Well-Posedness and Blow-Up for the Fractional Schrödinger-Choquard Equation

TAO Lu\textsuperscript{a}, ZHAO Yajuan\textsuperscript{b,∗} and LI Yongsheng\textsuperscript{a}

\textsuperscript{a} School of Mathematics, South China University of Technology, Guangzhou 510640, China.
\textsuperscript{b} Henan Academy of Big Data, Zhengzhou University, Zhengzhou 450000, China.

Received 10 January 2022; Accepted 19 April 2022

Abstract. In this paper, we study the well-posedness and blow-up solutions for the fractional Schrödinger equation with a Hartree-type nonlinearity together with a power-type subcritical or critical perturbations. For nonradial initial data or radial initial data, we prove the local well-posedness for the defocusing and the focusing cases with subcritical or critical nonlinearity. We obtain the global well-posedness for the defocusing case, and for the focusing mass-subcritical case or mass-critical case with initial data small enough. We also investigate blow-up solutions for the focusing mass-critical problem.

AMS Subject Classifications: 35R11, 35B44, 35A01, 35Q55

Chinese Library Classifications: O175

Key Words: Fractional Schrödinger equation; Hartree-type nonlinearity; well-posedness; blow-up.

1 Introduction

In this paper we consider the following Cauchy problem for the fractional nonlinear Schrödinger equation

\[
\begin{aligned}
    i\partial_t u &= (-\Delta)^{\alpha} u + \lambda \left( |u|^k u + \left( |x|^{-\gamma} * |u|^2 \right) u \right), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\
    u(0,x) &= \varphi(x), \quad x \in \mathbb{R}^N,
\end{aligned}
\]  

(1.1)

∗Corresponding author. Email addresses: zhaoyj91@163.com (Y. J. Zhao), taolu1998@126.com (L. Tao), yshli@scut.edu.cn (Y. S. Li)

http://www.global-sci.org/jpde/ 82
where $N \geq 1$, $0 < \alpha < 1$, $0 < \gamma < N$, $0 \leq k \leq \frac{4\alpha}{N}$, $\lambda = \pm 1$, $\ast$ denotes the convolution in $\mathbb{R}^N$, $i$ is the imaginary unit and $u = u(t, x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is the unknown complex-valued function. The fractional Laplace operator $(-\Delta)^{\alpha}$ is defined by
\[
(-\Delta)^{\alpha} u = \frac{1}{(2\pi)^N} \int e^{i x \cdot \xi} |\xi|^{2\alpha} \mathcal{F}[u](\xi) d\xi = \mathcal{F}^{-1} \left(|\xi|^{2\alpha} \mathcal{F}[u](\xi)\right),
\]
where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the Fourier inverse transform in $\mathbb{R}^N$, respectively. When $\lambda = 1$, (1.1) is referred to be defocusing fractional NLS, while $\lambda = -1$, (1.1) is referred to be focusing fractional NLS.

In recent years, there has been wide interest in applying fractional Laplacians to model physical phenomena. By extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, Laskin in [1, 2] used the theory of functionals over functional measure generated by the Lévy stochastic process to deduce the following nonlinear fractional Schrödinger equation
\[
\begin{align*}
&\partial_t u = (-\Delta)^{\alpha} u + f(u),
\end{align*}
\]
where $0 < \alpha < 1, f(u) = |u|^k u$. The parameter $0 < \alpha < 1$ is the corresponding index of the Lévy stable processes, see [1, 2]. Eq. (1.2) with $\alpha = \frac{1}{2}$ has been also used as models to describe Boson stars. Recently, an optical realization of the fractional Schrödinger equation was proposed by Longhi [3]. For the nonlinearity $|u|^p u$, the well-posedness and ill-posedness in the Sobolev space $H^s$ have been investigated in [4, 5]. In [6], Boullenger, Himmelsbach and Lenzmann have obtained a general criterion for blow-up of radial solution of (1.1) with $k \geq \frac{4\alpha}{N}$ and $N \geq 2$. Although a general existence theorem for blow-up solutions of this problem is still an open problem, it has been strongly supported by numerical evidence [7].

Also, Eq. (1.2) has attracted more and more attention in both physics and mathematics, see [4–6, 8–16]. For the Hartree-type nonlinearity
\[
f(u) = (|x|^{-\gamma} * |u|^2) u,
\]
Cho et al. in [8] proved existence and uniqueness of local and global solutions of (1.2). In [9] the authors showed the existence of blow-up solutions. The dynamical properties of blow-up solutions have been investigated in [10, 11]. The stability and instability of standing waves have been studied in [12]. For other kinds of fractional Schrödinger equations in which the Hartree-type nonlinearity being replaced by a sublinearity, the orbital stability of standing waves has been studied in [13, 14].

Recently, Bhattarai in [17] employed the concentration compactness techniques to prove existence and stability of standing waves for the following nonlinear fractional Schrödinger-Choquard equation
\[
\begin{align*}
\left\{ \begin{array}{ll}
&i\partial_t u = (-\Delta)^{\alpha} u - |u|^k u - (|x|^{-\gamma} * |u|^p) |u|^{p-2} u, \quad (t, x) \in [0, T) \times \mathbb{R}^N, \\
&u(0, x) = \varphi(x),
\end{array} \right.
\end{align*}
\]
where $u: [0, T) \times \mathbb{R}^N \to \mathbb{C}$ is the complex-valued function, $N \geq 2$, $\varphi \in H^a$, $0 < a < 1$, $0 < T \leq \infty$, $0 < k < \frac{4a}{N}$, $2 \leq p < 1 + \frac{2a + N - 1}{N}$, $\gamma \in (0, N)$. Feng and Zhang in [18] proved existence and stability of standing waves with $k = -\frac{1}{\gamma}$. We remark that when $p = 2$, it reduces to (1.1); and when $0 < k < \frac{4a}{N}$, $k = -\frac{1}{\gamma}$ and $k > \frac{4a}{N}$ the equation is referred as the $L^2$-subcritical, $L^2$-critical and $L^2$-supercritical.

In this paper, we are going to prove the well-posedness and blow-up solutions of the above fractional Schrödinger-Choquard equation with $p = 2$, i.e. (1.1).

We write the Cauchy problem (1.1) in the following integral form

$$u(t) = U(t) \varphi - i \int_0^t U(t-t') (F(u) + G(u)) (t') dt',$$

where

$$G(u) = \lambda |u|^k u, \quad F(u) = \lambda \left( |x|^{-\gamma} \ast |u|^2 \right) u \equiv \lambda K_{\gamma}(u) u,$$

and $U(t)$ is the unitary group defined by

$$U(t) \varphi(x) = \left( e^{-it(-\Delta)^a} \varphi \right)(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i(x-\xi \cdot t)|x|^2} \hat{\varphi}(\xi) d\xi.$$

Here $\hat{\varphi}$ denotes the Fourier transform i.e. $\hat{\varphi}(\xi) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} \varphi(x) dx$.

