Well-Posedness and Blow-Up for the Fractional Schrödinger-Choquard Equation

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Abstract. In this paper, we study the well-posedness and blow-up solutions for the fractional Schrödinger equation with a Hartree-type nonlinearity together with a power-type subcritical or critical perturbations. For nonradial initial data or radial initial data, we prove the local well-posedness for the defocusing and the focusing cases with subcritical or critical nonlinearity. We obtain the global well-posedness for the defocusing case, and for the focusing mass-subcritical case or mass-critical case with initial data small enough. We also investigate blow-up solutions for the focusing mass-critical problem.

1 Introduction

In this paper we consider the following Cauchy problem for the fractional nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u = (-\Delta)^\alpha u + \lambda \left(|u|^k u + \left(|x|^{-\gamma} * |u|^2 \right) u \right), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $N \geq 1$, $0 < \alpha < 1$, $0 < \gamma < N$, $0 \leq k \leq \frac{4\alpha}{N}$, $\lambda = \pm 1$, $*$ denotes the convolution in $\mathbb{R}^N$, i is the imaginary unit and $u = u(t,x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is the unknown complex-valued function. The fractional Laplace operator $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u = \frac{1}{(2\pi)^N} \int e^{ix\cdot\xi} |\xi|^{2\alpha} \mathcal{F}[u](\xi) d\xi = \mathcal{F}^{-1} \left[|\xi|^{2\alpha} \mathcal{F}[u](\xi) \right], 20$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and the Fourier inverse transform in $\mathbb{R}^N$, respectively. When $\lambda = 1$, (1.1) is referred to be defocusing fractional NLS, while $\lambda = -1$, (1.1) is referred to be focusing fractional NLS.

In recent years, there has been wide interest in applying fractional Laplacians to model physical phenomena. By extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths, Laskin in [1,2] used the theory of functionals over functional measure generated by the Lévy stochastic process to deduce the following nonlinear fractional Schrödinger equation

$$i\partial_t u = (-\Delta)^\alpha u + f(u), \tag{1.2}$$

where $0 < \alpha < 1$, $f(u) = |u|^k u$. The parameter $0 < \alpha < 1$ is the corresponding index of the Lévy stable processes, see [1,2]. Eq. (1.2) with $\alpha = \frac{1}{2}$ has been also used as models to describe Boson stars. Recently, an optical realization of the fractional Schrödinger equation was proposed by Longhi [3]. For the nonlinearity $|u|^p u$, the well-posedness and ill-posedness in the Sobolev space H^α have been investigated in [4,5]. In [6], Boulenger, Himmelsbach and Lenzmann have obtained a general criterion for blow-up of radial solution of (1.1) with $k \geq \frac{4\alpha}{N}$ and $N \geq 2$. Although a general existence theorem for blow-up solutions of this problem is still an open problem, it has been strongly supported by numerical evidence [7].

Also, Eq. (1.2) has attracted more and more attention in both physics and mathematics, see [4–6,8–16]. For the Hartree-type nonlinearity

$$f(u) = \left(|x|^{-\gamma} * |u|^2 \right) u,$$

Cho et al. in [8] proved existence and uniqueness of local and global solutions of (1.2). In [9] the authors showed the existence of blow-up solutions. The dynamical properties of blow-up solutions have been investigated in [10,11]. The stability and instability of standing waves have been studied in [12]. For other kinds of fractional Schrödinger equations in which the Hartree-type nonlinearity being replaced by a sublinearity, the orbital stability of standing waves has been studied in [13,14].

Recently, Bhattarai in [17] employed the concentration compactness techniques to prove existence and stability of standing waves for the following nonlinear fractional Schrödinger-Choquard equation

$$\begin{cases} i\partial_t u = (-\Delta)^\alpha u - |u|^k u - \left(|x|^{-\gamma} * |u|^p \right) |u|^{p-2} u, & (t,x) \in [0,T) \times \mathbb{R}^N, \\ u(0,x) = \varphi(x), \end{cases}$$

where $u:[0,T)\times\mathbb{R}^N\to\mathbb{C}$ is the complex-valued function, $N\geq 2$, $\varphi\in H^\alpha$, $0<\alpha<1$, $0<T\leq\infty$, $0<k<\frac{4\alpha}{N}$, $2\leq p<1+\frac{2\alpha+N-\gamma}{N}$, $\gamma\in(0,N)$. Feng and Zhang in [18] proved existence and stability of standing waves with $k=\frac{4\alpha}{N}$. We remark that when $p=2$, it reduces to (1.1); and when $0<k<\frac{4\alpha}{N}$, $k=\frac{4\alpha}{N}$ and $k>\frac{4\alpha}{N}$ the equation is referred as the L^2-subcritical, L^2-critical and L^2-supercritical.

In this paper, we are going to prove the well-posedness and blow-up solutions of the above fractional Schrödinger-Choquard equation with $p=2$, i.e. (1.1).

We write the Cauchy problem (1.1) in the following integral form

$$u(t)=U(t)\varphi-i\int_0^t U\left(t-t'\right)\left(F(u)+G(u)\right)\left(t'\right)dt', \tag{1.3}$$

where

$$G(u)=\lambda|u|^k u, \quad F(u)=\lambda\left(|x|^{-\gamma}*|u|^2\right)u\equiv\lambda K_\gamma(u)u,$$

and $U(t)$ is the unitary group defined by

$$U(t)\varphi(x)=\left(e^{-it(-\Delta)^\alpha}\varphi\right)(x)=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{i\left(x\cdot\xi-t|\xi|^{2\alpha}\right)}\hat{\varphi}(\xi)d\xi.$$

Here $\hat{\varphi}$ denotes the Fourier transform i.e. $\hat{\varphi}(\xi)=\int_{\mathbb{R}^N}e^{-ix\cdot\xi}\varphi(x)dx$.

We note that (1.1) possesses some conservation laws. If the solution u of (1.1) has sufficient decay at infinity and smoothness, it satisfies the conservation of mass

$$M(u(t)):=\|u(t)\|_{L^2}=\|\varphi\|_{L^2}, \tag{1.4}$$

and the conservation of energy

$$E(u(t))=E(\varphi), \tag{1.5}$$

where $E(u(t))$ is defined by

$$\begin{aligned}
E(u(t))&=\frac{1}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{\alpha}{2}}u(t,x)\right|^2 dx+\frac{\lambda}{k+2}\int_{\mathbb{R}^N}|u(t,x)|^{k+2}dx\\
&\quad+\frac{\lambda}{4}\int_{\mathbb{R}^N}\left(|x|^{-\gamma}*|u(t,x)|^2\right)|u(t,x)|^2 dx\\
&=\frac{1}{2}\left\|(-\Delta)^{\frac{\alpha}{2}}u(t,\cdot)\right\|_{L^2}^2+\frac{\lambda}{k+2}\|u(t,\cdot)\|_{L^{k+2}}^{k+2}\\
&\quad+\frac{\lambda}{4}\int_{\mathbb{R}^N}\left(|x|^{-\gamma}*|u(t,x)|^2\right)|u(t,x)|^2 dx
\end{aligned} \tag{1.6}$$

and the energy space is H^α.

A pair (p,q) is said to be (general) admissible if they satisfy

$$p\in[2,\infty], \quad q\in[2,\infty), \quad (p,q,N)\neq(2,\infty,2), \quad \frac{2}{p}+\frac{N}{q}\leq\frac{N}{2}. \tag{1.7}$$

For any pair (g,h), we define

$$\gamma_{gh} \equiv \frac{N}{2} - \frac{N}{h} - \frac{2\alpha}{g}. \tag{1.8}$$

Specially, for radially symmetric data, (p,q) satisfy the radial admissible condition,

$$p \in [2,\infty], \quad q \in [2,\infty), \quad (p,q) \neq \left(2, \frac{4N-2}{2N-3}\right), \quad \frac{2\alpha}{p} + \frac{N}{q} = \frac{N}{2}. \tag{1.9}$$

We denote that $[s]$ is the integer part of s, i.e. $s = [s] + \sigma$, where $0 \leq \sigma < 1$.

We will use Strichartz estimates in (general) admissible condition and radial admissible condition to prove the well-posedness.

Our first result is the following local well-posedness for (1.1) with nonradial data in the sub-critical and critial nonlinearities.

