following divergence degenerate elliptic equation with bounded coefficients:

$$X_j^* \left(a^{ij}(x) X_i u \right) + b_i(x) X_i u + c(x) u = f(x) - X_i^* f^i(x) \quad \text{in } \Omega, \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^n , $X_i = \sum_{k=1}^n b_{ik}(x) \frac{\partial}{\partial x_k}$ ($b_{ik}(x) \in C^\infty(\Omega)$, $i = 1, \dots, q$, $q \leq n$) are smooth vector fields satisfying the finite rank condition, and the summation symbols in (1.1) are omitted. We assume that there exists $\Lambda > 0$, such that

$$\Lambda^{-1} |\tilde{\xi}|^2 \leq a^{ij}(x) \tilde{\xi}_i \tilde{\xi}_j \leq \Lambda |\tilde{\xi}|^2, \quad \text{for } x \in \Omega, \tilde{\xi} \in \mathbb{R}^q, \quad (1.2)$$

$$\sum_{i,j} \|a^{ij}\|_{L^\infty(\Omega)} + \sum_i \|b^i\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)} \leq \Lambda, \tag{1.3}$$

$$f \in L^{\frac{pQ}{Q+p}}(\Omega) \quad \text{and} \quad f^i \in L^p(\Omega), \quad (i=1, \dots, q), \tag{1.4}$$

for $p > Q$, Q is the local homogeneous dimension relative to Ω (see (2.2) below). Here we refer the corresponding results of (1.1) when $X_i = \frac{\partial}{\partial x_i}$, $i=1, \dots, n$ to [27].

Let $H_{loc}^1(\Omega; X)$ be the local Sobolev space of Hörmander’s vector fields.

A function $u \in H_{loc}^1(\Omega; X)$ is said a weak sub-solution (super-solution) to Eq. (1.1), if

$$\int_{\Omega} \left(a^{ij} X_i u X_j \varphi + b_i X_i u \cdot \varphi + c u \varphi \right) dx \leq (\geq) \int_{\Omega} \left(f \cdot \varphi + f^i X_i \varphi \right) dx, \tag{1.5}$$

for any $\varphi \in H_0^1(\Omega; X)$ with $\varphi \geq 0$. If $u \in H_{loc}^1(\Omega; X)$ is both a sub-solution and a super-solution to (1.1), then u is said a weak solution to (1.1), where (1.5) becomes an integral equality and $\varphi \geq 0$ is not needed.

Let us describe the De Giorgi class in the frame of Carnot-Carathéodory distance induced by Hörmander’s vector fields.

We say that $u \in H_{loc}^1(\Omega; X)$ belongs to the De Giorgi class, denoted by $DG(\Omega)$, if $\|u\|_{L^\infty(\Omega)} \leq M$ and there exists $\delta \in (0, 1)$, such that for any real k satisfying

$$0 < \text{esssup}_{B_R} (u - k)_{\pm} \leq \delta, \quad B_R \subset \Omega, \quad (0 < R < 1),$$

the following inequalities

$$\int_{A_{k,\sigma R}^{\pm}} |Xu|^2 dx \leq \frac{\gamma_0}{(1-\sigma)^2 R^2} \int_{B_R} (u - k)_{\pm}^2 dx + \gamma_1 |A_{k,R}^{\pm}|^{1-\frac{2}{\eta}} \tag{1.6}$$

are valid, where $\sigma \in (0, 1]$, γ_0 and γ_1 are positive constants, $\eta > Q$, and

$$A_{k,R}^+ = \{x \in B_R \mid u(x) \geq k\}, \quad A_{k,R}^- = \{x \in B_R \mid u(x) < k\}.$$

If (1.6)₊ (or (1.6)₋) is valid, then we say $u \in DG^+(\Omega)$ (or $u \in DG^-(\Omega)$). Obviously,

$$DG(\Omega) = DG^+(\Omega) \cap DG^-(\Omega).$$

Clearly, $A_{k,R}^+$ is decreasing and $A_{k,R}^-$ is increasing on k . Let us note that (1.6) here is not same as one in [21].

Now we state the main results of this paper.

Theorem 1.1. (Hölder Regularity) *Let $u \in H_{loc}^1(\Omega; X)$ be the bounded weak solution to (1.1) with (1.2), (1.3) and (1.4). Then for any $B_R(x) \subset \subset \Omega$, $R \in (0, R_0]$, $0 < R_0 \leq 1$, there exists $0 < \alpha \leq 1 - \frac{Q}{\eta}$ in which $\eta > Q$, such that $u \in C_{loc}^\alpha(\Omega)$ and*

$$[u]_{\alpha; B_R} \leq C d_x^{-\alpha} \left(M + \gamma_1^{\frac{1}{2}} d_x |\Omega|^{-\frac{1}{\eta}} \right), \tag{1.7}$$

where $d_x = \text{dist}\{x, \partial\Omega\}$ and $C \geq 1$ depends on η, δ, Q and Λ .

Theorem 1.2. (Harnack Inequality) *Suppose that conditions in Theorem 1.1 are satisfied and $u \geq 0$ on $B_{8R}(x_0) \subset \Omega$ ($R \in (0, R_0)$, $0 < R_0 \leq 1$), then*

$$\inf_{B_R(x_0)} u \geq C^{-1} u(x_0) - C^2 \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \quad (1.8)$$

where $C > 0$ depends on η, δ, Q and Λ .

Remark 1.1. Because of the complexity of geometry structure caused by Hörmander's vector fields, the Lebesgue measure $|B_R|$ and R^Q are not equivalent (see [17]). We do not use the equivalence in our proofs of main results. It is a lightspot in the paper.

The paper is organized as follows: Section 2 contains notions related to Hörmander's vector fields, the Campanato space, the Hölder space, the Sobolev inequality and Poincaré inequality of Hörmander's vector fields. A De Giorgi type lemma is inferred which is not same as in [4]. In Section 3 we prove that weak solutions to (1.1) are actually in the De Giorgi class $DG(\Omega)$ and derive oscillation estimates for functions in $DG(\Omega)$. Section 4 is devoted to proofs of main results. The proof of Theorem 1.1 is based on oscillation estimates of weak solutions and the isomorphism between the Campanato space and Hölder space. Theorem 1.2 is proved by giving an extension lemma of positivity of weak solutions and using Theorem 1.1.

2 Preliminaries

Let X_1, \dots, X_q ($q < n$) be C^∞ vector fields in the domain $\Omega \subset \mathbb{R}^n$. We say that these vector fields satisfy the finite rank condition in Ω , if the vector space spanned by X_1, \dots, X_q and their commutators up to r step is the whole \mathbb{R}^N at every point in Ω (more definitions are referred to [5]), i.e.,

$$\text{rank Lie}[X_1, \dots, X_q] \equiv N.$$

An absolutely continuous curve $\gamma : [0, T] \rightarrow \Omega$ is called sub-unitary with respect to vector fields X_1, \dots, X_q , if $\gamma'(t)$ exists and for every $\xi \in \mathbb{R}^N$ and $t \in [0, T]$ a.e., it holds

$$\langle \gamma'(t), \xi \rangle^2 \leq \sum_{j=1}^q \langle X_j(\gamma(t)), \xi \rangle^2.$$

Denote by $\Phi(x, y)$ the collection of sub-unitary curves connecting x and y .

Definition 2.1 (Carnot-Carathéodory distance, [17, 28, 29]). *The Carnot-Carathéodory distance (C-C distance) is defined by*

$$d(x, y) = \inf\{T \geq 0 : \gamma \in \Phi(x, y)\}.$$

The C-C ball of the centre $x_0 \in \Omega$ and radius $R \geq 0$ is the set

$$B(x_0, R) = B_R(x_0) = \{x \in \Omega : d(x_0, x) < R\}.$$

The Lebesgue measure of $B(x_0, R)$ is expressed as $|B_R(X_0)|$. From [17], there exist positive constants c_1 and c_2 such that for any $x, y \in \Omega$,

$$c_1|x-y| \leq d(x, y) \leq c_2|x-y|^{\frac{1}{r}},$$

where $|\cdot|$ means the Euclidean norm, and there are positive constants c_3 and R_0 , such that for $x_0 \in \Omega, B(x_0, 2R) \subset \subset \Omega, 0 \leq 2R \leq R_0$, the doubling property holds:

$$|B(x_0, 2R)| \leq c_3|B(x_0, R)|. \quad (2.1)$$

We have by (2.1) that for any $0 \leq R \leq R_0$ and $\theta \in (0, 1)$,

$$|B_{\theta R}| \geq c_3^{-1}\theta^Q|B_R|, \quad (2.2)$$

where $Q = \log_2 c_3$ acts as a dimension. It knows from [22, 30] that $B(x_0, R)$ is a homogeneous space which allows us to apply known statements in the homogeneous space in the sequel.

We define the Sobolev spaces $W^{1,p}(\Omega; X)$ and $W_0^{1,p}(\Omega; X)$ ($p \geq 1$), which are the closures of $C^1(\Omega)$ and $C_0^1(\Omega)$ under the norm

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_{\Omega} (|u|^p + |Xu|^p) dx \right]^{\frac{1}{p}}, \quad (2.3)$$

respectively, where $Xu = (X_1u, \dots, X_qu)$. If $p=2$, we simply denote $H^1(\Omega; X) = W^{1,2}(\Omega; X)$, $H_0^1(\Omega; X) = W_0^{1,2}(\Omega; X)$, and $H_{loc}^1(\Omega; X) = \{u \in H^1(\Omega'; X); \Omega' \subset \subset \Omega\}$.