We note that (1.1) possesses some conservation laws. If the solution $u$ of (1.1) has sufficient decay at infinity and smoothness, it satisfies the conservation of mass

$$M(u(t)) := \|u(t)\|_{L^2} = \|\varphi\|_{L^2},$$

and the conservation of energy

$$E(u(t)) = E(\varphi),$$

where $E(u(t))$ is defined by

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{a}{2}} u(t,x) \right|^2 dx + \frac{\lambda}{k+2} \int_{\mathbb{R}^N} |u(t,x)|^{k+2} dx \quad \text{and the energy space is } H^a.$$

A pair $(p, q)$ is said to be (general) admissible if they satisfy

$$p \in [2, \infty], \quad q \in [2, \infty), \quad (p, q, N) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{N}{q} \leq \frac{N}{2}.$$
For any pair \((g,h)\), we define
\[
\gamma_{gh} = \frac{N}{2} - \frac{N - 2}{h} - \frac{2\alpha}{g}. \tag{1.8}
\]
Specially, for radially symmetric data, \((p,q)\) satisfy the radial admissible condition,
\[
p \in [2,\infty), \quad q \in [2,\infty), \quad (p,q) \neq \left(\frac{4N-2}{2N-3}, \frac{2N}{p} + \frac{N}{q} = \frac{N}{2}\right). \tag{1.9}
\]
We denote that \([s]\) is the integer part of \(s\), i.e. \(s = [s] + \sigma\), where \(0 \leq \sigma < 1\).

We will use Strichartz estimates in (general) admissible condition and radial admissible condition to prove the well-posedness.

Our first result is the following local well-posedness for (1.1) with nonradial data in the sub-critical and critical nonlinearities.

**Theorem 1.1.** (Nonradial Local Well-posedness) Let \(N \geq 1\), \(\alpha \in (0,1)\setminus\{\frac{1}{2}\}\), \(0 < \gamma < N\) and \(0 < k \leq \frac{4\alpha}{N}\). Let \(s \in \left[\frac{\gamma^2}{2}, \frac{N}{2}\right)\) such that
\[
\left\{ \begin{array}{ll}
  s > \frac{1}{2} - \frac{\alpha}{4}, & N = 1, \\
  s > \frac{1}{2} - \frac{\alpha}{p}, & N \geq 2,
\end{array} \right.
\]
and if \(k\) is not an even integer, \([s] \leq k\) holds. Then for all \(\varphi \in H^s\), there exist \(T > 0\) and a unique solution \(u\) to (1.1) satisfying
\[
u \in C([0,T],H^s) \cap L^p_{loc}([0,T],W^{s-\gamma pq, q})
\]
for \(p > 4\) when \(N = 1\), \(p > 2\) when \(N \geq 2\) and \(s > N/2 - 2\alpha/p\), where \((p,q)\) is an admissible pair satisfying (1.7) and \(\gamma_{pq}\) satisfy (1.8).

Moreover, if \(T < \infty\), then \(\|u\|_{H^s} \to \infty\) when \(t \to T^-\).

Next theorem is about the global well-posedness for the defocusing fractional NLS and focusing fractional NLS with subcritical and critical nonlinearities.

**Theorem 1.2.** (Nonradial Global Well-posedness) Let \(N \geq 1\), \(0 < \alpha < 1\), \(0 < \gamma < N\), \(0 < k \leq \frac{4\alpha}{N}\) and \(s = \alpha \geq \gamma / 2\) with
\[
\left\{ \begin{array}{ll}
  \alpha > \frac{1}{2}, & N = 1 \\
  \alpha > \frac{\gamma}{2}, & 2 \leq N < 4.
\end{array} \right.
\]
Then for any \(\varphi \in H^s\), the solution \(u\) to (1.1) exists globally if one of the following conditions is satisfied:

I. \(\lambda = 1\), i.e. defocusing case,

II. \(\lambda = -1\), \(N \geq 2\), \(0 < k < \frac{4\alpha}{N}\) and \(\gamma < 2\alpha\), i.e. focusing subcritical nonlinearity case,

III. \(\lambda = -1\), \(N \geq 2\), \(k = \frac{4\alpha}{N}\), \(\gamma = 2\alpha\) and \(\|\varphi\|_2\) is small enough, i.e. focusing critical nonlinearity case.
Then, we give the local and global well-posedness for (1.1) with radial data in the sub-critical and critical nonlinearities.

**Theorem 1.3.** (Radial Local Well-posedness) Let \( N \geq 2, 0 < \gamma < N, 0 < k \leq \frac{4\alpha}{N} \) and \( s \in \left[ \frac{\gamma}{2}, \frac{N}{2} \right) \). If \( k \) is not an even integer, \( |s| \leq k \) holds. Define
\[
p = \frac{4\alpha(k+2)}{k(N-2s)}, \quad q = \frac{N(k+2)}{N+sk}.
\]
Then for any \( \varphi \in H^s_{\text{rad}} \), there exist \( T > 0 \) and a unique solution \( u \) to (1.1) satisfying
\[
u \in C([0,T],H^s_{\text{rad}}) \cap L^p_{\text{loc}}([0,T],W^{s,q}_{\text{rad}}).
\]
Moreover, if \( T < \infty \), then \( \| u \|_{H^s} \to \infty \) when \( t \to T^- \).

**Theorem 1.4.** (Radial Global Well-posedness) Let \( N \geq 2, 0 < \alpha < 1, 0 < \gamma < N, 0 < k \leq \frac{4\alpha}{N} \) and \( s = \alpha \geq \gamma / 2 \). Then for any \( \varphi \in H^s_{\text{rad}} \), the solution \( u \) to (1.1) exists globally if one of the following conditions is satisfied:

I. \( \lambda = 1 \), i.e. defocusing case,
II. \( \lambda = -1, 0 < k < \frac{4\alpha}{N} \) and \( \gamma < 2\alpha \), i.e. focusing subcritical nonlinearity case,
III. \( \lambda = -1, k = \frac{4\alpha}{N}, \gamma = 2\alpha \) and \( \| \varphi \|_{L^2} \) is small enough, i.e. focusing critical nonlinearity case.

Finally, we consider the blow-up dynamics of focusing fractional NLS (1.1) with \( \lambda = -1 \).

**Theorem 1.5.** Let \( N \geq 2, \alpha \in \left( \frac{1}{2}, 1 \right), k = \frac{4\alpha}{N}, \gamma = 2\alpha, \lambda = -1 \) and \( \varphi \in H^s_{\text{rad}} \). Let \( u \) be such that the corresponding solution to (1.1) exists on the maximal time \( T^* \). If \( E(\varphi) < 0 \), then one of the following statements holds true:

1. \( u(t) \) blows up in finite time in the sense that \( T^* < +\infty \) must hold.
2. \( u(t) \) blows up infinite time such that
\[
\sup_{t \geq 0} \left\| \left( -\Delta \right)^{\gamma / 4} u(\cdot, t) \right\|_{L^2} = \infty.
\]

**Remark 1.1.** Content of your remarks. As explained by Boulenger, Himmelsbach and Lenzmann [6], it is hard to expulse the possibility that the solutions may blow up at infinite time. We can only show that the solutions can not be uniformly bounded if they exist globally.

The rest of the paper is organized as follow. In Section 2, we introduce some important facts and tools. In Section 3, we prove the local and global existence of nonradial solutions via standard Strichartz estimates when \( 0 < k \leq \frac{4\alpha}{N} \) (for \( k = 0 \), we refer to [8]). In Section 4, we obtain the local and global existence of radial solutions via radial Strichartz estimates when \( 0 < k \leq \frac{4\alpha}{N} \). In the last section, we study the blow-up of radial solutions for the focusing fractional NLS in the critical case, i.e. \( k = \frac{4\alpha}{N} \) and \( \gamma = 2\alpha \).
We will use the notations $|\nabla| = \sqrt{-\Delta}$, $W^{s,p} = |\nabla|^{-s}L^p(\mathbb{R}^N)$ $(H^s = W^{s,2})$ and $W^{s,r} = (1-\Delta)^{-s/2}L^r(\mathbb{R}^N)$ $(H^s = W^{s,2})$. The norm $\|F\|_{L^q(X)}$ means $\left(\int_X |F(t,\cdot)|^q dt\right)^{1/q}$. We denote the space $L^q_t(B)$ by $L^q(0,T;B)$ and its norm by $\|\cdot\|_{L^q_tB}$ for some Banach space $B$, and also $L^q(B)$ with the norm $\|\cdot\|_{L^q(B)}$ for $L^q(0,\infty;B), 1 \leq q \leq \infty$. If not specified, throughout this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \leq CB$ and $A \geq C^{-1}B$, respectively. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$. $C$ is a generic positive constant possibly depending on $N, \alpha$ and $\gamma$.