Theorem 1.1. (Nonradial Local Well-posedness) *Let $N \geq 1$, $\alpha \in (0,1) \backslash \{\frac{1}{2}\}$, $0 < \gamma < N$ and $0 < k \leq \frac{4\alpha}{N}$. Let $s \in \left[\frac{\gamma}{2}, \frac{N}{2}\right)$ such that*

$$\begin{cases} s > \frac{1}{2} - \frac{\alpha}{2}, & N=1, \\ s > \frac{N}{2} - \alpha, & N \geq 2, \end{cases}$$

and if k is not an even integer, $[s] \leq k$ holds. Then for all $\varphi \in H^s$, there exist $T > 0$ and a unique solution u to (1.1) satisfying

$$u \in C\left([0,T], H^s\right) \cap L^p_{\text{loc}}\left([0,T], W^{s-\gamma_{pq},q}\right)$$

for $p > 4$ when $N=1$, $p > 2$ when $N \geq 2$ and $s > N/2 - 2\alpha/p$, where (p,q) is an admissible pair satisfying (1.7) and γ_{pq} satisfy (1.8).

Moreover, if $T < \infty$, then $\|u\|_{H^s} \to \infty$ when $t \to T^-$.

Next theorem is about the global well-posedness for the defocusing fractional NLS and focusing fractional NLS with subcritical and critical nonlinearities.

Theorem 1.2. (Nonradial Global Well-posedness) *Let $N \geq 1$, $0 < \alpha < 1$, $0 < \gamma < N$, $0 < k \leq \frac{4\alpha}{N}$ and $s = \alpha \geq \gamma/2$ with*

$$\begin{cases} \alpha > \frac{1}{3}, & N=1 \\ \alpha > \frac{N}{4}, & 2 \leq N < 4. \end{cases}$$

Then for any $\varphi \in H^s$, the solution u to (1.1) exists globally if one of the following conditions is satisfied:

I. $\lambda = 1$, *i.e. defocusing case,*

II. $\lambda = -1$, $N \geq 2$, $0 < k < \frac{4\alpha}{N}$ *and $\gamma < 2\alpha$, i.e. focusing subcritical nonlinearity case,*

III. $\lambda = -1$, $N \geq 2$, $k = \frac{4\alpha}{N}$, $\gamma = 2\alpha$ *and $\|\varphi\|_{L^2}$ is small enough, i.e. focusing critical nonlinearity case.*

Then, we give the local and global well-posedness for (1.1) with radial data in the sub-critical and critial nonlinearities.

Theorem 1.3. (Radial Local Well-posedness) *Let $N \geq 2$, $0 < \gamma < N$, $0 < k \leq \frac{4\alpha}{N}$, $\alpha \in \left[\frac{N}{2N-1}, 1\right)$ and $s \in \left[\frac{\gamma}{2}, \frac{N}{2}\right)$. If k is not an even integer, $[s] \leq k$ holds. Define*

$$p = \frac{4\alpha(k+2)}{k(N-2s)}, \quad q = \frac{N(k+2)}{N+sk}.$$

Then for any $\varphi \in H^s_{rad}$, there exist $T > 0$ and a unique solution u to (1.1) satisfying

$$u \in C\left([0,T], H^s_{rad}\right) \cap L^p_{loc}\left([0,T], W^{s,q}_{rad}\right).$$

Moreover, if $T < \infty$, then $\|u\|_{H^s} \to \infty$ when $t \to T^-$.

Theorem 1.4. (Radial Global Well-posedness) *Let $N \geq 2$, $0 < \alpha < 1$, $0 < \gamma < N$, $0 < k \leq \frac{4\alpha}{N}$ and $s = \alpha \geq \gamma/2$. Then for any $\varphi \in H^s_{rad}$, the solution u to (1.1) exists globally if one of the following conditions is satisfied:*

    I. $\lambda = 1$, *i.e. defocusing case,*
    II. $\lambda = -1$, $0 < k < \frac{4\alpha}{N}$ *and* $\gamma < 2\alpha$, *i.e. focusing subcritical nonlinearity case,*
    III. $\lambda = -1$, $k = \frac{4\alpha}{N}$, $\gamma = 2\alpha$ *and* $\|\varphi\|_{L^2}$ *is small enough, i.e. focusing critical nonlinearity case.*

Finally, we consider the blow-up dynamics of focusing fractional NLS (1.1) with $\lambda = -1$.

Theorem 1.5. *Let $N \geq 2$, $\alpha \in (\frac{1}{2}, 1)$, $k = \frac{4\alpha}{N}$, $\gamma = 2\alpha$, $\lambda = -1$ and $\varphi \in H^\alpha_{rad}$. Let u be such that the corresponding solution to (1.1) exists on the maximal time T^*. If $E(\varphi) < 0$, then one of the following statements holds true:*

    (1) $u(t)$ *blows up in finite time in the sense that $T^* < +\infty$ must hold.*
    (2) $u(t)$ *blows up infinite time such that*

$$\sup_{t \geq 0} \left\| (-\Delta)^{\frac{\alpha}{2}} u(\cdot, t) \right\|_{L^2} = \infty. \tag{1.10}$$

Remark 1.1. Content of your remarks. As explained by Boulenger, Himmelsbach and Lenzmann [6], it is hard to expulse the possibility that the solutions may blow up at infinite time. We can only show that the solutions can not be uniformly bounded if they exist globally.

The rest of the paper is organized as follow. In Section 2, we introduce some important facts and tools. In Section 3, we prove the local and global existence of nonradial solutions via standard Strichartz estimates when $0 < k \leq \frac{4\alpha}{N}$ (for $k = 0$, we refer to [8]). In Section 4, we obtain the local and global existence of radial solutions via radial Strichartz estimates when $0 < k \leq \frac{4\alpha}{N}$. In the last section, we study the blow-up of radial solutions for the focusing fractional NLS in the critical case, i.e. $k = \frac{4\alpha}{N}$ and $\gamma = 2\alpha$.

We will use the notations $|\nabla| = \sqrt{-\Delta}$, $\dot{W}^{s,r} = |\nabla|^{-s} L^r(\mathbb{R}^N)$ $\left(\dot{H}^s = \dot{W}^{s,2}\right)$ and $W^{s,r} = (1-\Delta)^{-s/2} L^r(\mathbb{R}^N)$ $\left(H^s = W^{s,2}\right)$. The norm $\|F\|_{L^q(X)}$ means $\left(\int_{\mathbb{R}} \|F(t,\cdot)\|_X^q dt\right)^{1/q}$. We denote the space $L_T^q(\mathbb{B})$ by $L^q(0,T;\mathbb{B})$ and its norm by $\|\cdot\|_{L_T^q \mathbb{B}}$ for some Banach space $\mathbb{B}$, and also $L^q(\mathbb{B})$ with the norm $\|\cdot\|_{L^q(\mathbb{B})}$ by $L^q(0,\infty;\mathbb{B}), 1 \leq q \leq \infty$. If not specified, throughout this paper, the notation $A \lesssim B$ and $A \gtrsim B$ denote $A \leq CB$ and $A \geq C^{-1}B$, respectively. $A \sim B$ means that both $A \lesssim B$ and $A \gtrsim B$. C is a generic positive constant possibly depending on N, α and γ.

2 Preliminaries

In this section, we present some facts which will be used to prove the local well-posedness. Firstly, we give the Strichartz estimates.

