Multiple Solutions for an Elliptic Equation with Hardy Potential and Critical Nonlinearity on Compact Riemannian Manifolds

MALIKI Youssef[*]and TERKI Fatima Zohra

Laboratoire Systèmes Dynamiques et Applications Department of Mathematics, University Aboubekr Belkaïd of Tlemcen, Tlemcen, Algeria.

Abstract. We prove the existence of multiple solutions of an elliptic equation with critical Sobolev growth and critical Hardy potential on compact Riemannian manifolds.

1 Introduction

Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. For a fixed point p in M, we define a function ρ_p on M as follows

$$\rho_p(x) = \begin{cases} \text{dist}_g(p,x), & x \in B(p,\delta_g), \\ \delta_g, & x \in M \setminus B(p,\delta_g), \end{cases} \tag{1.1}$$

where δ_g denotes the injectivity radius of M.

Let h and f be two regular functions on M. Consider on $M \setminus \{p\}$ the following Hardy-Sobolev equation:

$$\Delta_g u - \frac{h(x)}{\rho_p^2(x)} u = f(x)|u|^{2^*-2}u, \tag{$E_{f,h}$}$$

where $\Delta_g u = -\text{div}(\nabla_g u)$ is the Laplace-Beltrami operator and $2^* = \frac{2n}{n-2}$ is the Sobolev critical exponent.

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[*]Corresponding author. *Email addresses:* `malyouc@yahoo.fr` (Y. Maliki), `fatimazohra113@yahoo.fr` (F. Z. Terki)

As one may notice, when dropping the singular term $\frac{1}{\rho_p^2(x)}$ from equation $(E_{f,h})$ and putting $h = \frac{n-2}{4(n-1)}Scal_g$, where $Scal_g$ is the scalar curvature of (M,g), one falls in the celebrated prescribed scalar curvature equation whose origin comes from the study of conformal deformation of the metric to prescribed scalar curvature. A smooth positive solution u of the prescribed scalar curvature equation provides a conformal metric $g' = u^{\frac{4}{n-2}}g$ with scalar curvature the function f; when f is constant we fall n the famous Yamabe equation. The prescribed scalar curvature equation is largely studied and lot of results have been obtained. For those interested, good comprehensive references may be the books [1] and [2]. Equation $(E_{f,h})$ can be, then, seen as a singular prescribed scalar curvature equation.

The case where the function ρ_p, in equation $(E_{f,h})$, is of power $0 < \gamma < 2$ and $f \equiv 1$, has been studied in [3] and is related to the study of conformal deformation to constant scalar curvature of metrics which are smooth only in some geodesic ball $B(p,\delta)$ (see [3,4]). Note that the author in [3,4] considers also equation $(E_{f,h})$, with $f \equiv 1$, and shows existence of a solution on compact manifolds.

In this paper, we are interested in proving the existence of multiple solutions of equation $(E_{f,h})$. The tool used is a classical theorem from critical point theory (see Theorem 4.2 below). Note that the main difficulty in applying this theorem lies in satisfying the compactness assumption under which the critical points exist. This difficulty is due mainly to the presence of Sobolev exponent and Hardy potential. More explicitly, presence of Sobolev exponent and Hardy potential renders non-compact the inclusions $H_1^2(M) \subset L_{2^*}M$ and $H_1^2(M) \subset L(M, \rho_p^{-2})$ (see Section 2 for definition of the notation). This leads us to analyze compactness of Palais-Smale sequences which can be done by means of of a Struwe type decomposition formulas of Palais-Smale sequences.

2   Notation, useful results and statement of the main result

In this section, we introduce some notation and results that are useful in our study.

We denote by $\mathcal{D}^{1,2}(\mathbb{R}^n), (n \geq 3)$, the Euclidean Sobolev space which is the closure space of $\mathcal{C}_o(\mathbb{R}^n)$, the space of functions u with compact support in $\mathbb{R}^n$, with respect to the norm

$$||u||_{\mathcal{D}^{1,2}(\mathbb{R}^n)} = \sqrt{\int_{\mathbb{R}^n} |\nabla u|^2 dx}.$$

Let $K(n,2)$ denote the best constant in the sharp Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{2^*} dx\right)^{\frac{1}{2^*}} \leq K(n,2)\left(\int_{\mathbb{R}^n} |\nabla u|^2\right)^{\frac{1}{2}}.$$

The explicit value of $K(n,2)$ has been obtained in [5] and [6] (see also [2, Theorem 5.3.1])

$$K(n,2) = \sqrt{\frac{4}{n(n-2)w_n^{2/n}}},$$

where w_n is the volume of the unit sphere $S^n \subset \mathbb{R}^{n+1}$. It is well known that the extremal functions for the above Euclidean sobolev inequality are the family of functions

$$w_\xi(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{\xi}{\xi^2 + |x|^2} \right)^{\frac{n-2}{2}}, \qquad \xi > 0, \tag{2.1}$$

which classify all positive solutions of the Euclidean equation (see [7])

$$\Delta u = u^{2^*-1}. \tag{2.2}$$

Let (M,g) be a compact Riemannian manifold. Denote by $B(q,r)$ the geodesic ball of center $q \in M$ and radius r and by $B(r) \subset \mathbb{R}^n$ the Euclidean ball of center 0 and radius r.

Let $q \in M$. Denote by $\exp_q$ the exponential map at q which, for $0 < r < \delta_g$, where δ_g stands for the injectivity radius of (M,g), defines a diffeomorphism from $B(r)$ to $B(q,r)$.

Let $H_1^2(M)$ be the Sobolev space defined as the completion of $\mathcal{C}^\infty(M)$ with respect to the norm

$$||u||^2_{H_1^2(M)} = \int_M (|\nabla_g u|^2 + u^2) dv_g.$$

The manifold M is compact, the Sobolev space $H_1^2(M)$ is then compactly embedded in $L_q(M)$ for $q < 2^* = \frac{2n}{n-2}$ and continuously for $q = 2^*$.