## 2 Preliminaries

In this section, we present some facts which will be used to prove the local well-posedness. Firstly, we give the Strichartz estimates.

**Lemma 2.1.** ([19, 20]) For $N \geq 1$ and $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$, the following estimates hold

\[
\begin{align*}
\left\| e^{-it(-\Delta)^{\alpha}} \phi \right\|_{L^p_t(\mathbb{R}^N)} &\lesssim \left\| |\nabla|^{-\gamma_p} \phi \right\|_{L^2}, \quad (2.1) \\
\left\| \int_0^t e^{-i(t-\tau)(-\Delta)^{\alpha}} f(\tau) d\tau \right\|_{L^p_t(\mathbb{R}^N)} &\lesssim \left\| \nabla |\nabla|^{-\gamma_{p'}} f \right\|_{L^q_t(\mathbb{R}^N)} \quad (2.2)
\end{align*}
\]

where $(p,q), (a,b)$ are admissible pairs defined as (1.7), with

\[
\frac{1}{a} + \frac{1}{a'} = 1, \quad \frac{1}{b} + \frac{1}{b'} = 1,
\]

and $\gamma_{pq}$ and $\gamma_{p'q'}$ are defined as (1.8), i.e.

\[
\gamma_{pq} = \frac{N}{2} - \frac{N}{q} \frac{2\alpha}{p}, \quad \gamma_{p'q'} = \frac{N}{2} - \frac{N}{p'} \frac{2\alpha}{a'}. \quad (2.3)
\]

It is worthy noticing that for $\alpha \in (0,1) \setminus \{\frac{1}{2}\}$ the admissible condition $\frac{2}{p} + \frac{N}{q} \leq \frac{N}{2}$ implies $\gamma_{pq} > 0$ for all admissible pairs $(p,q)$ except $(p,q) = (\infty, 2)$. This means that the above Strichartz estimates have a loss of derivative. For radial data, the estimates (2.1) and (2.2) hold true for $N \geq 2, \alpha \in (0,1) \setminus \{\frac{1}{2}\}$ and $(p,q), (a,b)$ satisfy the radial admissible condition:

\[
\frac{2}{p} + \frac{2N-1}{q} \leq \frac{2N-1}{2}.
\]

This condition allows us to choose $(p,q)$ so that $\gamma_{pq} = 0$. More precisely, we have next lemma.

**Lemma 2.2.** ([20–22]) For $N \geq 2$ and $\frac{N}{2N-1} \leq \alpha < 1$,

\[
\left\| e^{-it(-\Delta)^{\alpha}} \phi \right\|_{L^p_t(\mathbb{R}^N)} \lesssim \left\| \phi \right\|_{L^2}, \quad (2.4)
\]
\[
\left\| \int_0^t e^{-i(t-\tau)(-\Delta)^a} f(\tau) d\tau \right\|_{L^p(R, L^q)} \lesssim \| f \|_{L^p(R, L^q)},
\]

(2.5)

where \( q \) and \( f \) are radially symmetric and \( (p, q), (a, b) \) satisfy the radial admissible condition (1.9).

The following is the Leibniz rule for fractional derivatives.

**Lemma 2.3.** ([8, 23]) Let \( \alpha \in (0, 1) \) and \( 1 < p, p_1 < \infty, 1 < q_1 \leq \infty \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} \). Then,

\[
\left\| \nabla^s (uv) \right\|_{L^p} \lesssim \left\| \nabla^s u \right\|_{L^{p_1}} \left\| v \right\|_{L^{q_1}} + \left\| \nabla^s v \right\|_{L^{p_2}} \left\| u \right\|_{L^{q_2}},
\]

\[
\left\| (1-\Delta)^{\frac{s}{2}} (uv) \right\|_{L^p} \lesssim \left\| (1-\Delta)^{\frac{s}{2}} u \right\|_{L^{p_1}} \left\| v \right\|_{L^{q_1}} + \left\| (1-\Delta)^{\frac{s}{2}} v \right\|_{L^{p_2}} \left\| u \right\|_{L^{q_2}}.
\]

The following fractional chain rule is basic for the estimates of the nonlinear terms.

**Lemma 2.4.** (Fractional Chain Rule [24]) Let \( F \in C^1(C, C) \) and \( \alpha \in (0, 1) \). Then for \( 1 < q \leq q_2 < \infty \) and \( 1 < q_1 \leq \infty \) satisfying \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \),

\[
\left\| \nabla^s F(u) \right\|_{L^q} \lesssim \left\| F'(u) \right\|_{L^{q_1}} \left\| \nabla^s u \right\|_{L^{q_2}}.
\]

Applying the above chain rule to \( F(z) = |z|^k \), we have

**Lemma 2.5.** ([23]) Let \( F(z) = |z|^k \) with \( k > 0, s \geq 0 \) and \( 1 < p, p_1 < \infty, 1 < q_1 \leq \infty \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \frac{k}{q_1} \). If \( k \) is an even integer or \( k \) is not an even integer with \( |s| \leq k \), then there exists \( C > 0 \) such that for all \( u \in \mathcal{X} \),

\[
\| F(u) \|_{W^{s,p}} \leq C \| u \|_{W^{s,p_1}}^k \| u \|_{W^{s,p_1}}.
\]

A similar estimate holds with \( W^{s,p}, W^{s,p_1} \) norms replaced by \( W^{s,p}, W^{s,p_1} \) norms.

Next, we recall a sharp Gagliardo-Nirenberg type inequality established by Boulenger, Himmelsbach and Lenzmann [6] and Zhu [14].

**Lemma 2.6.** Let \( N \geq 2, 0 < \alpha < 1 \) and \( 0 < p < \frac{4\alpha}{N - 2\alpha} \). Then, for all \( u \in H^\alpha \),

\[
\int_{\mathbb{R}^N} |u|^{p+2} dx \leq C_{opt} \| (-\Delta)^{\frac{\alpha}{2}} u \|_{L^2} \| u \|_{L^2}^{(p+2) - \frac{4N}{N - 2\alpha}},
\]

(2.6)

where the optimal constant \( C_{opt} \) is given by

\[
C_{opt} = \left( \frac{2\alpha (p+2) - pN}{pN} \right)^{\frac{Np}{Np - pN}} \left( \frac{2\alpha (p+2)}{2\alpha (p+2) - pN} \right)^{\frac{pN}{pN - pN}} Q \right\|_{L^2}^p,
\]

and \( Q \) is a ground state solution of the following elliptic equation

\[
(-\Delta)^{\alpha} Q + Q = |Q|^p Q \quad \text{in} \quad \mathbb{R}^N.
\]

In particular, in the \( L^2 \)-critical case \( p = \frac{4\alpha}{N} \), \( C_{opt} = \frac{2\alpha + N}{N |Q|_{L^2}} \).
Similarly, Feng and Zhang in [15] established the following sharp generalized Gagliardo-Nirenberg type inequality for Hartree-type nonlinearities.