Lemma 2.1. ([19, 20]) *For $N \geq 1$ and $\alpha \in (0,1) \backslash \{\frac{1}{2}\}$, the following estimates hold*

$$\left\| e^{-it(-\Delta)^\alpha} \varphi \right\|_{L^p(\mathbb{R}, L^q)} \lesssim \left\| |\nabla|^{\gamma_{pq}} \varphi \right\|_{L^2}, \tag{2.1}$$

$$\left\| \int_0^t e^{-i(t-\tau)(-\Delta)^\alpha} f(\tau) d\tau \right\|_{L^p(\mathbb{R}, L^q)} \lesssim \left\| |\nabla|^{\gamma_{pq} - \gamma_{a'b'} - 2\alpha} f \right\|_{L^{a'}(\mathbb{R}, L^{b'})}, \tag{2.2}$$

where (p,q), (a,b) are admissible pairs defined as (1.7), with

$$\frac{1}{a} + \frac{1}{a'} = 1, \qquad \frac{1}{b} + \frac{1}{b'} = 1,$$

and γ_{pq} and $\gamma_{a'b'}$ are defined as (1.8), i.e.

$$\gamma_{pq} = \frac{N}{2} - \frac{N}{q} - \frac{2\alpha}{p}, \qquad \gamma_{a'b'} = \frac{N}{2} - \frac{N}{b'} - \frac{2\alpha}{a'}. \tag{2.3}$$

It is worthy noticing that for $\alpha \in (0,1) \backslash \{\frac{1}{2}\}$ the admissible condition $\frac{2}{p} + \frac{N}{q} \leq \frac{N}{2}$ implies $\gamma_{pq} > 0$ for all admissible pairs (p,q) except $(p,q) = (\infty,2)$. This means that the above Strichartz estimates have a loss of derivative. For radial data, the estimates (2.1) and (2.2) hold true for $N \geq 2, \alpha \in (0,1) \backslash \{\frac{1}{2}\}$ and $(p,q), (a,b)$ satisfy the radial admissible condition:

$$\frac{2}{p} + \frac{2N-1}{q} \leq \frac{2N-1}{2}.$$

This condition allows us to choose (p,q) so that $\gamma_{pq} = 0$. More precisely, we have next lemma.

Lemma 2.2. ([20–22]) *For $N \geq 2$ and $\frac{N}{2N-1} \leq \alpha < 1$,*

$$\left\| e^{-it(-\Delta)^\alpha} \varphi \right\|_{L^p(\mathbb{R}, L^q)} \lesssim \|\varphi\|_{L^2}, \tag{2.4}$$

$$\left\|\int_0^t e^{-i(t-\tau)(-\Delta)^\alpha} f(\tau)d\tau\right\|_{L^p(\mathbb{R},L^q)} \lesssim \|f\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{2.5}$$

where φ and f are radially symmetric and $(p,q),(a,b)$ satisfy the radial admissible condition (1.9).

The following is the Leibniz rule for fractional derivatives.

Lemma 2.3. ([8,23]) *Let $\alpha \in (0,1)$ and $1 < p, p_i < \infty$, $1 < q_i \leq \infty$ satisfying $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$. Then,*

$$\||\nabla|^\alpha (uv)\|_{L^p} \lesssim \||\nabla|^\alpha u\|_{L^{p_1}} \|v\|_{L^{q_1}} + \||\nabla|^\alpha v\|_{L^{p_2}} \|u\|_{L^{q_2}},$$
$$\left\|(1-\Delta)^{\frac{\alpha}{2}}(uv)\right\|_{L^p} \lesssim \left\|(1-\Delta)^{\frac{\alpha}{2}}u\right\|_{L^{p_1}} \|v\|_{L^{q_1}} + \left\|(1-\Delta)^{\frac{\alpha}{2}}v\right\|_{L^{p_2}} \|u\|_{L^{q_2}}.$$

The following fractional chain rule is basic for the estimates of the nonlinear terms.

Lemma 2.4. (Fractional Chain Rule [24]) *Let $F \in C^1(\mathbb{C},\mathbb{C})$ and $\alpha \in (0,1)$. Then for $1 < q \leq q_2 < \infty$ and $1 < q_1 \leq \infty$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$*

$$\||\nabla|^\alpha F(u)\|_{L^q} \lesssim \|F'(u)\|_{L^{q_1}} \||\nabla|^\alpha u\|_{L^{q_2}}.$$

Applying the above chain rule to $F(z) = |z|^k z$, we have

Lemma 2.5. ([23]) *Let $F(z) = |z|^k z$ with $k > 0$, $s \geq 0$ and $1 < p, p_1 < \infty$, $1 < q_1 \leq \infty$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{k}{q_1}$. If k is an even integer or k is not an even integer with $[s] \leq k$, then there exists $C > 0$ such that for all $u \in \mathscr{S}$,*

$$\|F(u)\|_{\dot{W}^{s,p}} \leq C \|u\|_{L^{q_1}}^k \|u\|_{\dot{W}^{s,p_1}}.$$

A similar estimate holds with $\dot{W}^{s,p}, \dot{W}^{s,p_1}$-norms replaced by $W^{s,p}, W^{s,p_1}$ norms.

Next, we recall a sharp Gagliardo-Nirenberg type inequality established by Boulenger, Himmelsbach and Lenzmann [6] and Zhu [14].

Lemma 2.6. *Let $N \geq 2$, $0 < \alpha < 1$ and $0 < p < \frac{4\alpha}{N-2\alpha}$. Then, for all $u \in H^\alpha$,*

$$\int_{\mathbb{R}^N} |u|^{p+2} dx \leq C_{opt} \left\|(-\Delta)^{\frac{\alpha}{2}} u\right\|_{L^2}^{\frac{pN}{2\alpha}} \|u\|_{L^2}^{(p+2)-\frac{pN}{2\alpha}}, \tag{2.6}$$

where the optimal constant C_{opt} is given by

$$C_{opt} = \left(\frac{2\alpha(p+2)-pN}{pN}\right)^{\frac{Np}{4\alpha}} \frac{2\alpha(p+2)}{(2\alpha(p+2)-pN)\|Q\|_{L^2}^p},$$

and Q is a ground state solution of the following elliptic equation

$$(-\Delta)^\alpha Q + Q = |Q|^p Q \quad in \quad \mathbb{R}^N.$$

In particular, in the L^2-critical case $p = \frac{4\alpha}{N}$, $C_{opt} = \frac{2\alpha+N}{N\|Q\|_{L^2}^p}$.

Similarly, Feng and Zhang in [15] established the following sharp generalized Gagliardo-Nirenberg type inequality for Hartree-type nonlinearities.

Lemma 2.7. *Let* $\alpha \in (0,1)$, $N \geq 1$, $0 < \gamma < N$ *such that* $\frac{2N-\gamma}{N-2\alpha} > 2$, *the following generalized Gagliardo-Nirenberg inequality*

$$\int_{\mathbb{R}^N} \left(|x|^{-\gamma} * |u|^2 \right) |u|^2 dx \leq C_\gamma \left(\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\alpha}{2}} u \right|^2 dx \right)^{\frac{\gamma}{2\alpha}} \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{4\alpha-\gamma}{2\alpha}}$$

$$= C_\gamma \|u\|_{\dot{H}^\alpha}^{\frac{\gamma}{\alpha}} \|u\|_{L^2}^{4-\frac{\gamma}{\alpha}} \tag{2.7}$$

holds and the best constant is

$$C_\gamma = \frac{4\alpha^2}{4\alpha-\gamma} \left(\frac{4\alpha-\gamma}{\alpha\gamma} \right)^{\frac{\gamma}{2\alpha}} \|R\|_{L^2}^{-2},$$

where R is the ground state of the elliptic equation

$$(-\Delta)^\alpha R + R - \left(|x|^{-\gamma} * |R|^2 \right) R = 0.$$

Lemma 2.8. ([6]) *Let* $N \geqslant 1$ *and* $f : \mathbb{R}^N \to \mathbb{R}$ *satisfy* $\nabla f \in W^{1,\infty}(\mathbb{R}^N)$. *Then, for all* $u \in H^{\frac{1}{2}}(\mathbb{R}^N)$, *it holds that*

$$\left| \int_{\mathbb{R}^N} \bar{u}(x) \nabla f(x) \cdot \nabla u(x) dx \right| \leqslant C \left(\left\| |\nabla|^{\frac{1}{2}} u \right\|_{L^2}^2 + \|u\|_{L^2} \left\| |\nabla|^{\frac{1}{2}} u \right\|_{L^2} \right),$$

with some constant $C > 0$ *depending only on* $\|\nabla f\|_{W^{1,\infty}}$ *and N.*

3 Well-posedness for Nonradial Data

In this section, we show the local and global well-posedness for (1.1) in nonradially symmetric Sobolev spaces.

First we are going to prove Theorem 1.1, the local well-posedness, by applying the contraction mapping argument via Strichartz estimates (2.1) and (2.2). The proof is divided in two steps.

