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The Eigenvalues of a Class of Elliptic Differential Operators

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Abstract. Consider (M,g) as an n-dimensional compact Riemannian manifold. In this paper we are going to study a class of elliptic differential operators which appears naturally in the study of hypersurfaces with constant mean curvature and also the study of variation theory for 1-area functional.

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1 Introduction

Finding bounds for the eigenvalue of the Laplacian operator on a given manifold is a key aspects in Riemannian geometry. In the recent years, because of the theory of self-adjoint operators, the spectral properties of linear Laplacian studied extensively. Most of the mathematicians are generally interested in the spectrum of the Laplacian on compact manifolds with or without boundary or noncompact complete manifolds. Because in these cases, the linear Laplacian can be uniquely extended to self-adjoint operators (see [1,2]). For these purposes, mathematicians study the various extensions of such operators which are found in different theories in Riemannian and differential geometry.

As a first extension, consider M as a complete manifold. Let $f: M \longrightarrow \mathbb{R}$ be a smooth function on M or $f \in W^{1,p}(M)$ where $W^{1,p}(M)$ is the Sobolev space. The p-Laplacian of f for $1<p<\infty$ is defined as

$$\Delta_p f = \operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right) = |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}(\operatorname{Hess} f)(\nabla f, \nabla f),$$

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where

$$(\text{Hess} f)(X,Y) = \nabla(\nabla f)(X,Y) = X(Yf) - (\nabla_X Y)f, \qquad X,Y \in \chi(M).$$

The first eigenvalues of p-Laplace operator in both Dirichlet and Neumann cases have been studied in many papers (see [3–5]). If $\mu_{1,p}$ denotes the first Neumann eigenvalue of p-Laplace operator then as an example it was proved before in [3] that

Proposition 1.1 (Matei). *Let M be an n-dimensional complete Riemannian manifold and let K be a real constant such that $Ric^M \geq (n-1)K$. Then for any $x_0 \in M$, $r_0 \in (0,d_M)$ and $p \geq 2$,*

$$\mu_{1,p}(B(x_0,r_0)) \leq \mu_{1,p}(V_n(K,r_0)),$$

where $B(x_0,r_0)$ is geodesic ball in M centered at x_0 and radius r_0, $V_n(K,r_0)$ is geodesic ball with radius r_0 in model space, i.e. the n-dimensional simply connected space form with constant sectional curvature K. Also Ric^M and d_M denote the Ricci curvature tensor in M and the diameter of M respectively.

The other well known extension of Laplace operator is the weighted Laplace operator which is defined as $\Delta_f = \Delta - \nabla f \cdot \nabla$ and it acts as the Laplace operator in weighted manifolds, i.e. manifolds with density $e^{-f} dv$ (see [6,7]).

Another extension of Laplace operator is the elliptic divergence type operator

$$L_T f = \text{div}(T \nabla f),$$

where T is a positive definite symmetric (1,1)-tensor field on a complete Riemannian manifold M. This operator is studied in [8] by second author before. Also the general case of this operator is defined as

$$L_{T,\eta} f = \text{div}(T \nabla f) - \langle \nabla \eta, T(\nabla f) \rangle,$$

where $\eta \in C^2(M)$ and the eigenvalue problem for this operator is studied in [9]. As it was mentioned before, the first eigenvalue of these operators on a compact manifold M has been studied extensively in recent mathematical publications. Many connections between these invariants and other geometrical invariants have lead to some results of i-th eigenvalue of these operators. As an example in [9].

Proposition 1.2 (Gomez and Miranda). *Let Ω be a domain in an n-dimensional complete Riemannian manifold M isometrically immersed in $\mathbb{R}^m$, λ_i be the i-th eigenvalue of $L_{T,\eta}$ and f_t its corresponding normalized real-valued eigenfunction. Then*

$$\begin{aligned} &\text{tr}(T) \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \\ \leq &\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \Big((m-n)^2 A_0^2 T_*^2 + (T_0 + T_* \eta_0) + 4(T_0 + T_* \eta_0) ||T(\nabla f_i)||_{L^2} + 4\lambda_i \Big), \end{aligned}$$

where $A_0 = \max\{\sup_\Omega |A_{e_k}|, k = n+1, \ldots, m\}$, A_{e_k} is the Weingarten operator of the immersion with respect to e_k, $\eta_0 = \sup_{\bar{\Omega}} |\nabla \eta|$, $T_* = \sup_{\bar{\Omega}} |T|$ and $T_0 = \sup_{\bar{\Omega}} |\text{tr}(\nabla T)|$.