**Lemma 2.7.** Let \( \alpha \in (0,1) \), \( N \geq 1 \), \( 0 < \gamma < N \) such that \( \frac{2N-\gamma}{\gamma N-2} > 2 \), the following generalized Gagliardo-Nirenberg inequality

\[
\int_{\mathbb{R}^N} (|x|^{-\gamma}\ast|u|^2) |u|^2 dx \leq C_\gamma \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} u \right|^2 dx \right)^{\frac{2}{\alpha}} \left( \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{2N-\gamma}{\alpha N-2}}
\]

holds and the best constant is

\[
C_\gamma = \frac{4\alpha^2}{4\alpha - \gamma} \left( \frac{4\alpha - \gamma}{\alpha \gamma} \right)^{\frac{2}{\gamma}} \| R \|_{L^2}^{-2},
\]

where \( R \) is the ground state of the elliptic equation

\[
(-\Delta)^{\alpha} R + R - (|x|^{-\gamma}\ast|R|^2) R = 0.
\]

**Lemma 2.8.** ([6]) Let \( N \geq 1 \) and \( f: \mathbb{R}^N \to \mathbb{R} \) satisfy \( \nabla f \in W^{1,\infty}(\mathbb{R}^N) \). Then, for all \( u \in H^{\frac{1}{2}}(\mathbb{R}^N) \), it holds that

\[
\left| \int_{\mathbb{R}^N} \bar{u}(x) \nabla f(x) \cdot \nabla u(x) dx \right| \leq C \left( \| \nabla |^{\frac{1}{2}} u \|_{L^2}^2 + \| u \|_{L^2} \| \nabla |^{\frac{1}{2}} u \|_{L^2} \right),
\]

with some constant \( C > 0 \) depending only on \( \| \nabla f \|_{W^{1,\infty}} \) and \( N \).

### 3 Well-posedness for Nonradial Data

In this section, we show the local and global well-posedness for (1.1) in nonradially symmetric Sobolev spaces.

First we are going to prove Theorem 1.1, the local well-posedness, by applying the contraction mapping argument via Strichartz estimates (2.1) and (2.2). The proof is divided in two steps.

**Proof. Step 1.** First we note that, by the assumptions, \( s - \gamma pq > \frac{N}{q} \) and \( p > k \). Let

\[
X(T,\rho) := \left\{ u \in L^\infty([0,T],H^s) \cap L^p([0,T],W^{s-\gamma pq,\theta}) : \| u \|_{L^\infty_T H^s} + \| u \|_{L^p_T W^{s-\gamma pq,\theta}} \leq \rho \right\},
\]

equipped with the distance

\[
d_X(u,v) := \| u - v \|_{L^\infty_T L^2} + \| u - v \|_{L^p_T W^{s-\gamma pq,\theta}},
\]
where \( \rho, T > 0 \) to be chosen later. It is easy to see that \((X(T, \rho), d_X)\) is a complete metric space. Now we define a mapping \( \mathcal{N}: u \mapsto \mathcal{N}(u) \) on \( X(T, \rho) \) by

\[
\mathcal{N}(u)(t) = U(t) \varphi - i \int_0^1 U(t-t') (F(u)+G(u)) (t') dt'.
\]

By the Strichartz estimates (2.1) and (2.2),

\[
\|\mathcal{N}(u)\|_{L_T^p H^s \cap L_T^q W^{s-\gamma,p,q}} \lesssim \|\varphi\|_{H^s} + \|F(u)\|_{L_T^p H^s} + \|G(u)\|_{L_T^q H^s}. \tag{3.1}
\]

Now we estimate the right hand of (3.1). For \( s \geq \gamma/2 \), using the Hardy-Littlewood-Sobolev inequality, Lemma 2.3, the Hardy type inequality

\[
\sup_{x \in \mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{|u(x-y)|^2}{|y|^T} dy \right| \lesssim \|u\|_{H^{T/2}}^2,
\]

and the Sobolev embedding \( \dot{H}^{T/2} \hookrightarrow L^{2N/(N-\gamma)} \), we have

\[
\|F(u)\|_{L_T^p H^s} \lesssim T \|F(u)\|_{L_T^p H^s} \\
\lesssim T \left( \|K_\gamma(|u|^2)\|_{L_T^p L^\infty} \|u\|_{L_T^p H^s} + \|K_\gamma(|u|^2)\|_{L_T^p W^{2N/(N-\gamma)}} \|u\|_{L_T^p L^{2N/(N-\gamma)}} \right) \\
\lesssim T \left( \|u\|_{L_T^p H^{T/2}}^2 \|u\|_{L_T^p H^s} + \|u\|_{L_T^p L^{2N/(N-\gamma)}}^2 \|u\|_{L_T^p H^s} \right) \\
\lesssim T \|u\|_{L_T^p H^{T/2}}^2 \|u\|_{L_T^p H^s} \\
\lesssim T \rho^3. \tag{3.2}
\]

For the last term of (3.1), we apply the fractional chain rule given in Lemma 2.5, the Hölder inequality and the fact \( p > k \) to get

\[
\|G(u)\|_{L_T^p H^s} = \|u|^k \|_{L_T^p H^s} \\
\lesssim \|u|^k \|_{L_T^p L^\infty} \|u\|_{L_T^p H^s} \\
\lesssim \|u\|_{L_T^p L^\infty} \|u\|_{L_T^p H^s} \\
\lesssim T^{1-\frac{k}{p}} \|u\|_{L_T^p L^\infty} \|u\|_{L_T^p H^s}. \tag{3.3}
\]

Note the fact \( s - \gamma p/q > N/q \) and the Sobolev embedding \( W^{s-\gamma p/q} \hookrightarrow L^\infty \), we get

\[
\|G(u)\|_{L_T^p H^s} \lesssim T^{1-\frac{k}{p}} \|u\|_{L_T^p W^{s-\gamma p/q}} \|u\|_{L_T^p H^s} \lesssim T^{1-\frac{k}{p}} \rho^{k+1}. \tag{3.4}
\]

Hence, putting (3.3) and (3.4) into (3.1), we get

\[
\|\mathcal{N}(u)\|_{L_T^p H^s \cap L_T^q W^{s-\gamma,p,q}} \lesssim C(\|\varphi\|_{H^s} + T \rho^3 + T^{1-\frac{k}{p}} \rho^{k+1}).
\]
Let $\rho = 2C\|\varphi\|_{H^s}$ and choose $T$ such as $C(T\rho^2 + T^{1-\frac{1}{\gamma}}\rho^k) \leq \frac{1}{2}$, then $\mathcal{N}$ maps $X(T,\rho)$ into itself.