Definition 2.2 (Hölder space). For $\alpha \in (0, 1]$, the Hölder space $C^\alpha(\bar{\Omega})$ is a collection of α -Hölder continuous functions with the norm

$$\|u\|_{C^\alpha(\bar{\Omega})} = \sup_{\Omega} |u| + \sup_{\bar{\Omega}} \frac{|u(x) - u(y)|}{[d(x, y)]^\alpha} < +\infty.$$

Let $C_{loc}^\alpha(\Omega)$ stand for the local Hölder space $\{u \in C^\alpha(\bar{\Omega}'); \Omega' \subset \subset \Omega\}$.

Definition 2.3 (Campanato space). For $1 \leq p \leq +\infty$, $\lambda \geq 0$, $u \in L^p(\Omega)$, if

$$[u]_{p,\lambda} = \left\{ \sup_{x \in \Omega, 0 \leq R \leq d} \left(\frac{R^{-\lambda}}{|\Omega_R|} \int_{\Omega_R} |u(x) - u_{\Omega_R}|^p dx \right)^{\frac{1}{p}} \right\} < +\infty,$$

where $d = \text{diam} \Omega$, $\Omega_R = \Omega \cap B_R(y)$, $u_{\Omega_R} = \frac{1}{|\Omega_R|} \int_{\Omega_R} u(x) dx$, then we say that u belongs to the Campanato space $\mathcal{L}^{p,\lambda}(\Omega)$ with the norm

$$\|u\|_{p,\lambda} = [u]_{p,\lambda} + \|u\|_{L^p}.$$

Lemma 2.1 ([28,29]). For $0 < \lambda \leq p$ and $\alpha = \lambda/p$, it follows

$$\mathcal{L}^{p,\lambda}(\Omega) \cong C^\alpha(\overline{\Omega}).$$

Lemma 2.2 (Sobolev Inequality, [31,32]). For $1 \leq p < Q$, there exist $C > 0$ and $R_0 > 0$ such that for any $x \in \Omega$, $B_R = B(x, R)$, $0 < R \leq R_0$, it implies for any $u \in W_0^{1,p}(B_R)$,

$$\left(\frac{1}{|B_R|} \int_{B_R} |u|^{pk} dx \right)^{\frac{1}{pk}} \leq CR \left(\frac{1}{|B_R|} \int_{B_R} |Xu|^p dx \right)^{\frac{1}{p}}, \quad (2.4)$$

where $1 \leq k \leq Q/(Q-p)$.

Let $p=2$ and $k=Q/(Q-2)$ in (2.4), we immediately have

Corollary 2.1. For $u \in H_{loc}^1(\Omega; X)$, there exist $C > 0$ and $R_0 > 0$ such that for any $x \in \Omega$, $B_R = B(x, R) \subset \Omega$, $0 < R \leq R_0$,

$$\left(\int_{B_R} |u|^{2^*} dx \right)^{\frac{1}{2^*}} \leq C |B_R|^{\frac{1}{2^*} - \frac{1}{2}} \left(R^2 \int_{B_R} |Xu|^2 dx \right)^{\frac{1}{2}}. \quad (2.5)$$

Here $2^* = 2Q/(Q-2)$.

Lemma 2.3 (Poincaré Inequality, [13]). Let $u \in W^{1,p}(B_R)$ ($p \geq 1$), then for any $x \in \Omega$, $B_R = B(x, R) \subset \Omega$,

$$\int_{B_R} |u(x) - u_{B_R}|^p dx \leq CR^p \int_{B_R} |Xu|^p dx, \quad (2.6)$$

where $u_{B_R} = |B_R|^{-1} \int_{B_R} u(x) dx$.

Using (2.6), we can prove

Theorem 2.1 (De Giorgi type Lemma). Let $u \in W^{1,2}(B_R)$ and denote $A(k) = \{x \in B_R | u(x) > k\}$, then for $l > k$,

$$(l-k) |A(l)| \leq C \frac{|B_R|}{|B_R - A(k)|} \left(R^2 \int_{A(k) - A(l)} |Xu|^2 dx \right)^{\frac{1}{2}} |A(k) - A(l)|^{\frac{1}{2}}, \quad (2.7)$$

where C relies only on Q .

Proof. It is evident to see $A(l) \subset A(k)$ for $l > k$. Let $p=1$ in (2.6), it yields

$$\int_{B_R} |u(x) - u_{B_R}| dx \leq CR \int_{B_R} |Xu| dx. \quad (2.8)$$

Denote $N_0 = \{x \in B_R | u(x) = 0\}$, then

$$|N_0| |u_{B_R}| = \int_{N_0} |u_{B_R}| dx = \int_{N_0} |u(x) - u_{B_R}| dx \leq \int_{B_R} |u(x) - u_{B_R}| dx \leq CR \int_{B_R} |Xu| dx$$

and

$$\int_{B_R} |u_{B_R}| dx = |B_R| |u_{B_R}| \leq CR \frac{|B_R|}{|N_0|} \int_{B_R} |Xu| dx.$$

By using (2.8), we arrive at

$$\begin{aligned} \int_{B_R} |u(x)| dx &\leq \int_{B_R} |u(x) - u_{B_R}| dx + \int_{B_R} |u_{B_R}| dx \\ &\leq CR \int_{B_R} |Xu| dx + CR \frac{|B_R|}{|N_0|} \int_{B_R} |Xu| dx \\ &\leq CR \frac{|B_R|}{|N_0|} \int_{B_R} |Xu| dx. \end{aligned} \quad (2.9)$$

Applying (2.9) to function

$$\hat{u}(x) = \begin{cases} l-k, & x \in A(l), \\ u(x) - k, & x \in A(k) - A(l), \\ 0, & x \in B_R - A(k), \end{cases}$$

it follows that

$$\begin{aligned} (l-k) |A(l)| &\leq \int_{B_R} |\hat{u}(x)| dx \\ &\leq \frac{CR |B_R|}{|B_R - A(k)|} \int_{B_R} |X\hat{u}| dx \\ &\leq C \frac{|B_R|}{|B_R - A(k)|} \left(R^2 \int_{A(k) - A(l)} |Xu|^2 dx \right)^{\frac{1}{2}} |A(k) - A(l)|^{\frac{1}{2}} \end{aligned}$$

and (2.7) is proved. \square

Remark 2.1. We stress that (2.7) here is not same as the corresponding inequality in [21].

Lemma 2.4 (Existence of cut-off functions, [21, 32]). *If $0 < s < t$ and $x_0 \in \mathbb{R}^n$, then there exists a Lipschitz function ϕ such that $\phi = 1$ in $B_s(x_0)$, $\phi = 0$ in $\mathbb{R}^n \setminus B_t(x_0)$ and*

$$\sum_{i=1}^q |X_i \phi|^2 \leq \frac{C}{(t-s)^2}.$$

3 Some auxiliary lemmas

We have the following result for the weak sub-solution (super-solution) to (1.1):

Lemma 3.1. *Let $u \in H_{loc}^1(\Omega; X)$ be the bounded weak sub-solution (or super-solution) to (1.1) with (1.2), (1.3) and (1.4), then*

$$u \in DG^+(\Omega) \text{ (or } u \in DG^-(\Omega)\text{)}.$$

Proof. We only prove the conclusion for the bounded weak sub-solution, the proof for the other case is similar.

Taking the test function $\zeta^2(u-k)_+$ to (1.1), here ζ is the cut-off function between $B_{\sigma\rho}$ ($0 < \sigma < 1$) and B_ρ , it shows

$$\begin{aligned} & \int_{B_\rho} a^{ij} X_i u X_j (\zeta^2(u-k)_+) dx + \int_{B_\rho} (b^i X_i u + cu) \zeta^2(u-k)_+ dx \\ & \leq \int_{B_\rho} [f^i X_i (\zeta^2(u-k)_+) + f \zeta^2(u-k)_+] dx. \end{aligned} \quad (3.1)$$

Noting $X_i u = X_i(u-k)_+$ and

$$\sum_{i,j=1}^q X_i (\zeta(u-k)_+) (u-k)_+ X_j \zeta - X_j (\zeta(u-k)_+) (u-k)_+ X_i \zeta = 0,$$

it infers

$$\begin{aligned} & X_i u X_j (\zeta^2(u-k)_+) = X_i u X_j (\zeta \cdot \zeta(u-k)_+) \\ & = \zeta X_i u [(u-k)_+ X_j \zeta + X_j (\zeta(u-k)_+)] \\ & = [X_i (\zeta(u-k)_+) - (u-k)_+ X_i \zeta] [(u-k)_+ X_j \zeta + X_j (\zeta(u-k)_+)] \\ & = X_i (\zeta(u-k)_+) X_j (\zeta(u-k)_+) - (u-k)_+^2 X_i \zeta X_j \zeta \end{aligned}$$

and puts it into (3.1) to obtain

$$\begin{aligned} & \int_{B_\rho} a^{ij} X_i (\zeta(u-k)_+) X_j (\zeta(u-k)_+) - (u-k)_+^2 X_i \zeta X_j \zeta dx \\ & \leq \int_{B_\rho} (u-k)_+^2 a^{ij} X_i \zeta X_j \zeta dx - \int_{B_\rho} (b^i X_i u + cu) \zeta^2(u-k)_+ dx \\ & \quad + \int_{B_\rho} [f^i X_i (\zeta^2(u-k)_+) + f \zeta^2(u-k)_+] dx \\ & =: I_1 + I_2 + I_3. \end{aligned} \quad (3.2)$$