For all these operators, the major question is how we can extend the results of Laplace and p-Laplace operators and Ricci tensors to these operators. In this paper, our main aim is to improve these results for the special case operator

$$\Box_S f = \operatorname{div}(S\nabla f),$$

in n-dimensional manifold M, where $S_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}$. As a quick review, we are going to prove following theorems

Theorem 1.1. *Consider (M^n, g) as a compact Riemannian manifold. If Λ denotes the eigenvalue of $\Box_S^2$, then*

$$\Gamma\lambda \leq 2\sqrt{\Lambda\mu} - \frac{2}{\operatorname{tr}S}\Lambda,$$

where $\Box_S f = \operatorname{div}(S(\nabla f))$, $S_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}$, and

$$\Gamma = L_0^2 - \left(\frac{R}{2(n-1)} + K_0\right) L_0 + \frac{1}{2} K_0 R,$$

and K_0 and L_0 are the lower bounds of the sectional curvature and Ricci curvature of M, respectively. Also λ and μ denote the eigenvalues of Δ and Δ^2 respectively.

Theorem 1.2. *Let $x: M^n \to \bar{M}^{n+1}(\kappa)$ be an isometric immersion of a compact Riemannian manifold into a space form of constant sectional curvature κ. Also let the shape operator satisfies*

$$0 < \alpha I \leq A \leq a\alpha I,$$

where $\alpha > 0$ and $a > 1$ are constants. It is supposed that $L_1 f = \sum_{i,j=1}^n \left(H g_{ij} - \Pi_{ij}\right) f_{ij}$, H is mean curvature and Π is second fundamental form. If δ denotes the eigenvalue of

$$\begin{cases} L_1^2 f = \delta f & \text{in } M, \\ f = 0 & \text{on } \partial M, \end{cases}$$

then

$$\Gamma'_\kappa \lambda \leq 2\sqrt{\mu\delta} - \frac{2}{n(n-1)a\alpha}\delta,$$

where λ and μ are eigenvalues of Δ and Δ^2 respectively, Γ'_κ is

$$2\alpha^3(n-1)\left(n - a^2\right) + 2\kappa\alpha(n-1)^2 - \sigma,$$

when $\kappa > 0$ and

$$2\alpha^3(n-1)\left(n - a^2\right) + 2\kappa a\alpha(n-1)^2 - \sigma,$$

when $\kappa < 0$ and also

$$\sigma = \max_{(p,v)\in TM} \left(\operatorname{tr}(\operatorname{Hess} H)|_{v^\perp}(p)\right), \qquad v^\perp = \{u \in T_pM \mid \langle u, v\rangle = 0\}.$$

2 Preliminaries

Consider (M,g) as an n-dimensional Riemannian manifold. In this paper we are going to study the elliptic differential operator which is defined as

$$\Box_S f = \mathrm{div}(S(\nabla f)) = \langle \nabla^2 f, S\rangle, \tag{2.1}$$

where in coordinates we have

$$S_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}.$$

Where R_{ij} and R are Ricci curvature tensor and scalar curvature tensor respectively.

Generally, let $\{e_1, e_2, ..., e_n\}$ be a local coframe field defined on a Riemannian manifold (M,g). For a symmetric tensor $\phi = \sum_{i,j=1}^n \phi_{ij} e_i \otimes e_j$ on M, it is defined in [10] that

$$\Box f = \sum_{i,j=1}^n \phi_{ij} f_{ij}.$$

The Bochner-type formula for this operator was proved before in [12].