**Step 2.** Now we show that $\mathcal{N}$ is a contraction map for sufficiently small $T$. Let $u, v \in X(T,\rho)$, using (2.2), we have

$$d_X(\mathcal{N}(u), \mathcal{N}(v)) \lesssim \|F(u) - F(v)\|_{L_t^1 L_x^2} + \|G(u) - G(v)\|_{L_t^1 L_x^2}. \quad (3.5)$$

Now, we start to estimate the right hand of (3.5). For the first term, we have

$$\|F(u) - F(v)\|_{L_t^1 L_x^2} \lesssim T \|K_\gamma (|u|^2) u - K_\gamma (|v|^2) v\|_{L_t^1 L_x^2}$$

$$\lesssim T \left( \|K_\gamma (|u|^2) (u - v)\|_{L_t^1 L_x^2} + \|K_\gamma (|u|^2 - |v|^2) v\|_{L_t^1 L_x^2} \right)$$

$$\lesssim T \left( \|u\|_{L_t^{2s} H^s_x}^2 \|u - v\|_{L_t^1 L_x^2} + \|K_\gamma (|u|^2 - |v|^2)\|_{L_t^{1/(1-\gamma)} L_x^{2N/(N-\gamma)}} \|v\|_{L_t^{1/(1-\gamma)} L_x^{2N/(N-\gamma)}} \right)$$

$$\lesssim T \left( \|u\|_{L_t^{2s} H^s_x}^2 \|u - v\|_{L_t^1 L_x^2} + \|u^2 - |v|^2\|_{L_t^{1/(2N-\gamma)} L_x^{2N/(N-\gamma)}} \|v\|_{L_t^{1/(2N-\gamma)} L_x^{2N/(N-\gamma)}} \right)$$

$$\lesssim T \|u\|_{L_t^{2s} H^s_x}^2 \|u - v\|_{L_t^1 L_x^2} + \|u - v\|_{L_t^1 L_x^2} \|u + v\|_{L_t^{1/(2N-\gamma)} L_x^{2N/(N-\gamma)}} \|v\|_{L_t^{1/(2N-\gamma)} L_x^{2N/(N-\gamma)}}$$

$$\lesssim T \rho^2 d_X(u, v). \quad (3.6)$$

For the last term of (3.5), we get

$$\|G(u) - G(v)\|_{L_t^1 L_x^2} = \||u|^k u - |v|^k v\|_{L_t^1 L_x^2}$$

$$\lesssim \left( \|u\|_{L_t^4 L_x^\infty}^k + \|v\|_{L_t^4 L_x^\infty}^k \right) \|u - v\|_{L_t^1 L_x^2}$$

$$\lesssim T^{1-\frac{1}{\gamma}} \left( \|u\|_{L_t^4 L_x^\infty}^k + \|v\|_{L_t^4 L_x^\infty}^k \right) \|u - v\|_{L_t^1 L_x^2}$$

$$\lesssim T^{1-\frac{1}{\gamma}} \left( \|u\|_{L_t^{1/(1-\gamma)} H^s_x - \gamma \rho d_x}^k + \|v\|_{L_t^{1/(1-\gamma)} H^s_x - \gamma \rho d_x}^k \right) \|u - v\|_{L_t^1 L_x^2}$$

$$\lesssim T^{1-\frac{1}{\gamma}} \rho^k d_X(u, v). \quad (3.7)$$

From (3.5)–(3.7), we have

$$d_X(\mathcal{N}(u), \mathcal{N}(v)) \lesssim C \left( T \rho^2 + T^{1-\frac{1}{\gamma}} \rho^k \right) d_X(u, v).$$

Then we choose $T$ small enough such that

$$C(T \rho^2 + T^{1-\frac{1}{\gamma}} \rho^k) \leq \frac{1}{2},$$

the above estimate implies that the mapping $\mathcal{N}$ is a contraction. Then $\mathcal{N}$ admits a unique fixed point in $X(T,\rho)$, which is the solution of (1.1). This completes the proof of local well-posedness.
Next, we will prove Theorem 1.2, the global well-posedness for (1.1) in two cases: \( \lambda = 1 \), the defocusing case and \( \lambda = -1 \), the focusing case with subcritical/critical nonlinearities.

**Proof of Theorem 1.2.**

*Proof.* First we consider the defocusing case \( \lambda = 1 \). Let \( T^* \) be the maximal existence time. We will prove that \( T^* = \infty \) by contradiction. Suppose that \( T^* < \infty \), then the local well-posedness shows that

\[
\|u\|_{L^p_t W^{s-\gamma,p,q}} = \infty.
\]  

(3.8)

From Theorem 1.1, the conservation laws (1.4) and (1.5), for any \( t < T^* \), the solution \( u \) satisfies

\[
\frac{1}{2} \|u(t)\|_{H^s}^2 + T^* \|u\|_{L^p_t H^s}^{3-\frac{k}{2}} \|u\|_{L^p_t W^{s-\gamma,p,q}} \|u\|_{L^p_t H^s}^{1-\frac{k}{2}} \leq (\|\varphi\|_{L^2}^2 + E(\varphi))^\frac{3}{2} + T(\|\varphi\|_{L^2}^2 + E(\varphi))^\frac{3}{2} + T^{1-\frac{k}{2}} (\|\varphi\|_{L^2}^2 + E(\varphi))^\frac{3}{2}
\]

Thus for sufficiently small \( T \) depending only on \( \|\varphi\|_{L^2}^2 + E(\varphi) \), we have

\[
\|u\|_{L^p_t(0,T;W^{s-\gamma,p,q})} \leq C(\|\varphi\|_{L^2}^2 + E(\varphi))^\frac{3}{2},
\]

where \( T_j - T_{j-1} = T \). If \( T^* < \infty \), then there exists \( M \in \mathbb{N} \) such that \( (M-1)T < T^* \leq MT \). Let \( T_j = (j-1)T, j = 1, \ldots, M-1 \) and \( T_M = T^* \). Then we have

\[
\|u\|_{L^p_t(0,T;W^{s-\gamma,p,q})} \leq \sum_{1 \leq j \leq M} \|u\|_{L^p_t(T_{j-1},T_j;W^{s-\gamma,p,q})}^p \leq \left(MC(\|\varphi\|_{L^2}^2 + E(\varphi))^\frac{3}{2}\right)^p < \infty.
\]

This contradicts to (3.8), and thus completes the proof of the defocusing case. \( \lambda = -1 \). From Lemmas 2.6 and 2.7,

\[
\int_{\mathbb{R}^N} |u|^{k+2} dx \leq C_{opt} \|u\|_{L^p} \|\varphi\|_{L^2}^{(k+2)-\frac{k}{2}},
\]

\[
\int_{\mathbb{R}^N} (|x|^{-\gamma} * |u|^2) |u|^2 dx \leq C_{\gamma} \|u\|_{L^p}^\frac{3}{2} \|u\|_{L^2}^{4-\frac{3}{2}},
\]
we get
\[
E(u(t)) = \frac{1}{2} \|u\|_{H^s}^2 - \frac{1}{k+2} \|u\|_{L^2}^{k+2} - \frac{1}{4} \int_{\mathbb{R}^N} (|x|^{-\frac{\gamma}{s}} |u|^2) |u|^2 \, dx
\]
\[
\geq \frac{1}{2} \|u\|_{H^s}^2 - \frac{C_{\text{opt}}}{k+2} \|u\|_{H^s}^{k+2} \|u\|_{L^2}^{k+2} - \frac{C\gamma}{4} \|u\|_{H^s}^{\frac{\gamma}{s}} \|u\|_{L^2}^{4-\frac{\gamma}{s}}
\]
\[
\geq \frac{1}{2} \|u\|_{H^s}^2 - \frac{C_{\text{opt}}}{k+2} \|u\|_{H^s}^{k+2} \|\varphi\|_{H^s}^{(k+2)-\frac{4\gamma}{s}} - \frac{C\gamma}{4} \|\varphi\|_{H^s}^{\frac{\gamma}{s}} \|\varphi\|_{L^2}^{4-\frac{\gamma}{s}}.
\]

For the condition II in Theorem 1.2, i.e. \(0 < k < \frac{4\alpha}{N}\) and \(\gamma < 2\alpha\), using the Young inequality, we can easily get
\[
\|u(t)\|_{H^s}^2 \lesssim |E(\varphi)|.
\]

For the condition III in Theorem 1.2, i.e. \(k = \frac{4\alpha}{N}, \gamma = 2\alpha\) and \(\|\varphi\|_{L^2}\) is sufficiently small, the above inequality still holds true. Therefore,
\[
\|u(t)\|_{H^s}^2 \leq \|u(t)\|_{L^2}^2 + E(u) = \|\varphi\|_{L^2}^2 + E(\varphi).
\]

Similar to the defocusing case, we can apply the contradiction argument to prove the global well-posedness. This completes the proof of Theorem 1.2. \(\square\)

## 4 Well-posedness for Radial Data

In this section, we will show the local and global well-posedness for (1.1) with radial initial data in Sobolev spaces. The proof is also based on the contraction mapping argument via Strichartz estimates (2.4) and (2.5). In the radial case, thanks to Strichartz estimates without loss of derivatives, we have better local well-posedness comparing to the nonradial case.