It is known that for any $u \in H_1^2(M)$, there exists a constant $B > 0$ such that(see [8, Theorem 4.6])

$$\left(\int_M |u|^{2^*} \right)^{\frac{2}{2^*}} \leq K^2(n,2) \int_M |\nabla_g u|^2 dv_g + B \int_M u^2 dv_g. \tag{2.3}$$

Let ρ_p be the function defined by (1.1) and denote by $L_2(M, \rho_p^{-2})$ the weighted space of functions u such that $\frac{u^2}{\rho_p^2}$ is integrable. It is a Banach space endowed with norm

$$||u||^2_{2,\rho_p^{-2}} = \int_M \frac{|u|^2}{\rho_p^2} dv_g.$$

In [3], the author proved the following Hardy inequality: let (M,g) be a compact Riemannian manifold, for every $\varepsilon > 0$ there exists a positive constant $A(\varepsilon)$ such that for any $u \in H_1^2(M)$,

$$\int_M \frac{u^2}{\rho_p^2} dv_g \leq (K^2(n,-2) + \varepsilon) \int_M |\nabla_g u|^2 dv_g + A(\varepsilon) \int_M u^2 dv_g, \tag{2.4}$$

with $K(n,-2)$ being the best constant in the Euclidean Hardy inequality

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq K(n,-2)^2 \int_{\mathbb{R}^n} |\nabla u|^2 dx, \qquad u \in \mathcal{C}_0^\infty(\mathbb{R}^n).$$

The constant $K(n,-2)$ is equal to $\frac{2}{n-2}$ and is not attained.

If u is supported in some ball $B(p,\delta)$, $0 < \delta < \delta_g$, then there exists a positive constant $K_\delta(n,-2)$

$$\int_{B(p,\delta)} \frac{u^2}{\rho_p^2} dv_g \leq K_\delta(n,-2) \int_{B(p,\delta)} |\nabla_g u|^2 dv_g, \tag{2.5}$$

with $K_\delta(n,-2)$ goes to $K(n,-2)$ as δ goes to 0.

On the Euclidean space $\mathbb{R}^n$, the author in [9] considered the equation

$$\Delta u - \frac{\lambda}{|x|^2} u = |u|^{\frac{4}{n-2}} u, \qquad \lambda > 0, \tag{2.6}$$

and proved that for $0 < \lambda < \frac{(n-2)^2}{4} = \frac{1}{K^2(n,-2)}$, Eq. (2.6) has a one parameter family of radially symmetric positive solutions

$$U_{\lambda,w}(x) = w^{\frac{2-n}{2}} U_\lambda\left(\frac{x}{w}\right), \qquad w > 0, \ x \in \mathbb{R}^n, \tag{2.7}$$

where

$$U_\lambda(x) = (n(n-2))^{\frac{n-2}{4}} \left(\frac{a_\lambda |x|^{a_\lambda-1}}{1+|x|^{2a_\lambda}}\right)^{\frac{n}{2}-1}, \qquad x \in \mathbb{R}^n,$$

where $a_\lambda = \sqrt{1-\lambda K^2(n,-2)}$. Note that for $\lambda = 0$, we meet the functions w_ξ defined by (2.1). Furthermore, if we denote by S_λ the infimum

$$S_\lambda = \inf_{u \in D^{1,2}, u \neq 0} \frac{\int_{\mathbb{R}^n}\left(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\right) dx}{\left(\int_{\mathbb{R}^n}|u|^{2^*} dx\right)^{\frac{2}{2^*}}},$$

then the functions defined by (2.7) are extremal for this infimum. That is

$$S_\lambda = \frac{\int_{\mathbb{R}^n}\left(|\nabla w_{\lambda,\xi}|^2 - \lambda\frac{w_{\lambda,\xi}^2}{|x|^2}\right) dx}{\left(\int_{\mathbb{R}^n}|w_{\lambda,\xi}|^{2^*} dx\right)^{\frac{2}{2^*}}}. \tag{2.8}$$

Moreover, it follows from [9] that

$$S_\lambda = \frac{(1-\lambda K^2(n,-2))^{\frac{n-1}{n}}}{K^2(n,2)}. \tag{2.9}$$

Let h and f be smooth functions on M such that f is positive everywhere on M and $1 - h(p)K(n, -2)^2 > 0$. Denote by D^* the constant

$$D^* = \frac{(S_{h(p)})^{\frac{n}{2}}}{n(f(p))^{\frac{n-2}{2}}} = \frac{(1 - h(p)K^2(n, -2))^{\frac{n-1}{2}}}{n(f(p))^{\frac{n-2}{2}}K^n(n, 2)}. \tag{2.10}$$

A weak solution of $(E_{f,h})$ is a function $u \in H_1^2(M)$ such that

$$\int_M \left(g(\nabla_g u, \nabla_g v) - \frac{h}{\rho_p^2} uv \right) dv_g - \int_M f(x)|u|^{2^*-2}uv \, dv_g = 0, \qquad \forall \, v \in H_1^2(M).$$

Weak solutions of $(E_{f,h})$ are in $\mathcal{C}^\infty(M \setminus \{p\})$. In fact, let $u \in H_1^2(M)$ be any weak solution of $(E_{f,h})$. for $\varepsilon > 0$ small, put $N_\varepsilon = M \setminus B(p, \varepsilon)$ and consider the problem

$$\begin{cases} \Delta_g v - \dfrac{h(x)}{(\rho_p(x))^2} v = f(x)|v|^{2^*-2}v, & x \in N_\varepsilon, \\ v|_{\partial B(p, \varepsilon)} = u|_{\partial B(p, \varepsilon)}, \\ v \in H_1^2(N_\varepsilon). \end{cases} \tag{2.11}$$

Since u is weak solution of the above problem, then by [2, Lemma 6.2.9], $u \in \mathcal{C}^\infty(N_\varepsilon)$. Since ε are arbitrary, we get that $u \in \mathcal{C}^\infty(M \setminus \{p\})$.