Applying (1.2), it gets

$$I_1 \leq \Lambda \int_{B_\rho} (u-k)_+^2 |X\zeta|^2 dx.$$

By using the Cauchy inequality to I_2 , we have

$$I_2 \leq \int_{B_\rho} |b^i X_i u \zeta^2(u-k)_+| dx + \int_{B_\rho} |cu \zeta^2(u-k)_+| dx =: I_{21} + I_{22};$$

since

$$I_{21} \leq \int_{B_\rho} |b^i \zeta(u-k)_+ \zeta X_i u| dx$$

$$\begin{aligned}
&\leq C(\epsilon) \int_{A_{k,\rho}^+} \zeta^2(u-k)_+^2 \left[\sum (b^i)^2 \right] dx + \epsilon \int_{A_{k,\rho}^+} |\zeta X_i u|^2 dx \\
&= C(\epsilon) \int_{A_{k,\rho}^+} \zeta^2(u-k)_+^2 \left[\sum (b^i)^2 \right] dx + \epsilon \int_{A_{k,\rho}^+} [X_i(\zeta(u-k)_+) - (u-k)_+ X_i \zeta]^2 dx \\
&\leq C(\epsilon) \int_{A_{k,\rho}^+} \zeta^2(u-k)_+^2 \left[\sum (b^i)^2 \right] dx + 2\epsilon \int_{A_{k,\rho}^+} |X_i(\zeta(u-k)_+)|^2 + (u-k)_+^2 |X\zeta|^2 dx
\end{aligned}$$

and

$$\begin{aligned}
I_{22} &\leq \int_{B_\rho} \left| \sqrt{|c|} \zeta(u-k)_+ \sqrt{|c|} u \zeta \right| dx \\
&\leq C(\epsilon) \int_{A_{k,\rho}^+} |c| \zeta^2(u-k)_+^2 dx + \epsilon \int_{A_{k,\rho}^+} |c| ((u-k)_+ + k)^2 \zeta^2 dx \\
&\leq C(\epsilon) \int_{A_{k,\rho}^+} |c| \zeta^2(u-k)_+^2 dx + 2\epsilon \int_{A_{k,\rho}^+} |c| ((u-k)_+^2 + k^2) \zeta^2 dx \\
&\leq C(\epsilon) \int_{A_{k,\rho}^+} |c| \zeta^2(u-k)_+^2 + |c| k^2 \zeta^2 dx,
\end{aligned}$$

where $A_{k,\rho}^+ = \{x \in B_\rho \mid u(x) > k\}$, it follows

$$\begin{aligned}
I_2 &\leq C(\epsilon) \int_{A_{k,\rho}^+} \left\{ \zeta^2(u-k)_+^2 \left[\sum (b^i)^2 + |c| \right] + |c| k^2 \zeta^2 \right\} dx \\
&\quad + 2\epsilon \int_{A_{k,\rho}^+} |X_i(\zeta(u-k)_+)|^2 + (u-k)_+^2 |X\zeta|^2 dx.
\end{aligned}$$

Also

$$\begin{aligned}
I_3 &\leq C(\epsilon) \int_{A_{k,\rho}^+} \left\{ \sum (f^i)^2 + |f| \zeta^2(u-k)_+ \right\} dx + \epsilon \int_{A_{k,\rho}^+} |X_i(\zeta \cdot \zeta(u-k)_+)|^2 dx \\
&= C(\epsilon) \int_{A_{k,\rho}^+} \left\{ \sum (f^i)^2 + |f| \zeta^2(u-k)_+ \right\} dx + \epsilon \int_{A_{k,\rho}^+} [(u-k)_+ X_j \zeta + X_j(\zeta(u-k)_+)]^2 dx \\
&\leq C(\epsilon) \int_{A_{k,\rho}^+} \left\{ \sum (f^i)^2 + |f| \zeta^2(u-k)_+ \right\} dx + 2\epsilon \int_{A_{k,\rho}^+} (u-k)_+^2 |X\zeta|^2 + |X_i(\zeta(u-k)_+)|^2 dx.
\end{aligned}$$

Now substituting estimates of I_1, I_2 and I_3 into (3.2), it derives

$$\begin{aligned}
&\Lambda^{-1} \int_{B_\rho} |X(\zeta(u-k)_+)|^2 dx \\
&\leq \Lambda \int_{B_\rho} (u-k)_+^2 |X\zeta|^2 dx + C(\epsilon) \int_{A_{k,\rho}^+} \left\{ \zeta^2(u-k)_+^2 \left[\sum (b^i)^2 + |c| \right] + |c| k^2 \zeta^2 \right\} dx \\
&\quad + C(\epsilon) \int_{A_{k,\rho}^+} \left\{ \sum (f^i)^2 + |f| \zeta^2(u-k)_+ \right\} dx
\end{aligned}$$

$$+4\epsilon \int_{A_{k,\rho}^+} (u-k)_+^2 |X\zeta|^2 + |X(\zeta(u-k)_+)|^2 dx. \quad (3.3)$$

Choosing ϵ small enough, we have

$$\begin{aligned} & \int_{B_\rho} |X(\zeta(u-k)_+)|^2 dx \\ & \leq C \left\{ \int_{B_\rho} (u-k)_+^2 |X\zeta|^2 dx + \int_{A_{k,\rho}^+} \left[\zeta^2 (u-k)_+^2 \left[\sum (b^i)^2 + |c| \right] + |c|k^2 \zeta^2 \right] dx \right. \\ & \quad \left. + \int_{A_{k,\rho}^+} \left[\sum (f^i)^2 + |f| \zeta^2 (u-k)_+ \right] dx \right\} \\ & =: C(J_1 + J_2 + J_3). \end{aligned} \quad (3.4)$$

As $|X\zeta| \leq \frac{C}{(1-\theta)\rho}$, it knows

$$J_1 \leq \frac{C}{(1-\theta)^2 \rho^2} \int_{B_\rho} (u-k)_+^2 dx.$$

By using (1.3),

$$J_2 \leq \Lambda \int_{B_\rho} |\zeta(u-k)_+|^2 dx + k^2 \Lambda |A_{k,\rho}^+| \leq \Lambda \int_{B_\rho} (u-k)_+^2 dx + k^2 \Lambda |A_{k,\rho}^+|.$$

Employing the Hölder inequality and (2.5),

$$\begin{aligned} J_3 & \leq C \|f^i\|_{L^p}^2 |A_{k,\rho}^+|^{1-\frac{2}{p}} + \|\zeta(u-k)_+\|_{L^{2Q/(Q-2)}} \|\zeta f\|_{L^{pQ/(p+Q)}} |A_{k,\rho}^+|^{\frac{1}{2}-\frac{1}{p}} \\ & \leq C \|f^i\|_{L^p}^2 |A_{k,\rho}^+|^{1-\frac{2}{p}} + \|X(u-k)_+\|_{L^2} \|f\|_{L^{pQ/(p+Q)}} |A_{k,\rho}^+|^{\frac{1}{2}-\frac{1}{p}} \\ & \leq C \|f^i\|_{L^p}^2 |A_{k,\rho}^+|^{1-\frac{2}{p}} + \epsilon \|X(u-k)_+\|_{L^2}^2 + C(\epsilon) \|f\|_{L^{pQ/(p+Q)}}^2 |A_{k,\rho}^+|^{1-\frac{2}{p}} \\ & \leq C \left(\|f^i\|_{L^p}^2 + \|f\|_{L^{pQ/(p+Q)}}^2 \right) |A_{k,\rho}^+|^{1-\frac{2}{p}} + \epsilon \|X(u-k)_+\|_{L^2}^2. \end{aligned}$$

Taking these estimates into (3.4) yields

$$\begin{aligned} & \int_{B_{\sigma\rho}} |X(\zeta(u-k)_+)|^2 dx = \int_{A_{k,\sigma\rho}^+} |Xu|^2 dx \\ & \leq \frac{C}{(1-\theta)^2 \rho^2} \int_{B_\rho} (u-k)_+^2 dx + \Lambda \int_{B_\rho} (u-k)_+^2 dx + k^2 \Lambda |A_{k,\rho}^+| \\ & \quad + C \left(\|f^i\|_{L^p}^2 + \|f\|_{L^{pQ/(p+Q)}}^2 \right) |A_{k,\rho}^+|^{1-\frac{2}{p}} + \epsilon \|X(u-k)_+\|_{L^2}^2 \\ & \leq \frac{C+\Lambda}{(1-\theta)^2 \rho^2} \int_{B_\rho} (u-k)_+^2 dx + C \left(\|f^i\|_{L^p}^2 + \|f\|_{L^{pQ/(p+Q)}}^2 + k^2 \Lambda \right) |A_{k,\rho}^+|^{1-\frac{2}{p}} + \epsilon \|X(u-k)_+\|_{L^2}^2. \end{aligned}$$

It implies (1.6) by choosing $\gamma_0 = C + \Lambda$, $\gamma_1 = C \left(\|f^i\|_{L^p}^2 + \|f\|_{L^{pQ/(p+Q)}}^2 + k^2 \Lambda \right)$ and $\eta = p$, and so $u \in DG^+(\Omega)$. Lemma 3.1 is proved. \square

Lemma 3.1 explains that the weak solution of (1.1) belongs to the De Giorgi class $DG(\Omega)$. The following Lemmas 3.2, 3.3, 3.4 and 3.5 give properties of functions in the De Giorgi class.

Lemma 3.2. *Let $u \in DG^+(\Omega)$ (or $u \in DG^-(\Omega)$), $B_R(x_0) \subseteq \Omega$, $0 < R \leq 1$. If there exists $\theta \in (0, 1)$ depending only on $Q, M, \eta, \delta, \gamma_0$, and γ_1 such that for any real k ,*

$$|A_{k,R}^+| \leq \theta |B_R| \quad (\text{or } |A_{k,R}^-| \leq \theta |B_R|), \quad (3.5)$$

$$\delta \geq H := \sup_{B_R} u - k \geq \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} \quad (\text{or } \delta \geq H := k - \inf_{B_R} u \geq \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}), \quad (3.6)$$

then

$$|A_{k+\frac{H}{2}, \frac{R}{2}}^+| = 0 \quad (\text{or } |A_{k-\frac{H}{2}, \frac{R}{2}}^-| = 0). \quad (3.7)$$

We explicitly note that (3.7) implies

$$u \leq k + \frac{H}{2} \quad (\text{or } u \geq k - \frac{H}{2}).$$

Before proving Lemma 3.2, let us recall a known result.