The proof of Theorem 1.3 is divided into two steps.

**Proof.** **Step 1.** It is easy to check that when \((p,q)\) satisfies the radial admissible condition (1.9), we can choose \((m,n)\) so that
\[
\frac{1}{p'} = \frac{1}{p} + \frac{k}{m}, \quad \frac{1}{q'} = \frac{1}{q} + \frac{k}{n},
\]
and we see that
\[
\theta = \frac{k}{m} - \frac{k}{p} = 1 - \frac{k(N-2s)}{4\alpha} \in (0,1), \quad q \leq n = \frac{qN}{N-sq}.
\]
The later fact gives the Sobolev embedding \(W^{s,q} \hookrightarrow L^n\). Let \((Y(T),d_T)\) be the complete metric space
\[
Y(T) = \left\{ C([0,T], H^s_{rad}) \cap L^p_T(W^{s,q}) : \|u\|_{L^p_t H^s} + \|u\|_{L_T^p W^{s,q}} \leq \rho \right\}
\]
equipped with the distance
\[ d_Y(u,v) := \|u-v\|_{L^p_t L^2} + \|u-v\|_{L^q_t L^q}. \]

As before we define a mapping \( \mathcal{N} : u \mapsto \mathcal{N}(u) \) on \( Y(T,\rho) \) by
\[
\mathcal{N}(u)(t) = U(t)\varphi - i \int_0^t U(t-t') (F(u) + G(u))(t') \, dt'.
\]

From the radial Strichartz estimates (2.4) and (2.5), we obtain
\[
\| \mathcal{N}(u) \|_{L^p_t H^s \cap L^q_t W^{\sigma,\theta}} \leq \| \varphi \|_{H^s} + \| F(u) \|_{L^p_t H^s} + \| G(u) \|_{L^q_t W^{\sigma,\theta}}, \tag{4.1}
\]

Now we estimate the right hand of (4.1). For \( s \geq \gamma/2 \),
\[
\| F(u) \|_{L^p_t H^s} \lesssim T \| F(u) \|_{L^p_t H^s}
\]
\[
\lesssim T \left( \| K_\gamma (|u|^2) \|_{L^p_t L^\infty} \| u \|_{L^p_t H^s} + \| K_\gamma(|u|^2) \|_{L^p_t W^{2N/(N-\gamma)}} \| u \|_{L^p_t L^{2N/(N-\gamma)}} \right)
\]
\[
\lesssim T \left( \| u \|_{L^p_t H^{s/2}}^2 \| u \|_{L^p_t H^s} + \| u \|_{L^p_t W^{2N/(N-\gamma)}}^2 \| u \|_{L^p_t H^s} \right)
\]
\[
\lesssim T \| u \|_{L^p_t H^{s/2}}^2 \| u \|_{L^p_t H^s}
\]
\[
\lesssim T \rho^3. \tag{4.2}
\]

Here we have used the Hardy-Littlewood-Sobolev inequality, Lemma 2.3, the Hardy type inequality (3.2) and the Sobolev embedding \( H^{\gamma/2} \hookrightarrow L^{2N/(N-\gamma)} \).

By the fractional chain rule in Lemma 2.5 and the Hölder inequality, we get
\[
\| G(u) \|_{L^p_t W^{\sigma,\theta}}^q = \left\| |u|^k u \right\|_{L^q_t W^{\sigma,\theta}}
\]
\[
\lesssim \| u \|_{L^p_t L^q}^k \| u \|_{L^p_t W^{\sigma,\theta}}
\]
\[
\lesssim T^q \| u \|_{L^p_t L^q}^k \| u \|_{L^p_t W^{\sigma,\theta}}
\]
\[
\lesssim T^q \| u \|_{L^p_t L^q}^{k+1} \| u \|_{L^p_t W^{\sigma,\theta}}
\]
\[
\lesssim T^q \rho^{k+1}. \tag{4.3}
\]

Hence, putting (4.2) and (4.3) into (4.1), we get
\[
\| \mathcal{N}(u) \|_{L^p_t H^s \cap L^q_t W^{\sigma,\theta}} \leq C(\| \varphi \|_{H^s} + T \rho^3 + T^q \rho^{k+1}).
\]

Let \( \rho = 2C\| \varphi \|_{H^s} \) and choose \( T \) such that \( C(T \rho^2 + T^q \rho^k) \leq \frac{1}{2} \), then \( \mathcal{N} \) maps \( Y(T,\rho) \) into itself.

**Step 2.** Now we show that \( \mathcal{N} \) is a contraction map for sufficiently small \( T \). Let \( u,v \in Y(T,\rho) \), then we have
\[
d_Y(\mathcal{N}(u),\mathcal{N}(v)) \lesssim \| F(u) - F(v) \|_{L^p_t L^2} + \| G(u) - G(v) \|_{L^q_t L^q}. \tag{4.4}
\]
Now we estimate the right hand of (4.4). For the first term, we have
\[
\| F(u) - F(v) \|_{L_t^1 L_x^2} \\
\lesssim T \| K_\gamma (|u|^2) u - K_\gamma (|v|^2) v \|_{L_t^1 L_x^2} \\
\lesssim T \left( \| K_\gamma (|u|^2) (u - v) \|_{L_t^1 L_x^2} + \| K_\gamma (|u|^2 - |v|^2) v \|_{L_t^1 L_x^2} \right) \\
\lesssim T \left( \| u \|_{L_t^q H^{1/2}}^2 \| (u - v) \|_{L_t^1 L_x^2} + \| K_\gamma (|u|^2 - |v|^2) \|_{L_t^p L_x^{2N/(N-\gamma)}} \| v \|_{L_t^p L_x^{2N/(N-\gamma)}} \right) \\
\lesssim T \left( \| u \|_{L_t^q H^{1/2}}^2 \| (u - v) \|_{L_t^1 L_x^2} + \| u \|_{L_t^2 L_x^{2N/(2N-\gamma)}} \| u - v \|_{L_t^1 L_x^2} \| v \|_{L_t^p L_x^{2N/(N-\gamma)}} \right) \\
\lesssim T \rho^2 d_Y (u,v).
\]
For the last term of (4.4), we have
\[
\| G(u) - G(v) \|_{L_t^p L_x^q} = \| u^k u - |v|^k v \|_{L_t^p L_x^q} \\
\lesssim \left( \| u \|_{L_t^p W^{\gamma/2}}^k + \| v \|_{L_t^p W^{\gamma/2}}^k \right) \| u - v \|_{L_t^p L_x^q} \\
\lesssim T^\theta \left( \| u \|_{L_t^p W^{\gamma/4}}^k + \| v \|_{L_t^p W^{\gamma/4}}^k \right) \| u - v \|_{L_t^p L_x^q} \\
\lesssim T^\theta \rho^k d_Y (u,v).
\]
Hence
\[
d_Y (\mathcal{N}(u), \mathcal{N}(v)) \lesssim C \left( T \rho^2 + T^\theta \rho^k \right) d_Y (u,v).
\]
Then we can choose \( T \) small enough such that \( C (T \rho^2 + T^\theta \rho^k) \lesssim \frac{1}{2} \), the above estimate implies that the mapping \( \mathcal{N} \) is a contraction. Then \( \mathcal{N} \) admits a fixed point in \( X(T, \rho) \), which is the solution of (1.1). This completes the proof of Theorem 1.3.