Lemma 2.1 (Bochner-type formula). *Let M^n be a Riemannian manifold and $\phi = \sum_{i,j=1}^n \phi_{ij} e_i \otimes e_j$ be a symmetric tensor defined on M. Then for any smooth function $f: M \longrightarrow \mathbb{R}$ and for any $C \in \mathbb{R}$*

$$\begin{aligned}\frac{1}{2}\Box\left(|\nabla f|^2\right) =& \langle \nabla f, \nabla(\Box f)\rangle + \langle \phi(\nabla f), \nabla(\Delta f)\rangle + 2\sum_{i,j,k=1}^n \phi_{ij} f_{jk} f_{ki} + 2\sum_{i,j,k,m=1}^n f_i f_j \phi_{im} R_{mkjk} \\ &+ C\sum_{i,j=1}^n (tr\phi)_{ij} f_i f_j - \sum_{i,j=1}^n f_i f_j \Delta\phi_{ij} + \sum_{i,j=1}^n f_i f_j \left(\sum_{k=1}^n \phi_{ikk} - C\sum_{k=1}^n \phi_{kki}\right)_j \\ &+ \sum_{k=1}^n \left(\sum_{i,j=1}^n f_i f_j (\phi_{jik} - \phi_{jki})\right)_k - \sum_{k=1}^n \left(\sum_{i,j=1}^n f_j \phi_{ij} f_{ik}\right)_k.\end{aligned} \tag{2.2}$$

It seems clear that if ϕ is equal to the metric g, then $\sum_{k=1}^n \left(\sum_{i,j=1}^n f_j \phi_{ij} f_{ik}\right)_k = \frac{1}{2}\Delta|\nabla f|^2$. In this case the above lemma returns to well-known Bochner formula for Laplacian. Also from [10], there are following basic properties. First of all,

$$\Box f = \mathrm{div}(\phi(\nabla f)) - \sum_{i=1}^n \left(\sum_{j=1}^n \phi_{ijj}\right) f_i.$$

And also, we say that ϕ is divergence free if $\mathrm{div}\phi \equiv 0$ or equivalently, $\sum_{j=1}^n \phi_{ijj} \equiv 0$, for all $1 \le j \le n$. Cheng and Yau in [10], discussed the operator $\Box$ extensively. If M is compact then $\square$ is self-adjoint if and only if ϕ is divergence free. If ϕ is symmetric and positive definite, then $\square$ is strictly elliptic. Therefore, if ϕ is divergence free, symmetric and positive definite, then $\square$ is strictly elliptic and self-adjoint (see also [11]).

We are going to study the system

$$\begin{cases} \mathcal{L}f=\Lambda f & \text{in M},\\ f=0 & \text{on } \partial\text{M}, \end{cases} \tag{2.3}$$

where $\mathcal{L}=\square_S^2$. As an important example under Codazzi Schouten operator i.e. $S_{ijk}=S_{ikj}$, it was proved before in [12] that

Proposition 2.1 (Alencar et al.). *Let M^n, $n\geq 4$ be a compact Riemannian manifold in which* $\operatorname{div}W\equiv 0$. *If M has constant scalar curvature R and the Einstein tensor is positive definite, then the first nonzero eigenvalue $\mu_1(\square_S,M)$ of the operator $\square_S$, satisfies*

$$\mu_1(\square_S,M)\geq\frac{n-2}{2(n-1)}\left(\frac{R}{R-2L_0}\right)\left[L_0^2-\left(\frac{R}{2(n-1)}+K_0\right)L_0\frac{1}{2}K_0R\right],$$

where K_0 and L_0 are the lower bounds of the sectional curvature and Ricci curvature of M, respectively and W is harmonic Weyl tensor. Furthermore, the equality holds if and only if M is the round sphere $\mathbb{S}^n(K_0)$.

Consider M^n, as an n-dimensional Riemannian manifold and $x:M^n\to\bar{M}^{n+1}$ as an isometric immersion of M to the model space of $\bar{M}$. Also A and H are shape operator and mean curvature of immersions respectively. In other words, if $\lambda_1,\lambda_2,\ldots,\lambda_n$ are eigenvalues of A, i.e. the principal curvatures of the immersions, then

$$H=\sum_{i=1}^{n}\lambda_i.$$

In this case, the first Newton transformation $P_1:TM\to TM$ associated with second fundamental form, is defined as

$$P_1=HI-A.$$

By this transformation, the new operator was first introduced in [13], as

$$L_1f=\sum_{i=1}^{n}(P_1)_{ij}f_{ij}=\sum_{i,j=1}^{n}\left(Hg_{ij}-\Pi_{ij}\right)f_{ij},$$

where Π_{ij} are the components of second fundamental form.