Lemma 3.3 ([33, 34]). *Assume that a nonnegative sequence $\{y_h\}$ ($h = 0, 1, 2, \dots$) satisfies*

$$y_{h+1} \leq C b^h y_h^{1+\epsilon}, \quad h = 0, 1, 2, \dots,$$

where $b > 1$ and $\epsilon > 0$. If

$$y_0 \leq \theta_0 = C^{-\frac{1}{\epsilon}} b^{-\frac{1}{\epsilon^2}}, \quad \theta_0 \in (0, 1),$$

then

$$\lim_{h \rightarrow \infty} y_h = 0.$$

Proof of Lemma 3.2. We only treat the case $u \in DG^+(\Omega)$, and the case $u \in DG^-(\Omega)$ is similarly treated. Set

$$R_m = \frac{R}{2} + \frac{R}{2^{m+1}}, \quad k_m = k + \frac{H}{2} - \frac{H}{2^{m+1}}, \quad m = 0, 1, 2, \dots, \quad (3.8)$$

where $H = \sup_{B_R} u - k$, then R_m is decreasing and k_m increasing. Denoting

$$B_0 = B_R, \quad A_0 = A_{k,R}^+, \quad B_m = B_{R_m}, \quad A_m = A_{k_m, R_m}^+ = \{u \in B_m \mid u(x) > k_m\}, \quad m = 1, 2, \dots,$$

it sees that B_m is decreasing and A_m increasing, and (3.5) is rewritten as

$$|A_0| \leq \theta |B_0|. \quad (3.9)$$

Using $k_m \rightarrow k + \frac{H}{2}$, $R_m \rightarrow \frac{R}{2}$ as $m \rightarrow \infty$, it has

$$|A_m| = |A_{k_m, R_m}^+| \rightarrow \left| A_{k + \frac{H}{2}, \frac{R}{2}}^+ \right|,$$

and (3.7) becomes

$$\lim_{m \rightarrow \infty} |A_m| = 0. \quad (3.10)$$

With Lemma 3.3, we promptly know that (3.10) is proved if one can check

$$\frac{|A_{m+1}|}{|B_R|} \leq C b^m \left(\frac{|A_m|}{|B_R|} \right)^{1+\kappa}, \quad m = 0, 1, \dots, \quad (3.11)$$

for some $C, \kappa > 0$, where $b > 1$ is independent of m, R, x_0 and $\theta_0 = C^{-1/\kappa} b^{-1/\kappa^2}$.

To check (3.11), we apply the function $(u - k_m)_+$ in (1.6) between the balls B_{m+1} and B_m and note

$$u - k_m \leq \sup_{B_m} u - k_m \leq \sup_{B_R} u - \left(k + \frac{H}{2} - \frac{H}{2^{m+1}} \right) \leq \frac{H}{2} - \frac{H}{2^{m+1}} \leq H$$

to get

$$\begin{aligned} \int_{B_{m+1}} |X(u - k_m)_+|^2 dx &= \int_{A_{m+1}} |Xu|^2 dx \\ &\leq \frac{\gamma_0}{(R_m - R_{m+1})^2} \int_{B_m} (u - k_m)_+^2 dx + \gamma_1 |A_m|^{1 - \frac{2}{\eta}} \\ &\leq 2^{2m} C \left\{ \frac{H^2}{R^2} |A_m| + \gamma_1 |A_m|^{1 - \frac{2}{\eta}} \right\} \\ &= 2^{2m} C \frac{H^2}{R^2} \left\{ |A_m| + \gamma_1 R^2 H^{-2} |A_m|^{1 - \frac{2}{\eta}} \right\}. \end{aligned} \quad (3.12)$$

It yields from (3.6) that

$$\gamma_1 R^2 H^{-2} \leq |B_R|^{\frac{2}{\eta}},$$

and then by putting it into (3.12) that

$$R^2 \int_{B_{m+1}} |X(u - k_m)_+|^2 dx \leq 2^{2m} C H^2 \left(|A_m| + |A_m|^{1 - \frac{2}{\eta}} |B_R|^{\frac{2}{\eta}} \right). \quad (3.13)$$

Using (2.5) to the function $\zeta(u - k_m)_+$, ζ is a cut-off function between B_{m+2} and B_{m+1} , we have

$$\begin{aligned} \left(\int_{A_{m+1}} (u - k_m)_+^{2^*} dx \right)^{\frac{1}{2^*}} &\leq C |B_R|^{\frac{1}{2^*} - \frac{1}{2}} \left(R^2 \int_{A_{m+1}} |X(u - k_m)_+|^2 dx \right)^{\frac{1}{2}} \\ &\leq C |B_R|^{\frac{1}{2^*} - \frac{1}{2}} \left(R^2 \int_{B_{m+1}} |X(u - k_m)_+|^2 dx \right)^{\frac{1}{2}} \end{aligned}$$

and then

$$\begin{aligned}
 |k_{m+1} - k_m|^2 |A_{m+1}|^{\frac{2}{2^*}} &\leq \left(\int_{A_{m+1}} (u - k_m)_+^{2^*} dx \right)^{\frac{2}{2^*}} \\
 &\leq C |B_R|^{\frac{2}{2^*} - 1} \left(R^2 \int_{B_{m+1}} |X(u - k_m)_+|^2 dx \right). \tag{3.14}
 \end{aligned}$$

Combining it with (3.13) and noting

$$|A_{m+1}| \leq |A_m| \leq |B_R| \quad \text{and} \quad u - k_m \leq H,$$

it follows

$$\begin{aligned}
 |k_{m+1} - k_m|^2 |A_{m+1}| &\leq \left(\int_{A_{m+1}} (u - k_m)_+^{2^*} dx \right)^{\frac{2}{2^*}} |A_{m+1}|^{1 - \frac{2}{2^*}} \\
 &\leq 2^{2m} C H^2 |A_m|^{1 - \frac{2}{2^*}} |B_R|^{\frac{2}{2^*} - 1} \left(|A_m| + |A_m|^{1 - \frac{2}{\eta}} |B_R|^{\frac{2}{\eta}} \right) \\
 &= 2^{2m} C H^2 |A_m|^{1 - \frac{2}{2^*}} |B_R|^{\frac{2}{2^*} - 1} |A_m|^{1 - \frac{2}{\eta}} \left(|A_m|^{\frac{2}{\eta}} + |B_R|^{\frac{2}{\eta}} \right) \\
 &\leq 2^{2m} C H^2 |A_m|^{1 - \frac{2}{2^*} - \frac{2}{\eta}} |B_R|^{\frac{2}{\eta} + \frac{2}{2^*} - 1}. \tag{3.15}
 \end{aligned}$$

We exploit $k_{m+1} - k_m = 2^{-m} \frac{H}{4}$ in (3.15) and show

$$\frac{|A_{m+1}|}{|B_R|} \leq 2^{4m} C |A_m|^{2 - \frac{2}{2^*} - \frac{2}{\eta}} |B_R|^{\frac{2}{\eta} + \frac{2}{2^*} - 2} = 2^{4m} C \left(\frac{|A_m|}{|B_R|} \right)^{1 + \kappa}, \tag{3.16}$$

where $\kappa = 1 - \frac{2}{\eta} - \frac{2}{2^*} > 0$ ($\eta > Q = \frac{2 \cdot 2^*}{2^* - 2}$). Now (3.11) is deduced. □

Lemma 3.4. For $u \in DG(\Omega)$ and any $B_R = B_R(x_0)$, $B_{2^{n_0+2}R} = B_{2^{n_0+2}R}(x_0) \subset \Omega$ for some $n_0 \in \mathbb{N}^+$, there exists $s \in \mathbb{N}^+$ large enough depending on δ, η, γ_0 and γ_1 , such that one of the following inequalities holds:

$$\text{osc}_{B_R} u \leq 2^s \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \tag{3.17}$$

$$\text{osc}_{B_R} u \leq \left(1 - \frac{1}{2^{s-1}} \right) \text{osc}_{B_{2^{n_0+2}R}} u, \tag{3.18}$$

where $\text{osc}_{B_R} u = \sup_{B_R} u - \inf_{B_R} u$.