Next, we will prove the global well-posedness for (1.1) with radial initial data in two cases: \( \lambda = 1 \), the defocusing case and \( \lambda = -1 \), the focusing case.

**Proof of Theorem 1.4.**

Proof. First we consider the defocusing case \( \lambda = 1 \). Let \( T^* \) be the maximal existence time. We will prove that \( T^* = \infty \) by contradiction. Suppose that \( T^* < \infty \), then the local well-posedness shows that
\[
\| u \|_{L_t^p W^{\gamma/4}} = \infty.
\]
(4.5)
From Theorem 1.3, the conservation laws (1.4) and (1.5), for any \( t < T^* \), the solution \( u \) satisfies
\[
\frac{1}{2} \| u(t) \|_{H^3}^2 + \frac{1}{2} \| u(t) \|_{L_x^2}^2 + E(u) = \frac{1}{2} \| \varphi \|_{L_x^2}^2 + E(\varphi).
\]
From the estimate (4.1) with $s = \alpha$, we have
\[ \|u\|_{L^p_t W^{k+1, q}_x} \lesssim \|\varphi\|_{H^{\alpha}} + T^\alpha \|u\|_{L^p_t W^{k\alpha, q}_x}^3 + T \|\varphi\|_{L^2}^2 + E(\varphi)^{\frac{3}{2}}. \]
Thus for sufficiently small $T$ depending only on $\|\varphi\|_{L^2}^2 + E(\varphi)$, we have
\[ \|u\|_{L^p_t (T_j-1, T_j; W^{k\alpha, q}_x)} \leq C \left( \|\varphi\|_{L^2}^2 + E(\varphi) \right)^{\frac{3}{2}}. \]
where $T_j - T_{j-1} = T$. If $T^* < \infty$, then there exists $M \in \mathbb{N}$ such that $(M-1)T < T^* \leq MT$. Let $T_j = (j-1)T$, $j = 1, \ldots, M-1$ and $T_M = T^*$. Then we have
\[ \|u\|_{L^p_t (0, T^*; W^{k\alpha, q}_x)} \leq \sum_{1 \leq j \leq M} \|u\|_{L^p_t (T_{j-1}, T_j; W^{k\alpha, q}_x)}^{p} \leq \left( MC \left( \|\varphi\|_{L^2}^2 + E(\varphi) \right)^{\frac{3}{2}} \right)^p < \infty. \]
This contradicts to (4.5), and thus completes the proof of the defocusing case.

Next we consider the focusing case, $\lambda = -1$. Similarly to the proof of the focusing case in Theorem 1.2, we have also
\[ \|u(t)\|_{H^\alpha} \leq \|u(t)\|_{L^2}^2 + E(u) = \|\varphi\|_{L^2}^2 + E(\varphi). \]
This completes the proof of Theorem 1.4.

\section{Blow-up Solutions}

In this section, we are going to give the proof of Theorem 1.5.

According to Theorem 1.3, for a radial initial function $\varphi \in H^\alpha_{\text{rad}}$ (1.1) admits a local solution $u \in H^\alpha_{\text{rad}}$. If $T^* < \infty$, then we are done. If $T^* = \infty$, we show (1.10).

Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be radial and satisfy
\[ \psi(r) = \begin{cases} \frac{1}{2} r^2, & \text{for } r \leq 1 \\ 0, & \text{for } r \geq 2 \end{cases} \quad \text{and} \quad \psi''(r) \leq 1 \quad \text{for } r = |x| > 0. \quad (5.1) \]
For a fixed $R > 0$, we define the rescaled function $\psi_R : \mathbb{R}^N \to \mathbb{R}$ by setting
\[ \psi_R(r) := R^2 \psi \left( \frac{r}{R} \right). \quad (5.2) \]
We readily verify the inequalities
\[ 1 - \psi_R(r) \geq 0, \quad 1 - \frac{\psi_R'(r)}{r} \geq 0, \quad N - \Delta \psi_R(r) \geq 0 \quad \text{for all } r \geq 0. \quad (5.3) \]
Indeed, this first inequality follows from $\psi''_R(r) = \psi''(r/R) \leq 1$. We obtain the second inequality by integrating the first inequality on $[0,r]$ and using that $\psi_R'(0) = 0$. Finally, we see that last inequality follows from

$$N - \Delta \psi_R(r) = 1 - \psi''_R(r) + (N - 1) \left\{ 1 - \frac{1}{r} \psi'_R(r) \right\} \geq 0.$$ 

Besides (5.3), $\psi_R$ admits the following properties, which can be easily checked.

$$\left\{ \begin{array}{l}
\nabla \psi_R(r) = R \psi'(r/R) \frac{x}{|x|} = \left\{ \begin{array}{ll}
x & \text{for } r \leq R, \\
0 & \text{for } r \geq 2R,
\end{array} \right.
\| \nabla^j \psi_R \|_{L^\infty} \leq R^{2-j} & \text{for } 0 \leq j \leq 4,
\text{supp} (\nabla^j \psi_R) \subset \left\{ \begin{array}{ll}
\{ |x| \leq 2R \} & \text{for } 1, 2,
\{ R \leq |x| \leq 2R \} & \text{for } 3 \leq j \leq 4.
\end{array} \right.
\end{array} \right.$$ 

We define

$$\mathcal{M}_{\psi_R}[u(t)] := 2\text{Im} \int_{\mathbb{R}^N} \bar{u}(t) \nabla \psi_R \cdot \nabla u(t) dx = 2\text{Im} \int_{\mathbb{R}^N} \bar{u}(t) \partial_t \psi_R \partial_t u(t) dx.$$ 

Define the self-adjoint differential operator

$$\Gamma_{\psi_R} := -i(\nabla \cdot \nabla \psi_R + \nabla \psi_R \cdot \nabla),$$

which acts on functions according to

$$\Gamma_{\psi_R} f = -i(\nabla \cdot ((\nabla \psi_R) f) + (\nabla \psi_R) \cdot (\nabla f)).$$

It’s easy to check that

$$\mathcal{M}_{\psi_R}[u(t)] = \langle u(t), \Gamma_{\psi_R} u(t) \rangle.$$ 

First, we give the following lemma.

**Lemma 5.1.** Let $N \geq 2$, $\alpha \in (\frac{1}{2}, 1)$, $k = \frac{N}{2}$, $\gamma = 2\alpha$, and $u \in H^2_{rad}$ is a solution of (1.1) with $\lambda = -1$. Let $\psi_R$ be as in (5.2), $T^*$ be the maximal existence time of solution $u(t)$ in $H^2_{rad}$. Then for sufficiently large $R$, we have

$$\frac{d}{dt} \mathcal{M}_{\psi_R}[u(t)] \leq 4\alpha E(u(t)), \quad t \in [0, T^*).$$

**Proof.** By taking the derivative of $\mathcal{M}_{\psi_R}[u(t)]$ with respect to time $t$ and using the equation of $u(t)$, for any $t \in [0, T)$, we get

$$\frac{d}{dt} \mathcal{M}_{\psi_R}[u(t)] = \langle u(t), \left[ (\Delta)^{\alpha}, i\Gamma_{\psi_R} \right] u(t) \rangle + \langle u(t), \left[ -|u|^k, i\Gamma_{\psi_R} \right] u(t) \rangle$$

$$+ \langle u(t), \left[ -K_{\gamma}(u), i\Gamma_{\psi_R} \right] u(t) \rangle$$

$$=: I_1 + I_2 + I_3,$$ (5.4)
where \([X,Y] \equiv XY - YX\) denotes the commutator of operators \(X\) and \(Y\).