Proof. **Step 1**. First we note that, by the assumptions, $s - \gamma_{pq} > \frac{N}{q}$ and $p > k$. Let

$$X(T,\rho) := \left\{ u \in L^\infty([0,T], H^s) \cap L^p([0,T], W^{s-\gamma_{pq},q}) : \|u\|_{L_T^\infty H^s} + \|u\|_{L_T^p W^{s-\gamma_{pq,q}}} \leq \rho \right\},$$

equipped with the distance

$$d_X(u,v) := \|u-v\|_{L_T^\infty L^2} + \|u-v\|_{L_T^p W^{-\gamma_{pq,q}}},$$

where $\rho, T > 0$ to be chosen later. It is easy to see that $(X(T,\rho), d_X)$ is a complete metric space. Now we define a mapping $\mathcal{N} : u \mapsto \mathcal{N}(u)$ on $X(T,\rho)$ by

$$\mathcal{N}(u)(t) = U(t)\varphi - i\int_0^t U(t-t')(F(u)+G(u))(t')dt'.$$

By the Strichartz estimates (2.1) and (2.2),

$$\|\mathcal{N}(u)\|_{L_T^\infty H^s \cap L_T^p W^{s-\gamma_{pq},q}} \lesssim \|\varphi\|_{H^s} + \|F(u)\|_{L_T^1 H^s} + \|G(u)\|_{L_T^1 H^s}. \tag{3.1}$$

Now we estimate the right hand of (3.1). For $s \geq \gamma/2$, using the Hardy-Littlewood-Sobolev inequality, Lemma 2.3, the Hardy type inequality

$$\sup_{x\in\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\frac{|u(x-y)|^2}{|y|^\gamma}dy\right| \lesssim \|u\|_{\dot{H}^{\gamma/2}}^2, \tag{3.2}$$

and the Sobolev embedding $\dot{H}^{\gamma/2} \hookrightarrow L^{2N/(N-\gamma)}$, we have

$$
\begin{aligned}
\|F(u)\|_{L_T^1 H^s} &\lesssim T\|F(u)\|_{L_T^\infty H^s}\\
&\lesssim T\left(\left\|K_\gamma(|u|^2)\right\|_{L_T^\infty L^\infty}\|u\|_{L_T^\infty H^s} + \left\|K_\gamma(|u|^2)\right\|_{L_T^\infty W^{s,2N/\gamma}}\|u\|_{L_T^\infty L^{2N/(N-\gamma)}}\right)\\
&\lesssim T\left(\|u\|_{L_T^\infty H^{\gamma/2}}^2\|u\|_{L_T^\infty H^s} + \|u\|_{L_T^\infty L^{2N/(N-\gamma)}}^2\|u\|_{L_T^\infty H^s}\right)\\
&\lesssim T\|u\|_{L_T^\infty H^{\gamma/2}}^2\|u\|_{L_T^\infty H^s}\\
&\lesssim T\rho^3. \tag{3.3}
\end{aligned}
$$

For the last term of (3.1), we apply the fractional chain rule given in Lemma 2.5, the Hölder inequality and the fact $p > k$ to get

$$
\begin{aligned}
\|G(u)\|_{L_T^1 H^s} = \left\||u|^k u\right\|_{L_T^1 H^s} &\lesssim \left\||u|^k\right\|_{L_T^1 L^\infty}\|u\|_{L_T^\infty H^s}\\
&\lesssim \|u\|_{L_T^k L^\infty}^k\|u\|_{L_T^\infty H^s}\\
&\lesssim T^{1-\frac{k}{p}}\|u\|_{L_T^p L^\infty}^k\|u\|_{L_T^\infty H^s}.
\end{aligned}
$$

Note the fact $s - \gamma_{pq} > N/q$ and the Sobolev embedding $W^{s-\gamma_{pq},q} \hookrightarrow L^\infty$, we get

$$\|G(u)\|_{L_T^1 H^s} \lesssim T^{1-\frac{k}{p}}\|u\|_{L_T^p W^{s-\gamma_{pq},q}}^k\|u\|_{L_T^\infty H^s} \lesssim T^{1-\frac{k}{p}}\rho^{k+1}. \tag{3.4}$$

Hence, putting (3.3) and (3.4) into (3.1), we get

$$\|\mathcal{N}(u)\|_{L_T^\infty H^s \cap L_T^p W^{s-\gamma_{pq},q}} \leq C(\|\varphi\|_{H^s} + T\rho^3 + T^{1-\frac{k}{p}}\rho^{k+1}).$$

Let $\rho = 2C\|\varphi\|_{H^s}$ and choose T such as $C(T\rho^2 + T^{1-\frac{k}{p}}\rho^k) \leq \frac{1}{2}$, then $\mathcal{N}$ maps $X(T,\rho)$ into itself.

Step 2. Now we show that $\mathcal{N}$ is a contraction map for sufficiently small T. Let $u, v \in X(T,\rho)$, using (2.2), we have

$$d_X(\mathcal{N}(u), \mathcal{N}(v)) \lesssim \|F(u) - F(v)\|_{L_T^1 L^2} + \|G(u) - G(v)\|_{L_T^1 L^2}. \tag{3.5}$$

Now, we start to estimate the right hand of (3.5). For the first term, we have

$$\|F(u) - F(v)\|_{L_T^1 L^2}$$
$$\lesssim T \left\|K_\gamma\left(|u|^2\right)u - K_\gamma\left(|v|^2\right)v\right\|_{L_T^\infty L^2}$$
$$\lesssim T \left(\left\|K_\gamma\left(|u|^2\right)(u-v)\right\|_{L_T^\infty L^2} + \left\|K_\gamma\left(|u|^2 - |v|^2\right)v\right\|_{L_T^\infty L^2} \right)$$
$$\lesssim T \left(\|u\|_{L_T^\infty H^{\gamma/2}}^2 \|(u-v)\|_{L_T^\infty L^2} + \left\|K_\gamma\left(|u|^2 - |v|^2\right)\right\|_{L_T^\infty L^{2N/\gamma}} \|v\|_{L_T^\infty L^{2N/(N-\gamma)}} \right)$$
$$\lesssim T \left(\|u\|_{L_T^\infty H^{\gamma/2}}^2 \|(u-v)\|_{L_T^\infty L^2} + \left\||u|^2 - |v|^2\right\|_{L_T^\infty L^{2N/(2N-\gamma)}} \|v\|_{L_T^\infty L^{2N/(N-\gamma)}} \right)$$
$$\lesssim T \left(\|u\|_{L_T^\infty H^{\gamma/2}}^2 \|(u-v)\|_{L_T^\infty L^2} + \|u-v\|_{L_T^\infty L^2} \|u+v\|_{L_T^\infty L^{2N/(N-\gamma)}} \|v\|_{L_T^\infty L^{2N/(N-\gamma)}} \right)$$
$$\lesssim T\rho^2 d_X(u,v). \tag{3.6}$$

For the last term of (3.5), we get

$$\|G(u) - G(v)\|_{L_T^1 L^2} = \left\||u|^k u - |v|^k v\right\|_{L_T^1 L^2}$$
$$\lesssim \left(\|u\|_{L_T^k L^\infty}^k + \|v\|_{L_T^k L^\infty}^k \right) \|u-v\|_{L_T^\infty L^2}$$
$$\lesssim T^{1-\frac{k}{p}} \left(\|u\|_{L_T^p L^\infty}^k + \|v\|_{L_T^p L^\infty}^k \right) \|u-v\|_{L_T^\infty L^2}$$
$$\lesssim T^{1-\frac{k}{p}} \left(\|u\|_{L_T^p W^{s-\gamma_{pq},q}}^k + \|v\|_{L_T^p W^{s-\gamma_{pq},q}}^k \right) \|u-v\|_{L_T^\infty L^2}$$
$$\lesssim T^{1-\frac{k}{p}} \rho^k d_X(u,v). \tag{3.7}$$

From (3.5)–(3.7), we have

$$d_X(\mathcal{N}(u), \mathcal{N}(v)) \lesssim C\left(T\rho^2 + T^{1-\frac{k}{p}}\rho^k\right) d_X(u,v).$$

Then we choose T small enough such that

$$C(T\rho^2 + T^{1-\frac{k}{p}}\rho^k) \leq \frac{1}{2},$$

the above estimate implies that the mapping $\mathcal{N}$ is a contraction. Then $\mathcal{N}$ admits a unique fixed point in $X(T,\rho)$, which is the solution of (1.1). This completes the proof of local well-posedness. □

Next, we will prove Theorem 1.2, the global well-posedness for (1.1) in two cases: $\lambda = 1$, the defocusing case and $\lambda = -1$, the focusing case with subcritical/critical nonlinearities.

Proof of Theorem 1.2.