It has been shown before in [14], that if $\bar{M}$ is a space form of constant sectional curvature, then $\operatorname{div}P_1\equiv 0$. Also under these assumptions L_1 is self adjoint. The new operator L_1 naturally appears in the study of variation theory for curvature functional $\mathcal{A}=\int_M H\mathrm{d}\mu$, which is called 1-area of M. Also it plays an important role in the study of stability for hypersurfaces with constant mean curvature (see as examples [15–17]). It was proved before in [12] that

Proposition 2.2. [15,16] *Let* $x:M^n\to\bar{M}^{n+1}(\kappa)$ *be an isometric immersion of a compact Riemannian manifold into a space form of constant sectional curvature* κ*. Suppose that shape operator* A *satisfies*

$$0<\alpha I\leq A\leq a\alpha I,$$

where $\alpha>0$ *and* $a>1$ *are constants. Then if* $\mu(L_1,M)$ *denotes the eigenvalue of* $L_1f=-\mu f$*, we see*

$$\mu(L_1,M)\geq\frac{1}{2}\left(\frac{na}{na-1}\right)\Gamma'_\kappa,$$

where Γ'_κ *are the same as in Theorem* 1.2.

3 Proof of main results

Consider the system

$$\Box_S^2 f=\Lambda f. \tag{3.1}$$

In this case Λ is called the eigenvalue of mentioned operator. By multiplying both sides of Eq. (3.1) we get

$$\int_M f\cdot\Box_S f\,\mathrm{d}\mu=\Lambda\int_M f^2\mathrm{d}\mu,$$

since $\int_M\langle f,\Box_S^2 f\rangle\mathrm{d}\mu=\int_M\langle\Box_S f,\Box_S f\rangle\mathrm{d}\mu$, by integrating by parts we have

$$\int_M(\Box_S f)^2\mathrm{d}\mu=\Lambda\int_M f^2\mathrm{d}\mu,$$

which concludes that

$$\Lambda=\frac{\int_M(\Box_S f)^2\mathrm{d}\mu}{\int_M f^2\mathrm{d}\mu}.$$

Also the operator

$$L_1^2 f=\delta f,$$

which was introduced before, in the similar context, because L_1 is self-adjoint we see

$$\delta=\frac{\int_M(L_1 f)^2\mathrm{d}\mu}{\int_M f^2\mathrm{d}\mu}.$$

Proof of Theorem 1.1. Since $S_{jik}=S_{jki}$ and S is divergence free, i.e. $\mathrm{div}(S)=0$, our Bochner formula (2.2) changes as

$$\frac{1}{2}\square_S\left(|\nabla f|^2\right)=\langle\nabla f,\nabla(\square_S f)\rangle+2\sum_{i,j,k=1}^{n}S_{ij}f_{jk}f_{ki}+\langle S(\nabla f),\nabla(\Delta f)\rangle$$
$$+2Ric(\nabla f,S(\nabla f))-\sum_{i,j=1}^{n}f_if_j\Delta S_{ij}-\sum_{k=1}^{n}\left(\sum_{i,j=1}^{n}f_jS_{ij}f_{ik}\right)_k.$$

Now, we are going to integrate and estimate each term, then our proof will be completed. It was known before that for two $n\times n$ symmetric matrices A and B, where B is positive definite, then

$$\mathrm{tr}\left(A^2B\right)\geq\frac{[\mathrm{tr}(AB)]^2}{\mathrm{tr}B}. \tag{3.2}$$

There is a suitable proof for the Eq. (3.2) in [12] which is known as a Newton inequality.