Let $J_{f,h}$ denote the energy functional defined on $H_1^2(M)$ by

$$J_{f,h}(u) = \frac{1}{2} \int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2 \right) dv_g - \frac{1}{2^*} \int_M f(x)|u|^{2^*} dv_g. \tag{2.12}$$

The functional $J_{f,h}$ is a C^2 functional on $H_1^2(M)$. Its Fréchet derivative is given by

$$(DJ_{f,h})(u) \cdot v = \int_M \left(g(\nabla_g u, \nabla_g v) - \frac{h}{\rho_p^2} uv \right) dv_g - \int_M f(x)|u|^{2^*-2}uv \, dv_g.$$

A critical point of the functional $J_{f,h}$ is a function $u \in H_1^2(M)$ such that $(DJ_{f,h})(u) \cdot v = 0, \forall v \in H_1^2(M)$. Weak solutions of equation $(E_{f,h})$ then coincide with critical points of the functional $J_{f,h}$. Now, put

$$\mu = \inf_{u \in H_1^2(M), u \neq 0} \frac{\displaystyle\int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2 \right) dv_g}{\left(\displaystyle\int_M f(x)|u|^{2^*} dv_g \right)^{\frac{2}{2^*}}}.$$

Denote by $\mathcal{E}_h$ the functional

$$\mathcal{E}_h(u) = \int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2 \right) dv_g. \tag{2.13}$$

The functional $\mathcal{E}_h(u)$ is said to be coercive if there exists a positive constant $\lambda > 0$ such that $\mathcal{E}_h(u) \geq \lambda \|u\|^2_{H_1^2(M)}$. If $\mathcal{E}_h(u)$ is coercive, by Sobolev inequality (2.3) and positivity of the function f, we get $\mu > 0$.

In [3], the authors showed, by the classical variational method, the existence of weak solution of $(E_{f,h})$, with $f \equiv 1$, under the condition

$$\mu < \frac{1-h(p)K^2(n,-2)}{K^2(n,2)} = (1-h(p)K^2(n,-2))^{\frac{1}{n}}(nD^*)^{\frac{2}{n}}.$$

In Proposition 4.1 below, we extend this existence result to equation $(E_{f,h})$ and we prove the existence of a weak solution under the condition

$$\mu < \frac{\left(1-h(p)K^2(n,-2)\right)^{\frac{n-1}{n}}}{(f(p))^{\frac{n-2}{n}}K^2(n,2)} = (nD^*)^{\frac{2}{n}}.$$

Now, for further use, we recall the notion of Lusternik-Schnirelmann category. For more details, the reader may consult, for example, the book [10].

Let X and Y be two topological spaces. The Lusternik-Schnirelmann category $Cat_Y(X)$ of a X with respect to Y with $X \subset Y$ is the least integer $k \leq \infty$ such that there exists an open covering of U_i of X with each U_i is contractible in Y. If $X = Y$, we put $Cat_X(X) = Cat(X)$.

In this paper, we prove the following main result

Theorem 2.1. *Let (M,g) be a compact Riemannian manifold of dimension n. Let f and h be two smooth functions on M such that f is positive everywhere on M and the function h is such that the operator $\mathcal{E}_h$, defined by (2.13), is coercive. Suppose that the following conditions are satisfied*

(a) $h(p) > 0$, $1 - h(p)\frac{4}{(n-2)^2} > 0$,

(b) $n = dim(M) > 2 + \frac{2}{a}$, $a = \sqrt{1 - h(p)\frac{4}{(n-2)^2}}$,

(c) $Scal_g(p) > 0$, $(\Delta_g h)(p) - \frac{1}{3}h(p)Scal_g(p) > 0$,

(d) $\sup_{x \in M} f(x) = f(p)$, $(\Delta_g f)(p) - \frac{1}{3}f(p)Scal_g(p) > 0$.

Then, equation $(E_{f,h})$ admits at least $Cat(M)$ weak solutions u with $0 < J_{f,h}(u) < D^*$ and at least one weak solution u with $J_{f,h}(u) > D^*$.

3   Compactness of Palais-Smale sequences

Consider again the energy function $J_{f,h}$ (defined in (2.12))

$$J_{f,h}(u) = \frac{1}{2}\int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2}u^2\right)dv_g - \frac{1}{2^*}\int_M f(x)|u|^{2^*}dv_g, \quad u \in H_1^2(M).$$

A Palais-Smale sequence u_m (P-S in short) of $J_{f,h}$ at a level d is a sequence that satisfies $J_{f,h}(u_m) \to d$ and $DJ_{f,h}(u_m)\varphi \to 0, \forall \varphi \in H_1^2(M)$.

The functional $J_{f,h}$ is said to satisfy P-S condition at level d if each P-S sequence at level d is relatively compact. Let us introduce on $\mathcal{D}^{1,2}(\mathcal{R}^n)$ the functionals

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n}|\nabla u|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^n}|u|^{2^*}dx, \text{ and}$$

$$J_\infty(u) = \frac{1}{2}\left(\int_{\mathbb{R}^n}|\nabla u|^2 dx - h(p)\int_{\mathbb{R}^n}\frac{u^2}{|x|^2}dx\right) - \frac{f(p)}{2^*}\int_{\mathbb{R}^n}|u|^{2^*}dx. \quad (3.1)$$

Let $0 < r < \frac{\delta_g}{2}$ be a constant and denote by η_r a cut-off function on $\mathbb{R}^n$ such that

$$0 \le \eta_r \le 1, \eta_r = 1 \quad \text{on} \quad B(r) \quad \text{and} \quad \eta = 0 \quad \text{on} \quad \mathbb{R}^n \setminus B(2r). \quad (3.2)$$

We state a theorem which is similar to Theorem 3.1 in [11]. This theorem describes the asymptotic behaviour of P-S sequences of the functional $J_{f,h}$.