Proof. Suppose

$$\text{osc}_{B_R} u > 2^s \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \tag{3.19}$$

where s satisfies $\frac{M}{2^{s-3}} \leq \delta$; we will prove (3.18). Set

$$\begin{aligned} \mu &= \sup_{B_{2R}} u, \quad \mu_1 = \sup_{B_{2^{n_0+2}R}} u, \quad \mu_2 = \inf_{B_{2^{n_0+2}R}} u, \quad \omega = \mu_1 - \mu_2, \quad \bar{\mu} = \frac{\mu_1 + \mu_2}{2}, \\ D_t &= A_{\mu_1 - \omega/2^t, 2^{n_0+1}R}^+ - A_{\mu_1 - \omega/2^{t+1}, 2^{n_0+1}R'}^+ \end{aligned}$$

where $t \in \mathbb{N}^+$ and

$$s \geq t \geq \log_2 \frac{2M}{\delta} \triangleq t_0.$$

Using (1.6) (R and $(1-\theta)^2 R^2$ in (1.6) are changed to $2^{n_0+2}R$ and $2^{2n_0+2}R^2$, respectively) to the function $(u-k)_+$ with $k = \mu_1 - \frac{\omega}{2^t}$ and denoting $A_{t,R} = A_{\mu_1 - \omega/2^t, R}^+$, it knows that for any $t_0 \leq t \leq s$,

$$\begin{aligned} \int_{A_{t, 2^{n_0+1}R}} |Xu|^2 dx &\leq \frac{\gamma_0}{2^{2n_0+2}R^2} \int_{B_{2^{n_0+2}R}} \left[u - \left(\mu_1 - \frac{\omega}{2^t} \right) \right]_+^2 dx + \gamma_1 |A_{t, 2^{n_0+2}R}|^{1-\frac{2}{\eta}} \\ &\leq C \left\{ R^{-2} \sup_{A_{t, 2^{n_0+2}R}} \left[u - \left(\mu_1 - \frac{\omega}{2^t} \right) \right]^2 |A_{t, 2^{n_0+2}R}| + \gamma_1 |A_{t, 2^{n_0+2}R}|^{1-\frac{2}{\eta}} \right\}. \end{aligned} \quad (3.20)$$

Furthermore, since

$$\sup_{A_{t, 2^{n_0+2}R}} \left[u - \left(\mu_1 - \frac{\omega}{2^t} \right) \right] \leq \sup_{B_{t, 2^{n_0+2}R}} u - \mu_1 + \frac{\omega}{2^t} = \frac{\omega}{2^t},$$

it follows

$$\int_{A_{t, 2^{n_0+1}R}} |Xu|^2 dx \leq C \left[R^{-2} \left(\frac{\omega}{2^t} \right)^2 |A_{t, 2^{n_0+2}R}| + \gamma_1 |A_{t, 2^{n_0+2}R}|^{1-\frac{2}{\eta}} \right]. \quad (3.21)$$

Noting by (3.19) that

$$\frac{\omega}{2^t} = \frac{\mu_1 - \mu_2}{2^t} \geq \frac{\text{osc } u}{2^t} > 2^{s-t} \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} \geq \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \quad t_0 \leq t \leq s,$$

it sees by putting it into (3.21) that

$$\begin{aligned} \int_{A_{t, 2^{n_0+1}R}} |Xu|^2 dx &\leq CR^{-2} \left(\frac{\omega}{2^t} \right)^2 \left[|A_{t, 2^{n_0+2}R}| + \gamma_1 R^2 \left(\frac{\omega}{2^t} \right)^{-2} |A_{t, 2^{n_0+2}R}|^{1-\frac{2}{\eta}} \right] \\ &\leq CR^{-2} \left(\frac{\omega}{2^t} \right)^2 \left(|A_{t, 2^{n_0+2}R}| + |B_R|^{\frac{2}{\eta}} |A_{t, 2^{n_0+2}R}|^{1-\frac{2}{\eta}} \right). \end{aligned} \quad (3.22)$$

We now consider the case

$$|A_{\bar{\mu}, 2R}^+| \leq \frac{1}{2} |B_{2R}|; \quad (3.23)$$

to the case $|A_{\mu,2R}^-| = |A_{\mu-\frac{\omega}{2^t},2R}^-| \leq \frac{1}{2}|B_{2R}|$, the following proof is also valid by taking the function $(u-k)_-$ in (3.20). Let $k = \mu_1 - \frac{\omega}{2^t}$ and $l = \mu_1 - \frac{\omega}{2^{t+1}}$, then $l - k = \frac{\omega}{2^{t+1}}$. Applying (2.7) and (3.23), it implies

$$\begin{aligned} \left(\frac{\omega}{2^{t+1}}\right) |A_{t+1,2R}| &\leq C \frac{|B_{2R}|}{|B_{2R} - A_{t,2R}|} \left(R^2 \int_{D_t} |Xu|^2 dx\right)^{\frac{1}{2}} |D_t|^{\frac{1}{2}} \\ &\leq C \frac{|B_{2R}|}{|B_{2R}| - \frac{1}{2}|B_{2R}|} \left(R^2 \int_{D_t} |Xu|^2 dx\right)^{\frac{1}{2}} |D_t|^{\frac{1}{2}} \\ &\leq C \left(R^2 \int_{D_t} |Xu|^2 dx\right)^{\frac{1}{2}} |D_t|^{\frac{1}{2}} \\ &\leq C \left(R^2 \int_{A_{t,2^{n_0+1}R}} |Xu|^2 dx\right)^{\frac{1}{2}} |D_t|^{\frac{1}{2}}. \end{aligned} \tag{3.24}$$

This leads to from (3.22) and (3.24) that

$$\begin{aligned} |A_{t+1,2R}| &\leq \left(\frac{\omega}{2^{t+1}}\right)^{-1} \left(\frac{\omega}{2^t}\right) \left(|A_{t,2^{n_0+2}R}| + |B_R|^{\frac{2}{\eta}} |A_{t,2^{n_0+2}R}|^{1-\frac{2}{\eta}}\right)^{\frac{1}{2}} |D_t|^{\frac{1}{2}} \\ &\leq C \left(|A_{t,2^{n_0+2}R}| + |A_{t,2^{n_0+2}R}|^{1-\frac{2}{\eta}} |B_R|^{\frac{2}{\eta}}\right)^{\frac{1}{2}} |D_t|^{\frac{1}{2}}. \end{aligned} \tag{3.25}$$

Replacing $|A_{t+1,2R}|$ in (3.25) by $|A_{s-2,2R}|$, $t \leq s-3$, and noting $|A_{t,2^{n_0+2}R}| \leq |B_{2^{n_0+2}R}|$, we have

$$|A_{s-2,2R}| \leq C \left(|B_{2^{n_0+2}R}| + |B_{2^{n_0+2}R}|^{1-\frac{2}{\eta}} |B_R|^{\frac{2}{\eta}}\right)^{\frac{1}{2}} |D_t|^{\frac{1}{2}} \leq C |B_{2^{n_0+2}R}|^{\frac{1}{2}} |D_t|^{\frac{1}{2}}. \tag{3.26}$$

Since (2.2) yields $|B_{2^{n_0+2}R}| \leq C|B_{2R}|$, it follows by (3.26) that

$$\begin{aligned} &(s-t_0-2) |A_{s-2,2R}|^2 \\ &\leq C |B_{2^{n_0+2}R}| \sum_{t=t_0}^{s-3} |D_t| \leq C |B_{2R}| \left|A_{\mu_1-\omega/2^t,2^{n_0+1}R}^+ - A_{\mu_1-\omega/2^{t+1},2^{n_0+1}R}^+\right| \\ &\leq C |B_{2R}| |B_{2^{n_0+1}R}| \leq C |B_{2R}|^2, \end{aligned}$$

hence

$$|A_{s-2,2R}| \leq \frac{C}{\sqrt{(s-t_0-2)}} |B_{2R}|. \tag{3.27}$$

Let

$$H = \mu - \left(\mu_1 - \frac{\omega}{2^{s-2}}\right), \quad \mu = \sup_{B_{2R}} u,$$

we need to treat two possibilities:

$$H < 2\gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \tag{3.28}$$

$$H \geq 2\gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}. \quad (3.29)$$

If (3.28) holds, it sees from (3.19) that $\omega > \operatorname{osc}_{B_R} u > 2^s \gamma_1^{1/2} R |B_R|^{-\frac{1}{\eta}}$, i.e.,

$$2\gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} < \frac{\omega}{2^{s-1}},$$

and then

$$\begin{aligned} \mu &= H + \left(\mu_1 - \frac{\omega}{2^{s-2}} \right) < \mu_1 - \frac{\omega}{2^{s-2}} + 2\gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} \\ &< \mu_1 - \frac{\omega}{2^{s-2}} + \frac{\omega}{2^{s-1}} = \mu_1 - \frac{\omega}{2^{s-1}}. \end{aligned} \quad (3.30)$$

Noting

$$\sup_{B_R} u \leq \sup_{B_{2R}} u = \mu \quad \text{and} \quad \inf_{B_R} u \geq \inf_{B_{2^{n_0+2}R}} u = \mu_2,$$

we obtain

$$\operatorname{osc}_{B_R} u = \sup_{B_R} u - \inf_{B_R} u \leq \mu - \mu_2 \leq \mu_1 - \frac{\omega}{2^{s-1}} - \mu_2 = \omega - \frac{\omega}{2^{s-1}} = \left(1 - \frac{1}{2^{s-1}} \right) \omega,$$

which indicates (3.18). If (3.29) holds, we note $2^{s-2} \geq 2^{s-3}$ and $\omega \leq \|u\|_{L^\infty(\Omega)} \leq M$ to get

$$H = \mu - \mu_1 + \frac{\omega}{2^{s-2}} \leq \frac{\omega}{2^{s-2}} \leq \frac{M}{2^{s-3}} \leq \delta. \quad (3.31)$$

Taking into account (3.27), (3.29) and (3.31), and applying (3.7) in Lemma 3.2 (here $\frac{C}{\sqrt{(s-t_0-2)}} < \theta < 1$, $t_0 = \log_2 \frac{2M}{\delta}$, so $s > t_0 + 2 + \max\{1, C^2/\theta^2\}$), it means

$$|A_{\mu_1 - (\omega/2^{s-2}) + (\omega/2^{s-1}), R}| = 0,$$

i.e.

$$\sup_{B_R} u \leq \mu_1 - \frac{\omega}{2^{s-1}}.$$

Hence

$$\operatorname{osc}_{B_R} u = \sup_{B_R} u - \inf_{B_R} u \leq \mu_1 - \frac{\omega}{2^{s-1}} - \inf_{B_R} u \leq \mu_1 - \mu_2 - \frac{\omega}{2^{s-1}} = \left(1 - \frac{1}{2^{s-1}} \right) \omega.$$

It also follows (3.18). \square

Lemma 3.5. *Suppose $u \in DG(\Omega)$, $B_{R_0} \subset \Omega$, $\lambda R \leq R_0$, $\lambda > 1$. If there exist positive constants C and $0 < \theta < 1$ such that one of the following inequalities is valid:*

$$\operatorname{osc}_{B_R} u \leq CR |B_R|^{-\frac{1}{\eta}}, \quad (3.32)$$

$$\text{osc}_{B_R} u \leq \theta \text{osc}_{B_{\lambda R}} u, \tag{3.33}$$

then

$$\text{osc}_{B_R} u \leq \bar{C} \left(\frac{R}{R_0} \right)^\alpha, \tag{3.34}$$

where $\alpha = \min \left\{ -\log_\lambda \theta, 1 - \frac{Q}{\eta} \right\}$, $\bar{C} = \lambda^\alpha \max \left\{ \omega_0, CR_0 |B_0|^{-\frac{1}{\eta}} \right\}$ and $\omega_0 = \text{osc}_{B_{R_0}} u$.