Proof. First we consider the defocusing case $\lambda = 1$. Let T^* be the maximal existence time. We will prove that $T^* = \infty$ by contradiction. Suppose that $T^* < \infty$, then the local well-posedness shows that

$$\|u\|_{L^q_{T^*}W^{s-\gamma pq,q}} = \infty. \tag{3.8}$$

From Theorem 1.1, the conservation laws (1.4) and (1.5), for any $t < T^*$, the solution u satisfies

$$\frac{1}{2}\|u(t)\|^2_{H^\alpha} \leq \frac{1}{2}\|u(t)\|^2_{L^2} + E(u) = \frac{1}{2}\|\varphi\|^2_{L^2} + E(\varphi).$$

From the estimate (3.1) with $s = \alpha$, we have

$$\|u\|_{L^p_T W^{\alpha-\gamma pq,q}} \leq \|\varphi\|_{H^s} + T\|u\|^3_{L^\infty_T H^\alpha} + T^{1-\frac{k}{p}}\|u\|^k_{L^p_T W^{\alpha-\gamma pq,q}}\|u\|_{L^\infty_T H^\alpha}$$

$$\lesssim (\|\varphi\|^2_{L^2} + E(\varphi))^{\frac{1}{2}} + T(\|\varphi\|^2_{L^2} + E(\varphi))^{\frac{3}{2}}$$

$$+ T^{1-\frac{k}{p}}(\|\varphi\|^2_{L^2} + E(\varphi))^{\frac{1}{2}}\|u\|^k_{L^p_T W^{\alpha-\gamma pq,q}}.$$

Thus for sufficiently small T depending only on $\|\varphi\|^2_{L^2} + E(\varphi)$, we have

$$\|u\|_{L^p\left(T_{j-1},T_j;W^{\alpha-\gamma pq,q}\right)} \leq C\left(\|\varphi\|^2_{L^2} + E(\varphi)\right)^{\frac{3}{2}},$$

where $T_j - T_{j-1} = T$. If $T^* < \infty$, then there exists $M \in \mathbb{N}$ such that $(M-1)T < T^* \leq MT$. Let $T_j = (j-1)T$, $j = 1,\ldots,M-1$ and $T_M = T^*$. Then we have

$$\|u\|^p_{L^p\left(0,T^*;W^{\alpha-\gamma pq,q}\right)} \leq \sum_{1 \leq j \leq M} \|u\|^p_{L^p\left(T_{j-1},T_j;W^{\alpha-\gamma pq,q}\right)}$$

$$\leq \left(MC\left(\|\varphi\|^2_{L^2} + E(\varphi)\right)^{\frac{3}{2}}\right)^p < \infty.$$

This contradicts to (3.8), and thus completes the proof of the defocusing case .

Next we consider the focusing case, $\lambda = -1$. From Lemmas 2.6 and 2.7,

$$\int_{\mathbb{R}^N} |u|^{k+2}dx \leq C_{opt}\|u\|^{\frac{kN}{2\alpha}}_{\dot{H}^\alpha}\|u\|^{(k+2)-\frac{kN}{2\alpha}}_{L^2},$$

$$\int_{\mathbb{R}^N} \left(|x|^{-\gamma} * |u|^2\right)|u|^2dx \leq C_\gamma\|u\|^{\frac{\gamma}{\alpha}}_{\dot{H}^\alpha}\|u\|^{4-\frac{\gamma}{\alpha}}_{L^2},$$

we get

$$
\begin{aligned}
E(u(t)) &= \frac{1}{2}\|u\|_{\dot{H}^\alpha}^2 - \frac{1}{k+2}\|u\|_{L^{k+2}}^{k+2} - \frac{1}{4}\int_{\mathbb{R}^N}\left(|x|^{-\gamma}*|u|^2\right)|u|^2 dx \\
&\geq \frac{1}{2}\|u\|_{\dot{H}^\alpha}^2 - \frac{C_{opt}}{k+2}\|u\|_{\dot{H}^\alpha}^{\frac{kN}{2\alpha}}\|u\|_{L^2}^{(k+2)-\frac{kN}{2\alpha}} - \frac{C_\gamma}{4}\|u\|_{\dot{H}^\alpha}^{\frac{\gamma}{\alpha}}\|u\|_{L^2}^{4-\frac{\gamma}{\alpha}} \\
&\geq \frac{1}{2}\|u\|_{\dot{H}^\alpha}^2 - \frac{C_{opt}}{k+2}\|u\|_{\dot{H}^\alpha}^{\frac{kN}{2\alpha}}\|\varphi\|_{L^2}^{(k+2)-\frac{kN}{2\alpha}} - \frac{C_\gamma}{4}\|u\|_{\dot{H}^\alpha}^{\frac{\gamma}{\alpha}}\|\varphi\|_{L^2}^{4-\frac{\gamma}{\alpha}}.
\end{aligned}
$$

For the condition II in Theorem 1.2, i.e. $0 < k < \frac{4\alpha}{N}$ and $\gamma < 2\alpha$, using the Young inequality, we can easily get

$$
\|u(t)\|_{\dot{H}^\alpha}^2 \lesssim |E(\varphi)|.
$$

For the condition III in Theorem 1.2, i.e. $k = \frac{4\alpha}{N}$, $\gamma = 2\alpha$ and $\|\varphi\|_{L^2}$ is suffciently small, the above inequality still holds true. Therefore,

$$
\|u(t)\|_{H^\alpha}^2 \leq \|u(t)\|_{L^2}^2 + E(u) = \|\varphi\|_{L^2}^2 + E(\varphi).
$$

Similar to the defocusing case, we can apply the contradiction argument to prove the global well-posedness. This completes the proof of Theorem 1.2. $\qquad\square$

4   Well-posedness for Radial Data

In this section, we will show the local and global well-posedness for (1.1) with radial initial data in Sobolev spaces. The proof is also based on the contraction mapping argument via Strichartz estimates (2.4) and (2.5). In the radial case, thanks to Strichartz estimates without loss of derivatives, we have better local well-posedness comparing to the nonradial case.

The proof of Theorem 1.3 is divided in two steps.

Proof. **Step 1**. It is easy to check that when (p,q) satisfies the radial admissible condition (1.9), we can choose (m,n) so that

$$
\frac{1}{p'} = \frac{1}{p} + \frac{k}{m}, \qquad \frac{1}{q'} = \frac{1}{q} + \frac{k}{n},
$$

and we see that

$$
\theta \equiv \frac{k}{m} - \frac{k}{p} = 1 - \frac{k(N-2s)}{4\alpha} \in (0,1), \qquad q \leq n = \frac{qN}{N-sq}.
$$

The later fact gives the Sobolev embedding $W^{s,q} \hookrightarrow L^n$. Let $(Y(T,\rho), d_Y)$ be the complete metric space

$$
Y(T,\rho) = \left\{ C([0,T], H_{rad}^s) \cap L_T^p(W^{s,q}) : \|u\|_{L_T^\infty H^s} + \|u\|_{L_T^p W^{s,q}} \leq \rho \right\}
$$

equipped with the distance

$$d_Y(u,v) := \|u-v\|_{L_T^\infty L^2} + \|u-v\|_{L_T^p L^q}.$$

As before we define a mapping $\mathcal{N}: u \mapsto \mathcal{N}(u)$ on $Y(T,\rho)$ by

$$\mathcal{N}(u)(t) = U(t)\varphi - i \int_0^t U(t-t')(F(u)+G(u))(t')\,dt'.$$

From the radial Strichartz estimates (2.4) and (2.5), we obtain

$$\|\mathcal{N}(u)\|_{L_T^\infty H^s \cap L_T^p W^{s,q}} \leq \|\varphi\|_{H^s} + \|F(u)\|_{L_T^1 H^s} + \|G(u)\|_{L_T^{p'} W^{s,q'}}, \tag{4.1}$$

Now we estimate the right hand of (4.1). For $s \geq \gamma/2$,

$$\begin{aligned}
\|F(u)\|_{L_T^1 H^s} &\lesssim T\|F(u)\|_{L_T^\infty H^s} \\
&\lesssim T\left(\left\|K_\gamma\left(|u|^2\right)\right\|_{L_T^\infty L^\infty} \|u\|_{L_T^\infty H^s} + \left\|K_\gamma\left(|u|^2\right)\right\|_{L_T^\infty W^{s,2N/\gamma}} \|u\|_{L_T^\infty L^{2N/(N-\gamma)}} \right) \\
&\lesssim T\left(\|u\|^2_{L_T^\infty H^{\gamma/2}} \|u\|_{L_T^\infty H^s} + \|u\|^2_{L_T^\infty L^{2N/(N-\gamma)}} \|u\|_{L_T^\infty H^s} \right) \\
&\lesssim T\|u\|^2_{L_T^\infty H^{\gamma/2}} \|u\|_{L_T^\infty H^s} \\
&\lesssim T\rho^3. \tag{4.2}
\end{aligned}$$

Here we have used the Hardy-Littlewood-Sobolev inequality, Lemma 2.3, the Hardy type inequality (3.2) and the Sobolev embedding $\dot{H}^{\gamma/2} \hookrightarrow L^{2N/(N-\gamma)}$.