Theorem 3.1. *Let (M,g) be a compact Riemannian manifold with $\dim(M) = n \ge 3$. Consider on M the distance function ρ_p defined by (1.1) and let h be a continuous function on M that at the point $p \in M$, it satisfies $0 < h(p) < \frac{1}{K(n,-2)^2}$.*

Let u_m be a P-S sequence of the functional $J_{f,h}$ at level d. Then, there exist $k \in \mathbb{N}$, sequences $R_m^i > 0, R_m^i \underset{m\to\infty}{\to} 0, \ell \in \mathbb{N}$ sequences $\tau_m^j > 0, \tau_m^j \underset{m\to\infty}{\to} 0$, converging sequences $x_m^j \to x_o^j \ne p$ in M, a solution $u \in H_1^2(M)$ of $(E_{f,h})$, solutions $v_i \in D^{1,2}(\mathbb{R}^n)$ of

$$\Delta u - \frac{h(p)}{|x|^2}u = f(p)|u|^{2^*-2}u, \qquad x \in \mathbb{R}^n, \quad (3.3)$$

and nontrivial solutions $v_j \in D^{1,2}(\mathbb{R}^n)$ of (2.2) such that up to a subsequence

$$u_m = u + \sum_{i=1}^{k}(R_m^i)^{\frac{2-n}{2}}\eta_r(\exp_p^{-1}(x))v_i((R_m^i)^{-1}\exp_p^{-1}(x))$$

$$+ \sum_{j=1}^{\ell}(\tau_m^i)^{\frac{2-n}{2}}\eta_r(\exp_{x_m^j}^{-1}(x))(f(x_o^j))^{\frac{2-n}{4}}v_j((\tau_m^j)^{-1}\exp_{x_m^j}^{-1}(x)) + \mathcal{W}_m, \quad (3.4)$$

with $\mathcal{W}_m \to 0$ in $H_2^1(M)$, and

$$J_{f,h}(u_m) = J_{f,h}(u) + \sum_{i=1}^{k}J_\infty(v_i) + \sum_{j=1}^{\ell}(f(x_o^j))^{\frac{2-n}{2}}J(v_j) + o(1). \quad (3.5)$$

Proof. The proof is identical to the proof of Theorem 3.1 in [11]. □

A direct consequence of the above theorem is the following corollaries.

Corollary 3.1. *Suppose that* $\sup_{x\in M} f(x) = f(p)$. *Let* u_m *be a P-S sequence of* $J_{f,h}$ *at a level* d. *Then, if* $0 < d < D^*$, *the sequence* u_m *converges, up to a subsequence, strongly in* $H_1^2(M)$, *to a non-trivial critical point of* $J_{f,h}$.

Proof. By the above theorem, there exists a critical point u of $J_{f,h}$, a sequence of solutions v_i of (2.6) and sequence of non trivial solutions v_j of (2.2) such that up to a subsequence, equalities (3.4) and (3.5) hold.

Suppose that $u \equiv 0$. Since every solution v of (3.3) satisfies $J_\infty(v) \geq D^*$ and for every solution ν of (2.6) satisfies $J(\nu) \geq \dfrac{1}{(\sup_{x\in M} f(x))^{\frac{n-2}{2}} K^n(n,2)}$, we get by (3.5) that

$$J_{f,h}(u_m) \geq \min\left(\frac{1}{(\sup_{x\in M} f(x))^{\frac{n-2}{2}} K^n(n,2)}, D^*\right) = D^*,$$

which is a contradiction. □

Corollary 3.2. *Let* u_m *be a P-S sequence of* $J_{f,h}$ *at level* D^*. *Then, up to a subsequence, either* u_m *converges strongly to a nontrivial critical point* u *of* $J_{f,h}$, *or there exists a sequence of functions* $w_m \in H_1^2(M)$ *such that* $w_m \to 0$ *strongly in* $H_1^2(M)$ *and*

$$u_m = w_m + \left(f(p)^{\frac{2-n}{4}}\right)\phi_{p,R_m},$$

where ϕ_{p,R_m} *is the function*

$$\phi_{p,R_m} = (n(n-2))^{\frac{n-2}{4}}\left(\frac{aR_m^a\left(d_g(p,x)\right)^{a-1}}{R_m^{2a} + \left(d_g(p,x)\right)^{2a}}\right)^{\frac{n}{2}-1},$$

with $a = \sqrt{1 - h(p)K^2(n,-2)}$ *and* $d_g(p,x)$ *the distance from* p *to* x.

Proof. By (3.5), we can have

$$D^* = J_{f,h}(u) + J_\infty(v) + o(1),$$

with v a solution of (3.3). Then, either $u \neq 0$ and $v = 0$, or $u = 0$ and

$$u_m = w_m + (R_m)^{\frac{2-n}{2}} \eta_r(\exp_p^{-1}(x)) v((R_m)^{-1}\exp_p^{-1}(x))$$

with v a positive solution of (3.3). As the solution v can be written as $v(x) = (f(p))^{\frac{2-n}{4}} \vartheta(x)$, such that ϑ is positive solution of (2.6) with $\lambda = h(p)$, we get

$$u_m = w_m + (R_m)^{\frac{2-n}{2}}(f(p))^{\frac{2-n}{4}} \eta_r(\exp_p^{-1}(x))\vartheta((R_m)^{-1}\exp_p^{-1}(x))$$

$$= w_m + (f(p))^{\frac{2-n}{4}} \eta_r(\exp_p^{-1}(x))U_{h(p),R_m}(\exp_p^{-1}(x)),$$

where $U_{h(p),R_m}$ is defined by (2.7). Then,

$$u_m = w_m + (f(p))^{\frac{2-n}{4}} \phi_{p,R_m}.$$

□

4   Construction of solutions

In this section, we construct weak solutions of $(E_{f,h})$ as critical points of the functional $J_{f,h}$. Let us cnsider the Nehari manifold $\mathcal{N}_{f,h}$ associated to the functional $J_{f,h}$

$$\mathcal{N}_{f,h} = \{u \in H_1^2(M) \setminus \{0\}, \ DJ_{f,h}(u) \cdot u = 0\}. \tag{4.1}$$