Proof. Denote

$$R_k = \lambda^{-k} R_0, \quad B_k = B_{R_k}, \quad \omega_k = \text{osc}_{B_k} u, \quad k = 0, 1, 2, \dots,$$

then R_k and ω_k are decreasing. Since (3.32) or (3.33) holds, it follows

$$\omega_k \leq \max \left\{ CR_k |B_k|^{-\frac{1}{\eta}}, \theta \omega_{k-1} \right\}, \quad k = 1, 2, \dots \tag{3.35}$$

Setting

$$\bar{C} = \lambda^\alpha \max \left\{ \omega_0, CR_0 |B_0|^{-\frac{1}{\eta}} \right\},$$

where $\alpha = \min \left\{ -\log_\lambda \theta, 1 - \frac{Q}{\eta} \right\} > 0$, and noting $R_k |B_k|^{-\frac{1}{\eta}} \leq CR_0 |B_0|^{-\frac{1}{\eta}}$ and $\omega_{k-1} \leq \omega_0$, we have

$$\text{osc}_{B_0} u = \omega_0 \leq \max \left\{ \omega_0, CR_0 |B_0|^{-\frac{1}{\eta}} \right\} \leq \bar{C} \lambda^{-\alpha}. \tag{3.36}$$

Using (2.1) and $R_k = \lambda^{-k} R_0$, it derives

$$|B_0| \leq c_3 \left(\frac{R_0}{R_k} \right)^Q |B_k|,$$

and then

$$|B_k|^{-\frac{1}{\eta}} \leq \left(c_3^{-1} |B_0| R_0^{-Q} R_k^Q \right)^{-\frac{1}{\eta}} = \left(c_3^{-1} |B_0| R_0^{-Q} \lambda^{-kQ} R_0^Q \right)^{-\frac{1}{\eta}} = c_3^{\frac{1}{\eta}} \lambda^{\frac{kQ}{\eta}} |B_0|^{-\frac{1}{\eta}}. \tag{3.37}$$

Let

$$y_k = \lambda^{k\alpha} \omega_k, \quad k = 0, 1, 2, \dots,$$

we obtain by (3.35)–(3.37) that

$$\begin{aligned} y_k &\leq \max \left\{ \lambda^{k\alpha} CR_k |B_k|^{-\frac{1}{\eta}}; \lambda^{k\alpha} \theta \omega_{k-1} \right\} \\ &\leq \max \left\{ C \lambda^{k\alpha} \lambda^{-k} R_0 \lambda^{\frac{kQ}{\eta}} |B_0|^{-\frac{1}{\eta}}; \lambda^{k\alpha} \theta \lambda^{-(k-1)\alpha} y_{k-1} \right\} \\ &= \max \left\{ C \lambda^{k(\alpha-1+\frac{Q}{\eta})} R_0 |B_0|^{-\frac{1}{\eta}}; \lambda^\alpha \theta y_{k-1} \right\} \\ &\leq \max \left\{ \bar{C} \lambda^{-\alpha}; y_{k-1} \right\}, \end{aligned} \tag{3.38}$$

where we used $CR_0|B_0|^{-\frac{1}{\eta}} = \bar{C}\lambda^{-\alpha}$ and $\alpha = \min\left\{-\log_\lambda \theta, 1 - \frac{Q}{\eta}\right\}$, which yield $\lambda^{k(\alpha-1+\frac{Q}{\eta})} \leq 1$ and $\lambda^\alpha \theta < 1$. From (3.36), (3.38) and the induction, we conclude that for all $k=0,1,2,\dots$,

$$y_k \leq \bar{C}\lambda^{-\alpha}.$$

Noting $\lambda^{-k} = \frac{R_k}{R_0}$ and $\lambda^{-k\alpha} = \left(\frac{R_k}{R_0}\right)^\alpha$, it implies

$$\omega_k = \lambda^{-k\alpha} y_k \leq \bar{C}\lambda^{-k\alpha} \lambda^{-\alpha} = \bar{C}\lambda^{-\alpha} \left(\frac{R_k}{R_0}\right)^\alpha. \quad (3.39)$$

Since for any $0 < R \leq R_0$, there exists $k \geq 1$ such that $R_k \leq R \leq R_{k-1}$, we employ $R_k = \lambda^{-1}R_{k-1}$ and (3.39) to know

$$\operatorname{osc}_{B_R} u \leq \operatorname{osc}_{B_{R_{k-1}}} u \leq \bar{C}\lambda^{-\alpha} R_0^{-\alpha} R_{k-1}^\alpha = \bar{C}R_0^{-\alpha} R_k^\alpha \leq \bar{C}R_0^{-\alpha} R^\alpha,$$

which is (3.34). \square

4 Proofs of main results

The following Lemma is in preparation for the proof of Theorem 1.1.

Lemma 4.1. *Let $u \in H_{loc}^1(\Omega; X)$ be the bounded weak solution to (1.1) with (1.2), (1.3) and (1.4). Then for any $R \in (0, R_0]$, $0 < R_0 \leq 1$, $B_R(x) \subset \Omega$, we have*

$$\operatorname{osc}_{B_R} u \leq C \left(\frac{R}{R_0}\right)^\alpha \left(\operatorname{osc}_{B_{R_0}} u + \gamma_1^{\frac{1}{2}} R_0 |B_{R_0}|^{-\frac{1}{\eta}}\right), \quad (4.1)$$

where $\operatorname{osc}_{B_R} u = \operatorname{esssup}_{B_R} u - \operatorname{essinf}_{B_R} u$, $0 < \alpha \leq 1 - Q/\eta$ and $C \geq 1$ relies on η, δ, Q and Λ .

Proof. We denote $\omega(R) = \operatorname{osc}_{B_R} u = \sup_{B_R} u - \inf_{B_R} u$, then

$$\omega(R) \leq \omega(R_0), \quad \text{for } 0 < R \leq R_0. \quad (4.2)$$

It knows $u \in DG(\Omega)$ by Lemma 3.1. If $R = R_0$, it follows (4.1) from (4.2). If $0 < R < R_0$, then there exists $n^* \in \mathbb{N}^+$ such that $2^{n^*+2}R \leq R_0$. Using Lemma 3.4, we see that there exists a positive number s , such that one of the following inequalities holds:

$$\operatorname{osc}_{B_R} u \leq 2^s \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \quad (4.3)$$

$$\operatorname{osc}_{B_R} u \leq \left(1 - \frac{1}{2^{s-1}}\right) \operatorname{osc}_{B_{2^{n^*+2}R}} u. \quad (4.4)$$

It yields from Lemma 3.5 (at this time $\lambda = 2^{n^*+2}R$, $C = 2^s \gamma_1^{\frac{1}{2}}$ and $\theta = 1 - \frac{1}{2^s-1}$) that

$$\begin{aligned} \operatorname{osc}_{B_R} u &\leq \bar{C} \left(\frac{R}{R_0}\right)^\alpha = 2^{(n^*+2)\alpha} \left(\frac{R}{R_0}\right)^\alpha \max \left\{ \omega(R_0), 2^s \gamma_1^{\frac{1}{2}} R_0 |B_0|^{-\frac{1}{\eta}} \right\} \\ &\leq C \left(\frac{R}{R_0}\right)^\alpha \left(\omega(R_0) + 2^s \gamma_1^{\frac{1}{2}} R_0 |B_0|^{-\frac{1}{\eta}} \right) \\ &\leq C \left(\frac{R}{R_0}\right)^\alpha \left(\operatorname{osc}_{B_{R_0}} u + \gamma_1^{\frac{1}{2}} R_0 |B_0|^{-\frac{1}{\eta}} \right), \end{aligned}$$

where $0 < \alpha \leq 1 - \frac{Q}{\eta}$, and (4.1) is proved. □

Proof of Theorem 1.1. It follows from Lemma 4.1 that for any $x_0 \in \Omega$, $B_R(x_0) \subset \Omega$,

$$\operatorname{osc}_{B_R(x_0)} u \leq C \left(\frac{R}{d_{x_0}}\right)^\alpha \left(\operatorname{osc}_\Omega u + \gamma_1^{\frac{1}{2}} d_{x_0} |\Omega|^{-\frac{1}{\eta}} \right), \quad R \in (0, d_{x_0}], \tag{4.5}$$

where $d_{x_0} = \operatorname{dist}\{x_0, \partial\Omega\}$. We have by (4.5) and $\operatorname{osc}_\Omega u \leq M$ that

$$\frac{R^{-\alpha}}{|B_R(x_0)|} \int_{B_R(x_0)} |u(x) - u_{B_R(x_0)}| dx \leq R^{-\alpha} \operatorname{osc}_{B_R(x_0)} u \leq C d_{x_0}^{-\alpha} \left(M + \gamma_1^{\frac{1}{2}} d_{x_0} |\Omega|^{-\frac{1}{\eta}} \right),$$

which means $u \in \mathcal{L}_{loc}^{1,\alpha}(\Omega)$ and so $u \in C_{loc}^\alpha(\Omega)$ from Lemma 2.1.