Now consider $A=[f_{ij}]_{n\times n}$ and $B=[S_{ij}]_{n\times n}$, since S is positive definite, it implies that

$$\sum_{i,j,k=1}^{n}S_{ij}f_{jk}f_{ki}\geq\frac{(\square_S f)^2}{\mathrm{tr}S},$$

which finally gives us

$$2\int_M\sum_{i,j,k=1}^{n}S_{ij}f_{jk}f_{ki}\mathrm{d}\mu\geq\frac{2}{\mathrm{tr}S}\int_M(\square_S f)^2\mathrm{d}\mu.$$

Also it was proved in [12] that

$$\int_M\left[2Ric(\nabla f,S(\nabla f))-\sum_{i,j=1}^{n}f_if_j\Delta S_{ij}\right]\mathrm{d}\mu\geq\Gamma\int_M|\nabla f|^2\mathrm{d}\mu,$$

where Γ is as same as what mentioned before. Now since

$$\begin{aligned}&\int_M\langle\nabla f,\nabla(\square_S f)\rangle\mathrm{d}\mu+\int_M\langle S(\nabla f),\nabla(\Delta f)\rangle\mathrm{d}\mu\\=&-\int_M\langle\Delta f,\square_S f\rangle\mathrm{d}\mu-\int_M\langle\nabla(S(\nabla f)),\Delta f\rangle\mathrm{d}\mu\\=&-2\int_M\Delta f\cdot\square_S f\mathrm{d}\mu,\end{aligned}$$

and also by divergence theorem

$$\int_M\sum_{k=1}^{n}\left(\sum_{i,j=1}^{n}f_jS_{ij}f_{ik}\right)_k\mathrm{d}\mu=0.$$

Finally we get

$$-2\int_M \Delta f \Box_S f \mathrm{d}\mu + \Gamma\int_M |\nabla f|^2 \mathrm{d}\mu + \frac{2}{\mathrm{tr}\, S}\int_M (\Box_S f)^2 \mathrm{d}\mu \leq 0.$$

By simple calculation we have

$$\begin{aligned}\Gamma\int_M |\nabla f|^2 \mathrm{d}\mu &\leq 2\int_M \Delta f \Box_S f \mathrm{d}\mu - \frac{2}{\mathrm{tr} S}\int_M (\Box_S f)^2 \mathrm{d}\mu \\ &\leq 2\left(\int_M (\Delta f)^2 \mathrm{d}\mu\right)^{\frac{1}{2}}\left(\int_M (\Box_S f)^2 \mathrm{d}\mu\right)^{\frac{1}{2}} - \frac{2}{\mathrm{tr} S}\Lambda\int_M f^2 \mathrm{d}\mu \\ &= 2\left(\int_M (\Delta f)^2\right)^{\frac{1}{2}}\sqrt{\Lambda}\left(\int_M f^2 \mathrm{d}\mu\right)^{\frac{1}{2}} - \frac{2}{\mathrm{tr} S}\Lambda\int_M f^2 \mathrm{d}\mu,\end{aligned}$$

where in second inequality we used Hölder's inequality and Γ is as same as what mentioned in Theorem 1.1. By dividing both sides on $\int_M f^2 \mathrm{d}\mu$ we finally see

$$\Gamma\lambda \leq 2\sqrt{\Lambda\mu} - \frac{2}{\mathrm{tr} S}\Lambda. \qquad \Box$$

Proof of Theorem 1.2. If $A>0$ then P_1 is positive definite, therefore L_1 is elliptic operator. Since Codazzi equation i.e. $\Pi_{jik} = \Pi_{jki}$ holds for L_1, the Bochner formula becomes as

$$\begin{aligned}\frac{1}{2}L_1|\nabla f|^2 =& \langle\nabla f, \nabla(L_1 f)\rangle + \langle P_1(\nabla f), \nabla(\Delta f)\rangle + 2\sum_{i,j,k=1}^{n}\left(Hg_{ij} - \Pi_{ij}\right) f_{jk} f_{ki} \\ &+ 2Ric(\nabla f, P_1(\nabla f)) - \langle(\Delta P_1)(\nabla f), \nabla f\rangle \\ &- \sum_{k=1}^{n}\left(\sum_{i,j=1}^{n} f_i\left(Hg_{ij} - \Pi_{ij}\right) f_{ik}\right)_k + \sum_{k=1}^{n}\left(|\nabla f|^2 H_k - \langle\nabla H, \nabla f\rangle f_k\right)_k.\end{aligned}$$