It is easy to see that this manifold defines a natural constraint set for the functional $J_{f,h}$ in the sense that a P-S sequence of $J_{f,h}$ on $\mathcal{N}_{f,h}$ is also a P-S of $J_{f,h}$ on $H_1^2(M)$. Moreover, if h is such that $\int_M (|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2) dv_g > 0$ for all $u \in H_1^2(M) \setminus \{0\}$, then we have $\sup_{t>0}(tu) = t_o u$ with

$$t_o = \left(\frac{\int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2 \right) dv_g}{\left(\int_M |u|^{2^*} dv_g \right)^{\frac{2}{2^*}}} \right)^{\frac{n-2}{4}},$$

and $t_o u \in \mathcal{N}_{f,h}$. Consider the projection $\Phi : H_1^2(M) \setminus \{0\} \longrightarrow \mathcal{N}_{f,h}$ defined by

$$\Phi(u) = \left(\frac{\int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2 \right) dv_g}{\left(\int_M |u|^{2^*} dv_g \right)^{\frac{2}{2^*}}} \right)^{\frac{n-2}{4}} u. \tag{4.2}$$

We have the following existence result

Proposition 4.1. *Let f and h two smooth functions on M such that f is positive everywhere on M. Under the following conditions*

1. *The functional $\mathcal{E}_h(u) = \int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2 \right) dv_g$ is coercive,*

2. $h(p) > 0$, $1 - h(p)K^2(n,-2) > 0$,

3. $\sup_{x \in M} f(x) = f(p)$,

4. $\mu < (nD^*)^{\frac{2}{n}}$,

there exists a non trivial critical point of $J_{f,h}$.

Proof. Put $d = \inf_{\mathcal{N}_{f,h}} J_{f,h}$. By applying the Ekeland variational principle, we can obtain a P-S sequence on $\mathcal{N}_{f,h}$ at level d which is also a P-S sequence u_m on $H_1^2(M)$. It is clear that

$d \geq \frac{1}{n}\mu^{\frac{n}{2}} > 0$. Let $u \in H_1^2(M) \setminus \{0\}$, then by homogeneity of

$$I_h(u) = \frac{\int_M \left(|\nabla_g u|^2 - \frac{h}{\rho_p^2} u^2 \right) dv_g}{\left(\int_M f|u|^{2^*} dv_g \right)^{\frac{2}{2^*}}},$$

since $\Phi(u) \in \mathcal{N}_{f,h}$, where $\Phi(u)$ is defined by (4.2), we get that

$$I_h(u) = I_h(\Phi(u)) = (nJ_{f,h}(\Phi(u)))^{\frac{2}{n}} \geq (nd)^{\frac{2}{n}}.$$

Thus we get that $\mu \geq (nd)^{\frac{2}{n}}$ and hence $d = \frac{1}{n}\mu^{\frac{n}{2}}$ which means that, under the last condition of the corollary, that $d < D^*$. Hence the sequence u_m converges, up to a subsequence, strongly in $H_1^2(M)$ to a non trivial critical point of $J_{f,h}$. $\square$

In searching other critical points of the functional $J_{f,h}$, we apply the following classical theorem (see for example [10, 12]).

Theorem 4.1. *Let J be C^1 real functional defined on a $C^{1,1}$ Banach manifold N. For $c \in \mathbb{R}$, put $J^c = \{ u \in N : J(u) < c \}$. If J is bounded from below on N and satisfies the P-S condition, then it has at least $Cat(J^c)$ critical points in J^c.*
Moreover, if N is contractible and $Cat(J^c) > 1$ then there exists at least one critical point $u \notin J^c$.

The main difficulty in applying Theorem 4.1 above is that the P-S condition for the functional $J_{f,h}$ is not satisfied for any energy level because of the presence of the critical exponent 2^* and the critical singular term. We construct a subset of the manifold $\mathcal{N}_{f,h}$ on which the P-S condition is satisfied and then apply Theorem 4.1 on this subset to obtain critical pints of the functional $J_{f,h}$.

In the following part, we combine ideas from [13] and [14]. First, by the well-known Nash embedding theorem, without loss of generality, we can assume that the Riemannian manifold M is embedded in some Euclidean space $\mathbb{R}^N$.

Let M_r be the set

$$M_r = \{x \in \mathbb{R}^N : d(x, M) < r\}.$$

Define the radius of the topological invariance r_M of M by

$$r_M = \sup\{r > 0 : Cat(M_r) = Cat(M)\}.$$

Let Σ_ε be the subset of $\mathcal{N}_{f,h}$ defined by

$$\Sigma_\varepsilon = \{u \in \mathcal{N}_{f,h} : \text{s.t. } D^* - g(\varepsilon) < J_{f,h}(u) < D^*, \text{ for some } g(\varepsilon) > 0, \text{ with } g(\varepsilon) \to 0 \text{ as } \varepsilon \to 0\}.$$

By Corollary 3.3, the P-S condition is satisfied in the set Σ_ε. To prove the main theorem, we construct two continuous maps $\mathcal{I}_\varepsilon : M \to \Sigma_\varepsilon$ and $\beta : \Sigma_\varepsilon \to M_{r_M}$ such that the composition

$\beta o \mathcal{I}_\varepsilon$ is homotopic to the identity. This leads, by the Lusternik-Schnirelmann properties (see [10] for example) that $Cat(\Sigma_\varepsilon) \geq Cat(M)$. Thus, by applying Theorem 4.1 on the set Σ_ε, we obtain at least $Cat(M)$ critical points of the functional $J_{f,h}$ in Σ_ε.

Finally, we end the proof of the main theorem by proving the existence of another critical point $u \notin \Sigma_\varepsilon$. This can be done by constructing a set P_ε that contains $\mathcal{I}_\varepsilon(M)$ and is contractible in $\mathcal{N}_{f,h} \cap J^{C_\varepsilon}$ for C_ε such that $0 < C_\varepsilon < D^*$.

First, we have to prove that the set Σ_ε is not empty. This is achieved in Lemma 4.1 below.