Now let us estimate $[u]_{\alpha;\Omega}$. For any $x_1 \in \Omega$, if $d(x_1, x_0) \leq d_{x_0}$, then we take $x_{m_k} \in B_{d(x_1, x_0)}(x_0) \subset \Omega$, $x_{m_k} \rightarrow x_1$, and use (4.5) to give

$$|u(x_{m_k}) - u(x_0)| \leq \operatorname{osc}_{B_{d(x_1, x_0)}(x_0)} u \leq C \left(\frac{d(x_1, x_0)}{d_{x_0}}\right)^\alpha \left(M + \gamma_1^{\frac{1}{2}} d_{x_0} |\Omega|^{-\frac{1}{\eta}} \right).$$

Letting $x_{m_k} \rightarrow x_1$, it implies

$$|u(x_1) - u(x_0)| \leq \operatorname{osc}_{B_{d(x_1, x_0)}(x_0)} u \leq C \left(\frac{d(x_1, x_0)}{d_{x_0}}\right)^\alpha \left(M + \gamma_1^{\frac{1}{2}} d_{x_0} |\Omega|^{-\frac{1}{\eta}} \right). \tag{4.6}$$

If $d(x_1, x_0) > d_{x_0}$, then

$$|u(x_1) - u(x_0)| \leq 2M \leq 2M \left(\frac{d(x_1, x_0)}{d_{x_0}}\right)^\alpha. \tag{4.7}$$

Combining (4.6) and (4.7), it concludes (1.7). □

To prove Theorem 1.2, we need a lemma.

Lemma 4.2. *Let $u \in DG^-(\Omega)$, $u \geq 0$ on $B_{8R}(y) \subset \Omega$ and*

$$h \geq L\gamma_1^{\frac{1}{2}}R|B_R|^{-\frac{1}{\eta}} \quad (4.8)$$

for $h > 0$ and $L > 1$. If

$$|A_{h,R}^-| = 0, \quad (4.9)$$

then there exists λ_0 , $L^{-1} \leq \lambda_0 \leq 1/2$, such that for $\epsilon \in (0, 2\lambda_0]$, we have

$$|A_{\epsilon h, 2R}^-| = 0. \quad (4.10)$$

Proof. It sees that (4.9) actually implies $u > h$ on B_R , and then $|A_{h,4R}^+| = |\{x \in B_{4R} \mid u > h\}| \geq |B_R|$. Using (2.2), we observe

$$|A_{h,4R}^-| \leq |B_{4R}| - |A_{h,4R}^+| \leq |B_{4R}| - |B_R| \leq (1 - c_3^{-1}4^{-Q})|B_{4R}|. \quad (4.11)$$

To verify (4.10), we first claim that there exist $\tau > 0$ and $L^{-1} \leq \lambda_0 \leq \frac{1}{2}$, such that

$$|A_{\lambda_0 h, 4R}^-| \leq \tau |B_{4R}|. \quad (4.12)$$

In fact, it derives by choosing $\lambda_0, L^{-1} \leq \lambda_0 \leq 2^{-s_*}$ (s_* is to be determined), and using (4.8) that

$$h \geq 2^{s_*} \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}. \quad (4.13)$$

For $s = 0, 1, \dots, s_* - 1$, denote

$$A_{s,r} = A_{2^{-s}h,r}^-$$

and

$$(u - 2^{-s}h)_- = \begin{cases} 2^{-s}h - u, & u < 2^{-s}h, \\ 0, & u \geq 2^{-s}h. \end{cases}$$

Noting that $u \geq 0$ yields $(u - 2^{-s}h)_- = 2^{-s}h - u \leq 2^{-s}h$ and using (1.6) (k, R and $(1 - \sigma)^2 R^2$ in changed to $2^{-s}h, 8R$ and $16R^2$, respectively) to $(u - 2^{-s}h)_-$ on B_{4R} and B_{8R} , it implies

$$\begin{aligned} \int_{A_{s,4R}} |X(u - 2^{-s}h)_-|^2 dx &\leq \frac{\gamma_0}{16R^2} \int_{B_{8R}} (u - 2^{-s}h)_-^2 dx + \gamma_1 |A_{s,8R}|^{1 - \frac{2}{\eta}} \\ &= \frac{\gamma_0}{16R^2} \int_{A_{s,8R}} (u - 2^{-s}h)_-^2 dx + \gamma_1 |A_{s,8R}|^{1 - \frac{2}{\eta}} \\ &\leq 2^{-2s} \gamma_0 R^{-2} h^2 |A_{s,8R}| + \gamma_1 |A_{s,8R}|^{1 - \frac{2}{\eta}}. \end{aligned} \quad (4.14)$$

Take $k = -2^{-s}h$ and $l = -2^{-s-1}h$, then $l - k = \frac{h}{2^{s+1}}$. We apply (2.7) to $-u$ on B_{4R} ($A(k)$ and $A(l)$ in (2.7) becomes $A_{s,4R} = \{x \in B_{4R} \mid -u > -2^{-s}h\}$ and $A_{s+1,4R} = \{x \in B_{4R} \mid -u > -2^{-s-1}h\}$, respectively), and combine (4.11), (4.13) and (4.14) to derive

$$\frac{h}{2^{s+1}} |A_{s+1,4R}|$$

$$\begin{aligned} &\leq \frac{|B_{4R}|}{|B_{4R} - A_{s,4R}|} \left(R^2 \int_{A_{s,4R} - A_{s+1,4R}} |X(u - 2^{-s}h)|^2 dx \right)^{\frac{1}{2}} |A_{s,4R} - A_{s+1,4R}|^{\frac{1}{2}} \\ &\leq \frac{|B_{4R}|}{|B_{4R}| - (1 - c_3^{-1}4^{-Q})|B_{4R}|} \left(R^2 \int_{A_{s,4R} - A_{s+1,4R}} |X(u - 2^{-s}h)|^2 dx \right)^{\frac{1}{2}} |A_{s,4R} - A_{s+1,4R}|^{\frac{1}{2}} \\ &\leq \frac{|B_{4R}|}{c_3^{-1}4^{-Q}|B_{4R}|} \gamma_0^{\frac{1}{2}} \frac{h}{2^s} \left(|A_{s,8R}| + \gamma_0^{-1} \frac{2^{2s}}{h^2} R^2 \gamma_1 |A_{s,8R}|^{1 - \frac{2}{\eta}} \right)^{\frac{1}{2}} |A_{s,4R} - A_{s+1,4R}|^{\frac{1}{2}} \\ &\leq C \frac{h}{2^s} \left(|A_{s,8R}| + C |B_R|^{\frac{2}{\eta}} |A_{s,8R}|^{1 - \frac{2}{\eta}} \right)^{\frac{1}{2}} |A_{s,4R} - A_{s+1,4R}|^{\frac{1}{2}}. \end{aligned}$$

Furthermore, using $|A_{s,8R}| \leq |B_{8R}|$, $|B_R|^{\frac{2}{\eta}} |A_{s,8R}|^{1 - \frac{2}{\eta}} \leq |B_{8R}|$ and (2.2), the inequality above leads to

$$|A_{s+1,4R}| \leq C |B_{8R}|^{\frac{1}{2}} |D_s|^{\frac{1}{2}} \leq C |B_{4R}|^{\frac{1}{2}} |D_s|^{\frac{1}{2}}, \tag{4.15}$$

where $D_s = A_{s,4R} - A_{s+1,4R}$, $s = 0, 1, \dots, s_* - 1$. Since $|A_{s_*,4R}| \leq |A_{s+1,4R}|$, it follows from (4.15) that

$$|A_{s_*,4R}|^2 \leq C |B_{4R}| |D_s|.$$

Summing it with respect to s from 0 to $s_* - 1$, and noting $\sum_{s=0}^{s_*-1} |D_s| = |A_{0,4R} - A_{s_*,4R}|$, we have

$$s_* |A_{s_*,4R}|^2 \leq C |B_{4R}|^2. \tag{4.16}$$

Choosing $\tau = \sqrt{\frac{C}{s_*}} \in (0, 1)$, it gets from (4.16) that

$$|A_{s_*,4R}| \leq \tau |B_{4R}| \tag{4.17}$$

and s_* is also determined. As $\lambda_0 \leq 2^{-s_*}$ and so $\lambda_0 h \leq 2^{-s_*} h$, we see

$$|A_{\lambda_0 h, 4R}^-| \leq |A_{h 2^{-s_*}, 4R}^-| = |A_{s_*, 4R}| \leq \tau |B_{4R}|,$$

which is (4.12).

From $L \geq \lambda_0^{-1}$ and (4.8), it follows $\lambda_0 h \geq \gamma_1^{1/2} R |B_R|^{-\frac{1}{\eta}}$ and

$$4\lambda_0 h \geq \gamma_1^{\frac{1}{2}} (4R) |B_{4R}|^{-\frac{1}{\eta}}.$$

Combining it with (4.12) and employing Lemma 3.2, it obtains $|A_{2\lambda_0 h, 2R}^-| = 0$. For $0 < \epsilon \leq 2\lambda_0$, it sees $|A_{\epsilon h, 2R}^-| \leq |A_{2\lambda_0 h, 2R}^-| = 0$ and (4.10) is proved. \square

Proof of Theorem 1.2. Define two functions

$$K(r) = u(x_0) \left(1 - \frac{r}{R}\right)^{-s} \quad \text{and} \quad M(r) = \|u\|_{\infty, B_r(x_0)}, \tag{4.18}$$

where s is large enough from Lemma 3.4. Let r_0 be the largest root satisfying the equation $K(r) = M(r)$. Since $K(r) \rightarrow \infty$ as $r \rightarrow R-0$ and u is bounded and continuous on $B_R(x_0)$ by Theorem 1.1, we see that $r_0 < R$ is well defined and $K(r) > M(r)$, for $r_0 < r \leq R$. In addition, there exists $x_1 \in B_{r_0}(x_0)$ such that

$$u(x_1) = K(r_0) = M(r_0)$$

and $u(x_1) > 0$.