By the fractional chain rule in Lemma 2.5 and the Hölder inequality, we get

$$\begin{aligned}
\|G(u)\|_{L_T^{p'} W^{s,q'}} &= \left\| |u|^k u \right\|_{L_T^{p'} W^{s,q'}} \\
&\lesssim \|u\|^k_{L_T^m L^n} \|u\|_{L_T^p W^{s,q}} \\
&\lesssim T^\theta \|u\|^k_{L_T^p L^n} \|u\|_{L_T^p W^{s,q}} \\
&\lesssim T^\theta \|u\|^{k+1}_{L_T^p W^{s,q}} \\
&\lesssim T^\theta \rho^{k+1}. \tag{4.3}
\end{aligned}$$

Hence, putting (4.2) and (4.3) into (4.1), we get

$$\|\mathcal{N}(u)\|_{L_T^\infty H^s \cap L_T^p W^{s,q}} \leq C(\|\varphi\|_{H^s} + T\rho^3 + T^\theta \rho^{k+1}).$$

Let $\rho = 2C\|\varphi\|_{H^s}$ and choose T such as $C(T\rho^2 + T^\theta \rho^k) \leq \frac{1}{2}$, then $\mathcal{N}$ maps $Y(T,\rho)$ into itself.

Step 2. Now we show that $\mathcal{N}$ is a contraction map for sufficiently small T. Let $u,v \in Y(T,\rho)$, then we have

$$d_Y(\mathcal{N}(u),\mathcal{N}(v)) \lesssim \|F(u)-F(v)\|_{L_T^1 L^2} + \|G(u)-G(v)\|_{L_T^{p'} L^{q'}}. \tag{4.4}$$

Now we estimate the right hand of (4.4). For the first term, we have

$$
\begin{aligned}
&\|F(u)-F(v)\|_{L_T^1 L^2} \\
&\lesssim T \left\| K_\gamma\left(|u|^2\right)u - K_\gamma\left(|v|^2\right)v \right\|_{L_T^\infty L^2} \\
&\lesssim T \left(\left\| K_\gamma\left(|u|^2\right)(u-v) \right\|_{L_T^\infty L^2} + \left\| K_\gamma\left(|u|^2-|v|^2\right)v \right\|_{L_T^\infty L^2} \right) \\
&\lesssim T \left(\|u\|_{L_T^\infty H^{\gamma/2}}^2 \|(u-v)\|_{L_T^\infty L^2} + \left\| K_\gamma\left(|u|^2-|v|^2\right) \right\|_{L_T^\infty L^{2N/\gamma}} \|v\|_{L_T^\infty L^{2N/(N-\gamma)}} \right) \\
&\lesssim T \left(\|u\|_{L_T^\infty H^{\gamma/2}}^2 \|(u-v)\|_{L_T^\infty L^2} + \left\| |u|^2-|v|^2 \right\|_{L_T^\infty L^{2N/(2N-\gamma)}} \|v\|_{L_T^\infty L^{2N/(N-\gamma)}} \right) \\
&\lesssim T \left(\|u\|_{L_T^\infty H^{\gamma/2}}^2 \|(u-v)\|_{L_T^\infty L^2} + \|u-v\|_{L_T^\infty L^2} \|u+v\|_{L_T^\infty L^{2N/(N-\gamma)}} \|v\|_{L_T^\infty L^{2N/(N-\gamma)}} \right) \\
&\lesssim T\rho^2 d_Y(u,v).
\end{aligned}
$$

For the last term of (4.4), we have

$$
\begin{aligned}
\|G(u)-G(v)\|_{L_T^{p'} L^{q'}} &= \left\| |u|^k u - |v|^k v \right\|_{L_T^{p'} L^{q'}} \\
&\lesssim \left(\|u\|_{L_T^m L^n}^k + \|v\|_{L_T^m L^n}^k \right) \|u-v\|_{L_T^p L^q} \\
&\lesssim T^\theta \left(\|u\|_{L_T^p \dot{W}^{\gamma,q}}^k + \|v\|_{L_T^p \dot{W}^{\gamma,q}}^k \right) \|u-v\|_{L_T^p L^q} \\
&\lesssim T^\theta \rho^k d_Y(u,v).
\end{aligned}
$$

Hence

$$
d_Y(\mathscr{N}(u),\mathscr{N}(v)) \lesssim C\left(T\rho^2 + T^\theta \rho^k \right) d_Y(u,v).
$$

Then we can choose T small enough such that $C(T\rho^2 + T^\theta \rho^k) \leq \frac{1}{2}$, the above estimate implies that the mapping $\mathscr{N}$ is a contraction. Then $\mathscr{N}$ admits a fixed point in $X(T,\rho)$, which is the solution of (1.1). This completes the proof of Theorem 1.3. $\qquad\square$

Next, we will prove the global well-posedness for (1.1) with radial initial data in two cases: $\lambda = 1$, the defocusing case and $\lambda = -1$, the focusing case.

Proof of Theorem 1.4.

Proof. First we consider the defocusing case $\lambda = 1$. Let T^* be the maximal existence time. We will prove that $T^* = \infty$ by contradiction. Suppose that $T^* < \infty$, then the local well-posedness shows that

$$
\|u\|_{L_{T^*}^q W^{s,q}} = \infty. \tag{4.5}
$$

From Theorem 1.3, the conservation laws (1.4) and (1.5), for any $t < T^*$, the solution u satisfies

$$
\frac{1}{2}\|u(t)\|_{H^\alpha}^2 \leq \frac{1}{2}\|u(t)\|_{L^2}^2 + E(u) = \frac{1}{2}\|\varphi\|_{L^2}^2 + E(\varphi).
$$

From the estimate (4.1) with $s = \alpha$, we have

$$\|u\|_{L^p_T W^{\alpha,q}} \lesssim \|\varphi\|_{H^\alpha} + T^\theta \|u\|^{k+1}_{L^p_T W^{\alpha,q}} + T\|u\|^3_{L^\infty_T H^\alpha}$$

$$\lesssim \left(\|\varphi\|^2_{L^2} + E(\varphi)\right)^{\frac{1}{2}} + T^\theta \|u\|^{k+1}_{L^p_T W^{\alpha,q}} + T\left(\|\varphi\|^2_{L^2} + E(\varphi)\right)^{\frac{3}{2}}.$$

Thus for sufficiently small T depending only on $\|\varphi\|^2_{L^2} + E(\varphi)$, we have

$$\|u\|_{L^p\left(T_{j-1},T_j;W^{\alpha,q}\right)} \leq C\left(\|\varphi\|^2_{L^2} + E(\varphi)\right)^{\frac{3}{2}},$$

where $T_j - T_{j-1} = T$. If $T^* < \infty$, then there exists $M \in \mathbb{N}$ such that $(M-1)T < T^* \leq MT$. Let $T_j = (j-1)T, j = 1, \ldots, M-1$ and $T_M = T^*$. Then we have

$$\|u\|^p_{L^p(0,T^*;W^{\alpha,q})} \leq \sum_{1 \leq j \leq M} \|u\|^p_{L^p\left(T_{j-1},T_j;W^{\alpha,q}\right)}$$

$$\leq \left(MC\left(\|\varphi\|^2_{L^2} + E(\varphi)\right)^{\frac{3}{2}}\right)^p < \infty.$$

This contradicts to (4.5), and thus completes the proof of the defocusing case.