Let $0 < \delta < \frac{\delta_g}{2}$ be a constant and let φ be a smooth cut-off function defined on $\mathbb{R}$ such that $0 \leq \varphi < 1$, $\varphi \equiv 1$ on $(-\delta, \delta)$ and $\varphi \equiv 0$ on $\mathbb{R} \setminus (-2\delta, 2\delta)$.

For a given $\varepsilon \in (0,1)$ let us consider on M the functions

$$\phi_\varepsilon(x) = C(n,a)\varphi(\rho_p(x)) \left(\frac{\varepsilon^a}{((\rho_p(x))^{1-a}(\varepsilon^{2a} + (\rho_p(x))^{2a})} \right)^{\frac{n-2}{2}},$$

where

$$C(n,a) = (a^2 n(n-2))^{\frac{n-2}{4}}, \quad \text{and} \quad a = \sqrt{1 - h(p)K(n,-2)^2} \text{ with } 0 < h(p) < \frac{1}{K(n,-2)^2}. \quad (4.3)$$

By [15, Lemma 3], for each $\varepsilon \in (0,1)$, the function $\phi_\varepsilon(x)$ belongs to the Sobolev space $H_1^2(M)$. We prove the following

Lemma 4.1. *Suppose that*

(a) $n = \dim(M) > 2 + \dfrac{2}{a}$, $a = \sqrt{1 - h(p)K(n,-2)^2}$,

(b) $Scal_g(p) > 0$, $(\Delta_g h)(p) - \frac{1}{3}h(p)Scal_g(p) > 0$,

(c) $f(p) = \sup_{x \in M} f(x)$, $(\Delta_g f)(p) - \frac{1}{3}f(p)Scal_g(p) > 0$.

Then, there exists a function $g(\varepsilon) > 0$ with $g(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for ε small

$$D^* - g(\varepsilon) < J_{f,h}(\Phi(\phi_\varepsilon)) < D^*. \quad (4.4)$$

Proof. Put

$$I_n^{an+1} = \int_o^\infty \frac{t^{an+1}}{(1+t^{2a})^n} dt,$$

and

$$U(x) = C(n,a) \left(\frac{|x|^{a-1}}{1+|x|^{2a}} \right)^{\frac{n-2}{2}}, \qquad x \in \mathbb{R}^n, \quad (4.5)$$

where a and $c(n,a)$ are defined by (4.3).

By [15, Lemma 4], for $n > \frac{2}{a} + 2$, we have the following expansions

$$\int_M |\nabla_g \phi_\varepsilon|^2 dv_g = \int_{\mathbb{R}^n} |\nabla U|^2 dx - \frac{1}{6n} Scal_g(p) C_1(n,a) I_n^{an+1} \varepsilon^2 + o(\varepsilon^2), \tag{4.6}$$

with

$$C_1(n,a) = \left(\frac{n-2}{2}\right)^2 C(n,a)^2 w_{n-1} \left[(1-a)^2 \frac{an-2}{a(n-2)+2} + 2(1-a^2) \right.$$
$$\left. + (1+a)^2 \frac{an+2}{a(n-2)-2} \right], \tag{4.7}$$

$$\int_M \frac{h(x)}{\left(\rho_p(x)\right)^2} \phi_\varepsilon^2 dv_g = h(p) \int_{\mathbb{R}^N} \frac{U^2}{|x|^2} dx + C_2(n,a) I_n^{an+1}$$
$$\cdot \left(\frac{1}{2n} (\Delta_g h)(p) - \frac{1}{6n} h(p) Scal_g(p) \right) \varepsilon^2 + o(\varepsilon^2), \tag{4.8}$$

with

$$C_2(n,a) = C(n,a)^2 w_{n-1} \left(2 + \frac{an-2}{a(n-2)+2} + \frac{an+2}{a(n-2)-2} \right), \tag{4.9}$$

$$\int_M f |\phi_\varepsilon|^{2^*} dv_g = f(p) \int_{\mathbb{R}^n} |U(x)|^{2^*} dx + C(n,a)^{2^*} w_{n-1}$$
$$\cdot \left(\frac{1}{2n} (\Delta_g f)(p) - \frac{1}{6n} f(p) Scal_g(p) \right) I_n^{an+1} \varepsilon^2 + o(\varepsilon^2), \tag{4.10}$$

and

$$\left(\int_M f |\phi_\varepsilon|^{2^*} dv_g \right)^{-\frac{2}{2^*}} = \left(f(p) \int_{\mathbb{R}^n} |U(x)|^{2^*} dx \right)^{-\frac{2}{2^*}} \left[1 - \frac{C(n,a) w_{n-1}}{2^* n f(p) \int_{\mathbb{R}^n} |U(x)|^{2^*} dx} \right.$$
$$\left. \cdot \left((\Delta_g f)(p) - \frac{1}{3} f(p) Scal_g(p) \right) I_n^{an+1} \varepsilon^2 \right] + o(\varepsilon^2). \tag{4.11}$$

Now, put

$$A = \left[C_2(n,a) \left(\frac{1}{2n} (\Delta_g h)(p) - \frac{1}{6n} h(p) Scal_g(p) \right) + \frac{1}{6n} Scal_g(p) C_1(n,a) \right] I_n^{an+1},$$

$$B = \frac{C(n,a) w_{n-1}}{2^* n \left(f(p) \int_{\mathbb{R}^n} |U(x)|^{2^*} dx \right)^{\frac{2}{2^*}+1}} \left[(\Delta_g f)(p) - \frac{1}{3} f(p) Scal_g(p) \right] I_n^{an+1}.$$

By assumptions of the Lemma, we have $A > 0$ and $B > 0$. Put

$$E(\phi_\varepsilon) = \frac{\int_M \left(|\nabla_g \phi_\varepsilon|^2 - \frac{h(x)}{(\rho_p(x))^2} \phi_\varepsilon^2 \right) dv_g}{\left(\int_M f |\phi_\varepsilon|^{2^*} dv_g \right)^{\frac{2}{2^*}}}. \tag{4.12}$$