For any $x \in B_{(R-r_0)/2}(x_1)$, because of

$$d(x, x_0) \leq d(x, x_1) + d(x_1, x_0) \leq \frac{R-r_0}{2} + r_0 = \frac{R+r_0}{2},$$

we have $B_{(R-r_0)/2}(x_1) \subset B_{(R+r_0)/2}(x_0)$ and by the meaning of r_0 ,

$$\sup_{B_{(R-r_0)/2}(x_1)} u \leq \sup_{B_{(R+r_0)/2}(x_0)} u = M\left(\frac{R+r_0}{2}\right) \leq K\left(\frac{R+r_0}{2}\right) = u(x_0) \cdot 2^s \left(1 - \frac{r_0}{R}\right)^{-s} = 2^s K(r_0).$$

From it and (1.7) in Theorem 1.1 (R and R_0 in (1.6) are taken by $\frac{\epsilon(R-r_0)}{2}$ and $\frac{R-r_0}{2}$, respectively, ϵ is to be decided), it obtains

$$\begin{aligned} & |u(x_1) - \inf_{d(x, x_1) \leq \epsilon(R-r_0)/2} u(x)| \leq \operatorname{osc}_{B_{\epsilon(R-r_0)/2}(x_1)} u \\ & \leq C\epsilon^\alpha \left(\operatorname{osc}_{B_{(R-r_0)/2}(x_1)} u + \gamma_1^{\frac{1}{2}} \frac{R-r_0}{2} |B_{\frac{R-r_0}{2}}|^{-\frac{1}{\eta}} \right) \\ & \leq C\epsilon^\alpha \left(\sup_{B_{(R-r_0)/2}(x_1)} u + \gamma_1^{\frac{1}{2}} (R-r_0) |B_{R-r_0}|^{-\frac{1}{\eta}} \right) \\ & \leq C\epsilon^\alpha \left(2^s K(r_0) + \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} \right), \end{aligned}$$

where $0 < \alpha \leq 1 - \frac{Q}{\eta}$ and we used $R-r_0 < R$ and (2.1) in the last inequality. Therefore,

$$\inf_{d(x, x_1) \geq \epsilon(R-r_0)/2} u(x) \geq u(x_1) - C\epsilon^\alpha \left(2^s K(r_0) + \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} \right). \quad (4.19)$$

In what follows, let us suppose

$$u(x_0) \geq 2^{s+1+\alpha} C \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}; \quad (4.20)$$

otherwise, if (4.20) is invalid, it yields

$$\inf_{B_R(x_0)} u(x) \geq 0 > u(x_0) - 2^{s+1+\alpha} C \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \quad (4.21)$$

which implies (1.8). From (4.18) and (4.20), we have

$$K(r_0) \geq u(x_0) \geq 2^{s+1+\alpha} C \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}}, \tag{4.22}$$

and by (4.19),

$$\begin{aligned} \inf_{d(x,x_1) \leq \epsilon(R-r_0)/2} u(x) &\geq K(r_0) - C\epsilon^\alpha (2^s K(r_0) + K(r_0)) \\ &\geq K(r_0) - C\epsilon^\alpha 2^{s+1} K(r_0) = (1 - C\epsilon^\alpha 2^{s+1}) K(r_0). \end{aligned}$$

Choosing ϵ so small that $1 - C\epsilon^\alpha 2^{s+1} \geq \frac{1}{2}$, it implies

$$\inf_{d(x,x_1) \leq \epsilon(R-r_0)/2} u(x) \geq \frac{1}{2} K(r_0). \tag{4.23}$$

Setting $h := \frac{1}{2} K(r_0)$, we see by (4.22) and (4.23) that

$$h \geq 2^{s+\alpha} C \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} \tag{4.24}$$

and $u > h$ on $B_{\epsilon(R-r_0)/2}(x_1)$, then

$$\left| A_{h, \frac{\epsilon(R-r_0)}{2}} \right| = \left| \left\{ x \in B_{\epsilon(R-r_0)/2}(x_1) \mid u < h \right\} \right| = 0.$$

Noting by (2.2),

$$|B_{\epsilon(R-r_0)/2}| \geq c_3^{-1} \left(\frac{\epsilon(R-r_0)}{2R} \right)^Q |B_R|,$$

$\epsilon^\alpha \leq 2^{-s-2} C^{-1} \frac{R-r_0}{2R} < \frac{1}{2}$ and $c_3^{\frac{1}{\eta}} = 2^{\frac{Q}{\eta}} < 2$, we obtain

$$\begin{aligned} \frac{\epsilon(R-r_0)}{2} |B_{\epsilon(R-r_0)/2}|^{-\frac{1}{\eta}} &\leq \left(\frac{\epsilon(R-r_0)}{2R} \right) R c_3^{\frac{1}{\eta}} \left(\frac{\epsilon(R-r_0)}{2R} \right)^{-\frac{Q}{\eta}} |B_R|^{-\frac{1}{\eta}} \\ &= c_3^{\frac{1}{\eta}} \left(\frac{\epsilon(R-r_0)}{2R} \right)^{1-\frac{Q}{\eta}} R |B_R|^{-\frac{1}{\eta}} \\ &\leq c_3^{\frac{1}{\eta}} \left(\frac{\epsilon(R-r_0)}{2R} \right)^\alpha R |B_R|^{-\frac{1}{\eta}} \\ &\leq 2^{-s-1-\alpha} C^{-1} R |B_R|^{-\frac{1}{\eta}}. \end{aligned} \tag{4.25}$$

Combining (4.24) and (4.25), it shows

$$h \geq 2^{2s+2\alpha+1} C^2 \gamma_1^{\frac{1}{2}} \frac{\epsilon(R-r_0)}{2} |B_{\epsilon(R-r_0)/2}|^{-\frac{1}{\eta}}. \tag{4.26}$$

Now we use Lemma 4.2 to derive that there exists λ_0 , $2^{-2s-2\alpha}C^{-2} \leq 2\lambda_0 \leq 1$, such that if $\delta_0 \in (0, 2\lambda_0]$, then

$$\left| A_{\delta_0 h, \epsilon(R-r_0)}^- \right| = 0,$$

i.e.

$$\inf_{d(x, x_1) \leq \epsilon(R-r_0)} u(x) \geq \delta_0 h. \quad (4.27)$$

Adjust suitably $\epsilon > 0$ such that $\frac{\log \epsilon^{-1} R / (R-r_0)}{\log 2}$ is a positive integer and denote

$$N = \frac{\log \epsilon^{-1} R / (R-r_0)}{\log 2} + 2. \quad (4.28)$$

If

$$N > \frac{s \log \left(\frac{R-r_0}{2R} \right)}{\log 2^{-2s-2\alpha} C^{-2}},$$

then

$$1 > \left(\frac{R-r_0}{2R} \right)^{\frac{s}{N}} > 2^{-2s-2\alpha} C^{-2},$$

and we choose

$$\delta_0 = \left(\frac{R-r_0}{2R} \right)^{\frac{s}{N}};$$

if

$$N \leq \frac{s \log \left(\frac{R-r_0}{2R} \right)}{\log 2^{-2s-2\alpha} C^{-2}},$$

we choose

$$\delta_0 = 2^{-2s-2\alpha} C^{-2}$$

and then

$$\delta_0^N \geq \left(\frac{R-r_0}{2R} \right)^s.$$

Using $h = \frac{1}{2}K(r_0)$, (4.18), (4.20) and $R = 2^{N-1} \epsilon \frac{R-r_0}{2}$ from (4.28) we have

$$\begin{aligned} \delta_0^{N-1} h &\geq \delta_0^N h \geq \left(\frac{R-r_0}{2R} \right)^s \frac{K(r_0)}{2} = \frac{1}{2^{s+1}} u(x_0) \geq 2^\alpha C \gamma_1^{\frac{1}{2}} R |B_R|^{-\frac{1}{\eta}} \\ &= 2^\alpha C \gamma_1^{\frac{1}{2}} \left[2^{N-1} \frac{\epsilon(R-r_0)}{2} \right] \left| B_{2^{N-1} \epsilon(R-r_0)/2} \right|^{-\frac{1}{\eta}}, \end{aligned}$$

and apply Lemma 4.2 with N times to get

$$\inf_{d(x, x_1) \leq 2^{N-1} \epsilon(R-r_0)} u(x) \geq \delta_0^N h. \quad (4.29)$$

For any $x \in B_R(x_0)$, it derives

$$d(x, x_1) \leq d(x, x_0) + d(x_0, x_1) \leq R + r_0 \leq 2R = 2^{N-1} \epsilon (R - r_0),$$

and so $B_R(x_0) \subset B_{2^{N-1}\epsilon(R-r_0)}(x_1)$. We follow from (4.19) and (4.27) that

$$\inf_{B_R(x_0)} u(x) \geq \inf_{B_{2^{N-1}\epsilon(R-r_0)}(x_1)} u(x) \geq \delta_0^N h \geq \left(\frac{R-r_0}{2R}\right)^s \frac{K(r_0)}{2} = \frac{1}{2^{s+1}} u(x_0). \quad (4.30)$$

Combining (4.21) and (4.30) it proves (1.8). \square

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