Next we consider the focusing case, $\lambda = -1$. Similarly to the proof of the focusing case in Theorem 1.2, we have also

$$\|u(t)\|^2_{H^\alpha} \leq \|u(t)\|^2_{L^2} + E(u) = \|\varphi\|^2_{L^2} + E(\varphi).$$

This completes the proof of Theorem 1.4.

$$\square$$

5   Blow-up Solutions

In this section, we are going to give the proof of Theorem 1.5.

According to Theorem 1.3, for a radial initial function $\varphi \in H^\alpha_{rad}$, (1.1) admits a local solution $u \in H^\alpha_{rad}$. If $T^* < \infty$, then we are done. If $T^* = \infty$, we show (1.10).

Let $\psi \in C^\infty_0(\mathbb{R}^N)$ be radial and satisfy

$$\psi(r) = \begin{cases} \frac{1}{2}r^2, & \text{for } r \leqslant 1 \\ 0, & \text{for } r \geqslant 2 \end{cases} \quad \text{and} \quad \psi''(r) \leqslant 1 \quad \text{for } r = |x| \geqslant 0. \tag{5.1}$$

For a fixed $R > 0$, we define the rescaled function $\psi_R : \mathbb{R}^N \to \mathbb{R}$ by setting

$$\psi_R(r) := R^2 \psi\left(\frac{r}{R}\right). \tag{5.2}$$

We readily verify the inequalities

$$1 - \psi''_R(r) \geqslant 0, \quad 1 - \frac{\psi'_R(r)}{r} \geqslant 0, \quad N - \Delta\psi_R(r) \geqslant 0 \quad \text{for all } r \geqslant 0. \tag{5.3}$$

Indeed, this first inequality follows from $\psi_R''(r) = \psi''(r/R) \leqslant 1$. We obtain the second inequality by integrating the first inequality on $[0,r]$ and using that $\psi_R'(0) = 0$. Finally, we see that last inequality follows from

$$N - \Delta\psi_R(r) = 1 - \psi_R''(r) + (N-1)\left\{1 - \frac{1}{r}\psi_R'(r)\right\} \geqslant 0.$$

Besides (5.3), ψ_R admits the following properties, which can be easily checked.

$$\begin{cases} \nabla\psi_R(r) = R\psi'\left(\frac{r}{R}\right)\frac{x}{|x|} = \begin{cases} x & \text{for } r \leqslant R, \\ 0 & \text{for } r \geqslant 2R, \end{cases} \\ \|\nabla^j\psi_R\|_{L^\infty} \lesssim R^{2-j} & \text{for } 0 \leqslant j \leqslant 4, \\ \text{supp}\left(\nabla^j\psi_R\right) \subset \begin{cases} \{|x| \leqslant 2R\} & \text{for } j = 1,2, \\ \{R \leqslant |x| \leqslant 2R\} & \text{for } 3 \leqslant j \leqslant 4. \end{cases} \end{cases}$$

We define

$$\mathcal{M}_{\psi_R}[u(t)] := 2\text{Im}\int_{\mathbb{R}^N} \bar{u}(t)\nabla\psi_R \cdot \nabla u(t)dx = 2\text{Im}\int_{\mathbb{R}^N} \bar{u}(t)\partial_j\psi_R\partial_j u(t)dx.$$

Define the self-adjoint differential operator

$$\Gamma_{\psi_R} := -i(\nabla \cdot \nabla\psi_R + \nabla\psi_R \cdot \nabla),$$

which acts on functions according to

$$\Gamma_{\psi_R}f = -i(\nabla \cdot ((\nabla\psi_R)f) + (\nabla\psi_R) \cdot (\nabla f)).$$

It's easy to check that

$$\mathcal{M}_{\psi_R}[u(t)] = \langle u(t), \Gamma_{\psi_R}u(t)\rangle.$$

First, we give the following lemma.

Lemma 5.1. *Let* $N \geq 2$, $\alpha \in (\frac{1}{2}, 1)$, $k = \frac{4\alpha}{N}$, $\gamma = 2\alpha$, *and* $u \in H_{rad}^\alpha$ *is a solution of* (1.1) *with* $\lambda = -1$. *Let* ψ_R *be as in* (5.2), T^* *be the maximal existence time of solution* $u(t)$ *in* H_{rad}^α. *Then for sufficiently large R, we have*

$$\frac{d}{dt}\mathcal{M}_{\psi_R}[u(t)] \leq 4\alpha E(u(t)), \quad t \in [0,T^*).$$

Proof. By taking the derivative of $\mathcal{M}_{\psi_R}[u(t)]$ with respect to time t and using the equation of $u(t)$, for any $t \in [0,T)$, we get

$$\frac{d}{dt}\mathcal{M}_{\psi_R}[u(t)] = \langle u(t), [(-\Delta)^\alpha, i\Gamma_{\psi_R}]u(t)\rangle + \langle u(t), \left[-|u|^k, i\Gamma_{\psi_R}\right]u(t)\rangle$$
$$+ \langle u(t), \left[-K_\gamma(u), i\Gamma_{\psi_R}\right]u(t)\rangle$$
$$=: I_1 + I_2 + I_3, \tag{5.4}$$

where $[X,Y] \equiv XY - YX$ denotes the commutator of operators X and Y.

From [6] , $k = \frac{4\alpha}{N}$, we get

$$I_1 = \left\langle u(t), \left[(-\Delta)^{\alpha}, i\Gamma_{\psi_R} \right] u(t) \right\rangle \le 4\alpha \| (-\Delta)^{\frac{\alpha}{2}} u \|_{L^2}^2 + CR^{-2\alpha},$$

and

$$\begin{aligned}
I_2 &= \left\langle u, \left[-|u|^k, i\Gamma_{\psi_R} \right] u \right\rangle \\
&= -\left\langle u, \left[|u|^k, \nabla\psi_R \cdot \nabla + \nabla \cdot \nabla\psi_R \right] u \right\rangle \\
&= 2\int_{\mathbb{R}^N} |u|^2 \nabla\psi_R \cdot \nabla \left(|u|^k \right) dx \\
&= -\frac{2k}{k+2} \int_{\mathbb{R}^N} (\Delta\psi_R) |u|^{k+2} dx \\
&= -\frac{2kN}{k+2} \int_{\mathbb{R}^N} |u|^{k+2} dx - \frac{2k}{k+2} \int_{|x| \ge R} (\Delta\psi_R - N) |u|^{k+2} dx \\
&\le -\frac{2kN}{k+2} \int_{\mathbb{R}^N} |u|^{k+2} dx + CR^{\frac{-k(N-1)}{2} + \varepsilon_1 \alpha} \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2}^{(k/2\alpha) + \varepsilon_1},
\end{aligned}$$

where $0 < \varepsilon_1 < (2\alpha - 1)k/2\alpha$.

From [25] and $\gamma = 2\alpha$, we get

$$\begin{aligned}
I_3 &= \left\langle u(t), \left[-K_{\gamma}(u), i\Gamma_{\psi_R} \right] u(t) \right\rangle \\
&\le -\gamma \int \left(|x|^{-\gamma} * |u|^2 \right) |u(x)|^2 dx + \frac{C}{R^{\frac{1}{2}(N-1-\varepsilon_2)\left(\frac{\gamma}{N}\right)}} \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2}^{2 + \frac{\gamma(1+\varepsilon_2)}{2\alpha N}},
\end{aligned}$$

where $0 < \varepsilon_2 < 2\alpha - 1 < N$. Therefore,

$$\begin{aligned}
\frac{d}{dt} \mathcal{M}_{\psi_R}[u(t)] \le{}& 4\alpha \| (-\Delta)^{\frac{\alpha}{2}} u \|_{L^2}^2 - \frac{2kN}{k+2} \int_{\mathbb{R}^N} |u|^{k+2} dx \\
&- \gamma \int \left(|x|^{-\gamma} * |u|^2 \right) |u(x)|^2 dx \\
&+ CR^{-2\alpha} + CR^{\frac{-k(N-1)}{2} + \varepsilon_1 \alpha} \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2}^{(k/2\alpha) + \varepsilon_1} \\
&+ \frac{C}{R^{\frac{1}{2}(N-1-\varepsilon_2)\left(\frac{\gamma}{N}\right)}} \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2}^{2 + \frac{\gamma(1+\varepsilon_2)}{2\alpha N}}.
\end{aligned}$$

From (1.4) and (1.5), we have

$$\begin{aligned}
\frac{d}{dt} \mathcal{M}_{\psi_R}[u(t)] \le{}& 8\alpha E(u(t)) + CR^{-2\alpha} \\
&+ CR^{\frac{-k(N-1)}{2} + \varepsilon_1 \alpha} \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2}^{(k/2\alpha) + \varepsilon_1} \\
&+ \frac{C}{R^{\frac{1}{2}(N-1-\varepsilon_2)\left(\frac{\gamma}{N}\right)}} \left\| (-\Delta)^{\frac{\alpha}{2}} u \right\|_{L^2}^{2 + \frac{\gamma(1+\varepsilon_2)}{2\alpha N}},
\end{aligned} \tag{5.5}$$

where the constant $C > 0$ is independent of R. When $R > 1$ is sufficiently large, ε_1 and $\varepsilon_2 < 1$ are sufficiently small, then

$$\frac{d}{dt}\mathcal{M}_{\psi_R}[u(t)] \leq 4\alpha E(u(t)) = 4\alpha E(\varphi). \tag{5.6}$$

The proof of Lemma 5.1 is completed. $\qquad\square$

Now we begin to prove case (2) of Theorem 1.5. We suppose that $u(t)$ exists for all times $t \geq 0$, i.e. $T^* = \infty$.