Using the expansions (4.6), (4.8) and (4.11), we get

$$E(\phi_\varepsilon) = \frac{\int_{\mathbb{R}^n} \left(|\nabla U|^2 - h(p) \frac{U^2}{|x|^2} \right) dx}{\left(\int_{\mathbb{R}^n} f(p) |U|^{2^*} dx \right)^{\frac{2}{2^*}}} - \left[B \int_{\mathbb{R}^n} \left(|\nabla U|^2 - h(p) \frac{U^2}{|x|^2} \right) dx \right.$$

$$\left. + \frac{A}{\left(\int_{\mathbb{R}^n} f(p) |U|^{2^*} dx \right)^{\frac{2}{2^*}}} \right] \varepsilon^2 + AB\varepsilon^4 + o(\varepsilon^4). \tag{4.13}$$

Now, using the fact that

$$\frac{\int_{\mathbb{R}^n} \left(|\nabla U|^2 - h(p) \frac{U^2}{|x|^2} \right) dx}{\left(\int_{\mathbb{R}^n} f(p) |U|^{2^*} dx \right)^{\frac{2}{2^*}}} = \frac{(1 - h(p) K^2(n,2,-2))^{\frac{n-1}{n}}}{(f(p))^{\frac{n-2}{n}} K^2(n,2)} = (nD^*)^{\frac{2}{n}}. \tag{4.14}$$

Since $A > 0$ and $B > 0$, by taking

$$\mathcal{K}(\varepsilon) = \left[B \int_{\mathbb{R}^n} \left(|\nabla U|^2 - h(p) \frac{U^2}{|x|^2} \right) dx + \frac{A}{\left(\int_{\mathbb{R}^n} f(p) |U|^{2^*} dx \right)^{\frac{2}{2^*}}} \right] \varepsilon^2 > 0, \tag{4.15}$$

and by writing

$$\left(n J_{f,h}(\Phi(\phi_\varepsilon)) \right)^{\frac{2}{n}} = \frac{\int_M \left(|\nabla_g \phi_\varepsilon|^2 - \frac{h(x)}{(\rho_p(x))^2} \phi_\varepsilon^2 \right) dv_g}{\left(\int_M f |\phi_\varepsilon|^{2^*} dv_g \right)^{\frac{2}{2^*}}},$$

we get that for ε small

$$(nD^*)^{\frac{2}{n}} - \mathcal{K}(\varepsilon) < \left(n J_{f,h}(\Phi(\phi_\varepsilon)) \right)^{\frac{2}{n}} < (nD^*)^{\frac{2}{n}}.$$

Now, by (4.15), we have

$$\left((nD^*)^{\frac{2}{n}}-\mathcal{K}(\varepsilon)\right)^{\frac{n}{2}}=nD^*-\frac{n}{2}(nD^*)^{\frac{n-2}{n}}\mathcal{K}(\varepsilon)+o(\mathcal{K}(\varepsilon))$$

$$=nD^*-\frac{n}{2}(nD^*)^{\frac{n-2}{n}}\left[B\int_{\mathbb{R}^n}\left(|\nabla U|^2-h(p)\frac{U^2}{|x|^2}\right)\mathrm{d}x\right.$$

$$\left.+\frac{A}{\left(\int_{\mathbb{R}^n}f(p)|U|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}}\right]\varepsilon^2+o(\varepsilon^2).$$

Finally, by taking

$$g(\varepsilon)=\frac{n}{2}(nD^*)^{\frac{n-2}{n}}\left[B\int_{\mathbb{R}^n}\left(|\nabla U|^2-h(p)\frac{U^2}{|x|^2}\right)\mathrm{d}x+\frac{A}{\left(\int_{\mathbb{R}^n}f(p)|U|^{2^*}\mathrm{d}x\right)^{\frac{2}{2^*}}}\right]\varepsilon^2$$

$$+o(\varepsilon^2), \tag{4.16}$$

we get the desired conclusion. □

4.1   The map $\mathcal{I}_\varepsilon$

In this subsection, we construct a continuous map $\mathcal{I}_\varepsilon:M\to\Sigma_\varepsilon$. For a fixed point $q\in M$, we put $r_q(x)=\mathrm{dist}_g(q,x)$, $x\in M$ and let $\phi_{q,\xi}$ be the function

$$\phi_{q,\xi}(x)=C(n,a)\varphi(r_q(x))\left(\frac{\xi^a r_q(x)^{a-1}}{\xi^{2a}+r_q(x)^{2a}}\right)^{\frac{n}{2}-1},\qquad \xi>0, \tag{4.17}$$

where a and $c(n,a)$ are defined by (4.3). For $\varepsilon\in(0,1)$, define a function $\mathcal{I}_\varepsilon:M\to\mathcal{N}_{f,h}$ by

$$\mathcal{I}_\varepsilon(q)=\Phi((1-\varepsilon^2)\phi_{p,\varepsilon}+\varepsilon^2\phi_{q,\varepsilon}).$$

Let us prove the following lemmas

Lemma 4.2. *The function $\mathcal{I}_\varepsilon:M\to\mathcal{N}_{f,h}$ is continuous.*

Proof. By continuity of the projection $\Phi:H_1^2(M)(u)\setminus\{0\}\to\mathcal{N}_{f,h}$, in order to prove the continuity of the function $\mathcal{I}_\varepsilon(q)$, we need to prove the continuity of the function $\phi_{q,\varepsilon}$ with respect to q. We proceed as in the proof of [13, Proposition 4.2]. Let q_j be a sequence of points of M that converges to q and prove that

$$\phi_{q_j,\varepsilon}\to\phi_{q,\varepsilon} \text{ in } H_1^2(M) \text{ as } q_j\to q.$$