From [6], \(k = \frac{4\alpha}{N}\), we get
\[
I_1 = \langle u(t), \left[ \left(-\Delta\right)^\alpha, i\Gamma\psi_R \right] u(t) \rangle \leq 4\alpha \| (-\Delta)^\frac{\alpha}{2} u \|_{L^2}^2 + CR^{-2\alpha},
\]
and
\[
I_2 = \langle u, \left[ -|u|^k, i\Gamma\psi_R \right] u \rangle
= -\langle u, \left[ |u|^k, \nabla\psi_R \cdot \nabla + \nabla \cdot \nabla\psi_R \right] u \rangle
= 2 \int_{\mathbb{R}^N} |u|^2 \nabla\psi_R \cdot \nabla \left( |u|^k \right) dx
= \frac{2k}{k+2} \int_{\mathbb{R}^N} (\Delta\psi_R) |u|^{k+2} dx
= \frac{2kN}{k+2} \int_{\mathbb{R}^N} |u|^{k+2} dx - \frac{2k}{k+2} \int_{|x| > R} (\Delta\psi_R - N) |u|^{k+2} dx
\leq \frac{2kN}{k+2} \int_{\mathbb{R}^N} |u|^{k+2} dx + C R^{-\frac{k(N-1)}{2} + \epsilon_1 \alpha} \| (-\Delta)^\frac{\alpha}{2} u \|_{L^2}^{\frac{(k+1)\alpha + \epsilon_1}{2}},
\]
where \(0 < \epsilon_1 < (2\alpha - 1)k/2\alpha\).

From [25] and \(\gamma = 2\alpha\), we get
\[
I_3 = \langle u(t), \left[ -K_\gamma(u), i\Gamma\psi_R \right] u(t) \rangle
\leq -\gamma \int \left( |x|^{-\gamma} |u|^2 \right) |u(x)|^2 dx + \frac{C}{R^{\frac{1}{2}(N-1-\epsilon_2)(\frac{N}{2})}} \| (-\Delta)^\frac{\alpha}{2} u \|_{L^2}^{2 + \frac{(\gamma + 1)\alpha}{2N}},
\]
where \(0 < \epsilon_2 < 2\alpha - 1 < N\). Therefore,
\[
\frac{d}{dt} M_{\psi_R}[u(t)] \leq 4\alpha \| (-\Delta)^\frac{\alpha}{2} u \|_{L^2}^2 - \frac{2kN}{k+2} \int_{\mathbb{R}^N} |u|^{k+2} dx
- \gamma \int \left( |x|^{-\gamma} |u|^2 \right) |u(x)|^2 dx
+ CR^{-2\alpha} + C R^{-\frac{k(N-1)}{2} + \epsilon_1 \alpha} \| (-\Delta)^\frac{\alpha}{2} u \|_{L^2}^{\frac{(k+1)\alpha + \epsilon_1}{2}}
\]
From (1.4) and (1.5), we have
\[
\frac{d}{dt} M_{\psi_R}[u(t)] \leq 8\alpha E(u(t)) + CR^{-2\alpha}
+ C R^{-\frac{k(N-1)}{2} + \epsilon_1 \alpha} \| (-\Delta)^\frac{\alpha}{2} u \|_{L^2}^{\frac{(k+1)\alpha + \epsilon_1}{2}}
\]
\[
+ \frac{C}{R^{\frac{1}{2}(N-1-\epsilon_2)(\frac{N}{2})}} \| (-\Delta)^\frac{\alpha}{2} u \|_{L^2}^{2 + \frac{(\gamma + 1)\alpha}{2N}},
\]
(5.5)
where the constant $C > 0$ is independent of $R$. When $R > 1$ is sufficiently large, $\varepsilon_1$ and $\varepsilon_2 < 1$ are sufficiently small, then

$$\frac{d}{dt} M_{\psi_R}[u(t)] \leq 4\alpha E(u(t)) = 4\alpha E(\varphi).$$  (5.6)

The proof of Lemma 5.1 is completed. □

Now we begin to prove case (2) of Theorem 1.5. We suppose that $u(t)$ exists for all times $t \geq 0$, i.e. $T^* = \infty$.

It follows from Lemma 5.1 and conservation of mass, for $R \gg 1$ large enough,

$$\frac{d}{dt} M_{\psi_R}[u(t)] \leq 4\alpha E(\varphi) \overset{\triangle}{=} -A^* < 0, \quad t \geq 0. \quad (5.7)$$

From (5.7), we infer that

$$M_{\psi_R}[u(t)] \leq -\frac{1}{2}A^* t + M_{\psi_R}[\varphi], \quad t \geq 0. \quad (5.8)$$

On the one hand, let $T_0 = \frac{2|\mathcal{M}_{\psi_R}[\varphi]|}{A^*} > 0$, then for any $t \geq T_0$, we have

$$M_{\psi_R}[u(t)] \leq -\frac{1}{2}A^* t < 0.$$

On the other hand, by Lemma 2.8 and the conservation of mass, we see that for any $t \in [0, +\infty)$,

$$|M_{\psi_R}[u(t)]| \lesssim C(\psi_R) \left( \left\| \nabla \frac{1}{2} u(t) \right\|_{L^2}^2 + \|u(t)\|_{L^2} \left\| \nabla \frac{1}{2} u(t) \right\|_{L^2} \right).$$

$$\lesssim C(\psi_R) \left( \left\| \nabla \frac{1}{2} u(t) \right\|_{L^2}^2 + \|u(t)\|_{L^2}^2 \right).$$

$$\lesssim C(\psi_R) \left( \left\| \nabla \frac{1}{2} u(t) \right\|_{L^2}^2 + 1 \right).$$

$$\lesssim C(\psi_R) \left( \left\| (-\Delta)^{\frac{1}{2}} u(t) \right\|_{L^2}^\frac{1}{2} + 1 \right).$$

Here we have used the interpolation estimate

$$\left\| \nabla \frac{1}{2} u \right\|_{L^2} \lesssim \left\| (-\Delta)^{\frac{1}{4}} u \right\|_{L^2}^{\frac{1}{2}} \left\| u \right\|_{L^2}^{1-\frac{1}{4}} \quad \text{for} \quad \alpha > \frac{1}{2}.$$ 

This combined with (5.8) yields that for any $t \geq T_0$,

$$A^* t \leq -2M_{\psi_R}[u(t)] \lesssim C(\psi_R) \left( \left\| (-\Delta)^{\frac{1}{2}} u(t) \right\|_{L^2}^\frac{1}{2} + 1 \right).$$
This shows that

$$\left\| \left(-\Delta\right)^{\frac{\alpha}{2}} u(t) \right\|_{L^2} \geq C t^\alpha, \quad t \geq T_0.$$  

It means that

$$\sup_{t \geq 0} \left\| \left(-\Delta\right)^{\frac{\alpha}{2}} u(\cdot, t) \right\|_{L^2} = \infty.$$  

The proof of Theorem 1.5 is completed.

**Acknowledgement**

This research is supported by NSFC key project under the grant number 11831003, NSFC under the grant numbers 11971356 and 11571118, and also by Fundamental Research Founds for the Central Universities under the grant numbers 2019MS110 and 2019MS112.

**References**