It follows from Lemma 5.1 and conservation of mass, for $R \gg 1$ large enough,

$$\frac{d}{dt}\mathcal{M}_{\psi_R}[u(t)] \leq 4\alpha E(\varphi) \triangleq -A^* < 0, \quad t \geq 0. \tag{5.7}$$

From (5.7), we infer that

$$\mathcal{M}_{\psi_R}[u(t)] \leq -A^* t + \mathcal{M}_{\psi_R}[\varphi], \quad t \geq 0.$$

On the one hand, let $T_0 = \frac{2|\mathcal{M}_{\psi_R}[\varphi]|}{A^*} > 0$, then for any $t \geq T_0$, we have

$$\mathcal{M}_{\psi_R}[u(t)] \leq -\frac{1}{2}A^* t < 0. \tag{5.8}$$

On the other hand, by Lemma 2.8 and the conservation of mass, we see that for any $t \in [0, +\infty)$,

$$\begin{aligned}
\left|\mathcal{M}_{\psi_R}[u(t)]\right| &\lesssim C(\psi_R)\left(\left\||\nabla|^{\frac{1}{2}}u(t)\right\|_{L^2}^2 + \|u(t)\|_{L^2}\left\||\nabla|^{\frac{1}{2}}u(t)\right\|_{L^2}\right) \\
&\lesssim C(\psi_R)\left(\left\||\nabla|^{\frac{1}{2}}u(t)\right\|_{L^2}^2 + \|u(t)\|_{L^2}^2\right) \\
&\lesssim C(\psi_R)\left(\left\||\nabla|^{\frac{1}{2}}u(t)\right\|_{L^2}^2 + 1\right) \\
&\lesssim C(\psi_R)\left(\left\|(-\Delta)^{\frac{\alpha}{2}}u(t)\right\|_{L^2}^{\frac{1}{\alpha}} + 1\right).
\end{aligned}$$

Here we have used the interpolation estimate

$$\left\||\nabla|^{\frac{1}{2}}u\right\|_{L^2} \lesssim \left\|(-\Delta)^{\frac{\alpha}{2}}u\right\|_{L^2}^{\frac{1}{2\alpha}}\|u\|_{L^2}^{1-\frac{1}{2\alpha}} \qquad \text{for} \quad \alpha > \frac{1}{2}.$$

This combined with (5.8) yields that for any $t \geq T_0$,

$$A^* t \leq -2\mathcal{M}_{\psi_R}[u(t)] \lesssim C(\psi_R)\left(\left\|(-\Delta)^{\frac{\alpha}{2}}u(t)\right\|_{L^2}^{\frac{1}{\alpha}} + 1\right).$$

This shows that

$$\left\|(-\Delta)^{\frac{\alpha}{2}}u(t)\right\|_{L^2}\geq Ct^\alpha,\quad t\geq T_0.$$

It means that

$$\sup_{t\geq 0}\left\|(-\Delta)^{\frac{\alpha}{2}}u(\cdot,t)\right\|_{L^2}=\infty.$$

The proof of Theorem 1.5 is completed.

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References

[1] Laskin N., Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A*, **268** (2000), 298-305.

[2] Laskin N., Fractional Schrödinger equations. *Phys. Rev. E*, **66** (2002), 056108.

[3] Longhi S., Fractional Schrödinger equation in optics. *Opt. Lett.*, **40** (2015), 1117-1120.

[4] Cho Y., Hwang G., Kwon S., Lee S., Well-posedness and ill-posedness for the cubic fractional Schrödinger equations. *Discrete Contin. Dyn. Syst.*, **35** (2015), 2863-2880.

[5] Hong Y., Sire Y., On fractional Schrödinger equations in Sobolev spaces. *Commun. Pure Appl. Anal.* , **14** (2015), 2265-2282.

[6] Boulenger T., Himmelsbach D., Lenzmann E., Blow up for fractional NLS. *J. Funct. Anal*, **271** (2016), 2569-2603.

[7] Klein C., Sparber C., Markowich P., Numerical study of fractional nonlinear Schrödinger equations. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **470** (2014).

[8] Cho Y., Hajaiej H., Hwang G., Ozawa T., On the Cauchy problem of fractional Schrödinger equations with Hartree type nonlimearity. *Funkcial. Ekvac.*, **56** (2013), 193-224.

[9] Cho Y., Hwang G., Kwon S., Lee S., On finite time blow up for the mass-critical Hartree equations. *Proc. Roy. Soc. Edinburgh Sect. A*, **145** (2015), 467-479.

[10] Cho Y., Hwang G., Kwon S., Lee S., Profile decompositions and Blow-up phenomena of mass critical fractional Schrödinger equations. *Nonlinear Anal.*, **86** (2013), 12-29.

[11] Zhu S., On the blow-up solutions for the nonlinear fractional Schrödinger equation. *J. Differential Equations*, **261** (2016), 1506-1531.

[12] Zhang J., Zhu S., Stability of standing waves for the nonlinear fractional Schrödinger equation. *J. Dynam. Differential Equations*, **29** (2017) , 1017-1030.

[13] Cho Y., Hajaiej H., Hwang G., Ozawa T., On the orbital stability of fractional Schrödinger equations. *Commun. Pure Appl. Anal.*, **13** (2014), 1267-1282.

[14] Zhu S., Existence of stable standing waves for the fractional Schrödinger equations with combined nonlinearities. *J. Evol. Equ.*, **17** (2017), 1003-1021.

[15] Feng B., Zhang H., Stability of standing waves for the fractional Schrödinger-Hartree equation. *J. Math. Anal. Appl.*, **460** (2018), 352-364.

[16] Guo Q., Zhu S., Sharp threshold of blow-up and scattering for the fractional Hartree equation. *J. Differential Equations*, **264** (2018), 2802-2832.

[17] Bhattarai S., On fractional Schrödinger systems of Choquard type. *J. Differential Equations*, **263** (2017), 3197-3229.

[18] Feng B., Zhang H., Stability of standing waves for the fractional Schrödinger-Choquard equation. *Computers and Math. Appli. Math.*, **75** (2018), 2499-2507.

[19] Cho Y., Ozawa T., Xia S., Remarks on some dispersive estimates. *Commun. Pure Appl. Anal.*, **10** (2011), 1121-1128.

[20] Guo Z., Wang Y., Improved Strichartz estimates for a class of dispersive equations in the radial case and their applications to nonlinear Schrödinger and wave equations. *J. Anal. Math.*, **124** (2014), 1-38.

[21] Ke Y., Remark on the Strichartz estimates in the radial case. *J. Math. Anal. Appl.*, **387** (2012), 857-861.

[22] Cho Y., Lee S., Strichartz estimates in spherical coordinates. *Indiana Univ. Math. J.*, **62** (2013), 991-1020.

[23] Dinh V.D., Well-posedness of nonlinear fractional Schrödinger and wave equations in Sobolev spaces. *Int. J. Appl. Math.*, **31** (2018), 483-525.

[24] Christ M., Weinstein I., Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. *J. Funct. Anal*, **100** (1991), 87-109.

[25] Saanouni T., A note on the fractional Schrödinger equation of Choquard type. *J. Math. Anal. Appl.*, **470** (2019), 1004-1029.