By integrating from both sides and using divergence theorem, we get

$$\begin{aligned}0 =& \int_M \langle\nabla f, \nabla(L_1 f)\rangle \mathrm{d}\mu + \int_M \langle P_1(\nabla f), \nabla(\Delta f)\rangle \mathrm{d}\mu \\ &+ 2\int_M\left(\sum_{i,j,k=1}^{n}\left(Hg_{ij} - \Pi_{ij}\right) f_{jk} f_{ki}\right)\mathrm{d}\mu \\ &+ \int_M\Big[2Ric(\nabla f, P_1(\nabla f)) - \langle(\Delta P_1)(\nabla f), \nabla f\rangle\Big]\mathrm{d}\mu. \qquad (3.3)\end{aligned}$$

Now we are going estimate each quantity of Eq. (3.3) separately. Since

$$0 = \int_M \mathrm{div}(\Delta f P_1(\nabla f))\mathrm{d}\mu = \int_M \langle P_1(\nabla f), \nabla(\Delta f)\rangle \mathrm{d}\mu + \int_M \Delta f \cdot \mathrm{div}(P_1(\nabla f)),$$

thus

$$\int_M \langle \nabla f, \nabla (L_1 f)\rangle \mathrm{d}\mu + \int_M \langle P_1(\nabla f), \nabla(\Delta f)\rangle \mathrm{d}\mu = -2\int_M \Delta f \cdot L_1 f \mathrm{d}\mu.$$

Under the assumption

$$0 < \alpha I \leq A \leq a\alpha I,$$

it was proved before in [12] that

$$\int_M \Big[2Ric(\nabla f, P_1(\nabla f)) - \langle (\Delta P_1)(\nabla f), \nabla f\rangle\Big] \mathrm{d}\mu \geq \Gamma'_\kappa \int_M |\nabla f|^2 \mathrm{d}\mu,$$

where if $\kappa > 0$ then

$$\Gamma'_\kappa = 2\alpha^3 (n-1)\left(n-a^2\right) + 2\kappa\alpha(n-1)^2 - \sigma,$$

and also if $\kappa < 0$ then

$$\Gamma'_\kappa = 2\alpha^3 (n-1)\left(n-a^2\right) + 2\kappa a\alpha(n-1)^2 - \sigma,$$

and

$$\sigma = \max_{(p,v)\in TM} (\mathrm{tr}(Hess H)|_{v^\perp}(p)), \qquad v^\perp = \{u \in T_pM | \langle u, v\rangle = 0\}.$$

Also under $A > 0$, P is positive definite, thus

$$2\int_M \left(\sum_{i,j,k=1}^n (Hg_{ij} - \Pi_{ij}) f_{jk} f_{ki}\right) \mathrm{d}\mu \geq 2\int_M \frac{(L_1 f)^2}{(n-1)H} \mathrm{d}\mu \geq \frac{2}{n(n-1)a\alpha}\delta \int_M f^2 \mathrm{d}\mu,$$

where $L_1^2 f = \delta f^2$. Now by substituting into the Eq. (3.3), we see

$$\begin{aligned}
0 &\geq -2\int_M \Delta f \cdot L_1 f \mathrm{d}\mu + 2\int_M \frac{(L_1 f)^2}{(n-1)H}\mathrm{d}\mu + \Gamma'_\kappa \int_M |\nabla f|^2 \mathrm{d}\mu \\
&\geq -2\left(\int_M (\Delta f)^2 \mathrm{d}\mu\right)^{\frac{1}{2}} \left(\int_M (L_1 f)^2 \mathrm{d}\mu\right)^{\frac{1}{2}} + \frac{2}{n(n-1)a\alpha}\delta \int_M f^2 \mathrm{d}\mu + \Gamma'_\kappa \int_M |\nabla f|^2 \mathrm{d}\mu \\
&= -2\left(\int_M (\Delta f)^2 \mathrm{d}\mu\right)^{\frac{1}{2}} \sqrt{\delta} \left(\int_M f^2\right)^{\frac{1}{2}} + \frac{2}{n(n-1)a\alpha}\delta \int_M f^2 \mathrm{d}\mu + \Gamma'_\kappa \int_M |\nabla f|^2 \mathrm{d}\mu,
\end{aligned}$$

where in the second inequality we used Hölder's inequality. By dividing both sides on $\int_M f^2 \mathrm{d}\mu$ we finally get

$$\Gamma'_\kappa \lambda \leq 2\sqrt{\mu\delta} - \frac{2}{n(n-1)a\alpha}\delta.$$

□

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