Put $A_j = B(q_j, 2\delta) \cap B(q, 2\delta)$. Since $q_j \to q$ there exist j_o such that $A_j \neq \emptyset$ for all $j \geq j_o$. Then, for q_j close to q we have

$$\int_{A_j} |\phi_{q_j,\varepsilon}(x) - \phi_{q,\varepsilon}(x)|^2 dv_g$$

$$= \int_{\exp_q^{-1}(A_j)} |(\phi_{q_j,\varepsilon} - \phi_{q,\varepsilon})(\exp_q(z))|^2 \sqrt{|g_{\exp(z)}|} dz$$

$$= C(n,a)^2 \left[\int_{\exp_q^{-1}(A_j)} \eta_{q,\delta}(\exp_q(z))^2 |(U_{q_j} - U_q)(\exp_q(z))|^2 \sqrt{|g_{\exp(z)}|} dz \right.$$

$$+ \int_{\exp_q^{-1}(A_j)} U_{q_j}^2(\exp_q(z)) |\eta_{q,\delta}(\exp_q(z)) - \eta_{q_j,\delta}(\exp_q(z))|^2 \sqrt{|g_{\exp(z)}|} dz$$

$$\left. + 2 \int_{\exp_q^{-1}(A_j)} \eta_{q,\delta} U_{q_j,\varepsilon} |(\eta_{q,\delta} - \eta_{q_j,\delta})(\exp_q(z))| |(U_{q_j} - U_q)(\exp_q(z))|^2 \sqrt{|g_{\exp(z)}|} dz \right],$$

where

$$U_{q,\varepsilon}(x) = \left(\frac{\varepsilon^a r_q(x)^{a-1}}{\varepsilon^{2a} + r_q(x)^{2a}} \right)^{\frac{n}{2}-1}, \qquad q \in M.$$

Using the fact that $U_{q_j} \to U_q$ and $\eta_{q_j,\varepsilon} \to \eta_{q,\varepsilon}$ pointwise together with the boundedness of $\int_{\exp_q^{-1}(A_j)} U_{q_j}^2(\exp_q(z)) \sqrt{|g_{\exp(z)}|} dz$, we get that

$$\int_{A_j} |\phi_{q_j,\varepsilon}(x) - \phi_{q,\varepsilon}(x)|^2 dv_g \to 0.$$

Of course, outside the set A_j, $\int_{M \setminus A_j} |\phi_{q_j,\varepsilon}(x) - \phi_{q,\varepsilon}(x)|^2 dv_g \to 0$. Similarly, the same conclusion holds for $\int_M |\nabla_g \phi_{q_j,\varepsilon}(x) - \nabla_g \phi_{q,\varepsilon}(x)|^2 dv_g$. $\qquad \square$

Lemma 4.3. *Suppose that*

(a) $n = \dim(M) > 2 + \dfrac{2}{a}$, $a = \sqrt{1 - h(p)K(n,-2)^2}$,

(b) $Scal_g(p) > 0$, $(\Delta_g h)(p) - \dfrac{1}{3} h(p) Scal_g(p) \geq 0$,

(c) $f(p) = \sup_{x \in M} f(x)$, $(\Delta_g f)(p) - \frac{1}{3} f(p) Scal_g(p) \geq 0$.

Then, $\mathcal{I}_\varepsilon(q) \in \Sigma_\varepsilon$ for all $q \in M$.

Proof. First, if $q = p$, we have $\mathcal{I}_\varepsilon(p) = \Phi(\phi_{p,\varepsilon})$ and the conclusion follows from Lemma 4.1. If $q \neq p$, let $\delta > 0$ be small enough so that $B(q, 2\delta) \cap B(p, 2\delta) = \emptyset$. In this way, the functions $\phi_{p,\varepsilon}$ and $\phi_{q,\varepsilon}$ are of disjoint supports. Put

$$I\left((1 - \varepsilon^2) \phi_{p,\varepsilon} + \varepsilon^2 \phi_{q,\varepsilon} \right) \tag{4.18}$$

$$= \int_M \left(|\nabla_g((1-\varepsilon^2)\phi_{p,\varepsilon}+\varepsilon^2\phi_{q,\varepsilon}))|^2 - \frac{h}{r_p^2}((1-\varepsilon^2)\phi_{p,\varepsilon}+\varepsilon^2\phi_{q,\varepsilon})^2 \right) \mathrm{d}v_g.$$

Then, we have

$$I\left((1-\varepsilon^2)\phi_{p,\varepsilon}+\varepsilon^2\phi_{q,\varepsilon}\right)$$
$$=(1-\varepsilon^2)^2 \int_M \left(|\nabla_g\phi_{p,\varepsilon}|^2 - \frac{h}{r_p^2}\phi_{p,\varepsilon} \right) \mathrm{d}v_g + \varepsilon^4 \int_M \left(|\nabla_g\phi_{q,\varepsilon}|^2 - \frac{h}{r_p^2}\phi_{q,\varepsilon} \right) \mathrm{d}v_g.$$

We point out that by considering a normal geodesic coordinate system around the point q and following the proof of [15, Lemma 4] the expansion (4.6) remains the same for the point q, i.e.

$$\int_M |\nabla_g\phi_{q,\varepsilon}|^2 \mathrm{d}v_g = \int_{\mathbb{R}^n} |\nabla U|^2 \mathrm{d}x - \frac{1}{6n} Scal_g(q) C_1(n,a) I_n^{a(n-2)+1} \varepsilon^2 + o(\varepsilon^2),$$

with $c_1(n,a)$ is defined by (4.7). Then,

$$\varepsilon^4 \int_M |\nabla_g\phi_{q,\varepsilon}|^2 \mathrm{d}v_g = \varepsilon^4 \int_{\mathbb{R}^n} |\nabla U|^2 \mathrm{d}x + o(\varepsilon^4).$$

Moreover, always by considering a normal geodesic coordinate system around the point q, we have

$$\int_M \frac{h(x)}{r_p^2}\phi_{q,\varepsilon}^2 \mathrm{d}v_g = C(n,a)^2 w_{n-1} \frac{h(q)}{(r_p(q))^2}\varepsilon^2 \int_0^\infty \frac{t^{a(n-2)+1}}{(1+t^{2a})^{n-2}}\mathrm{d}t + O(\varepsilon^{a(n-2)}),$$

with w_{n-1} is the volume of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Since $a(n-2) > 2$, we get that

$$\varepsilon^4 \int_M \frac{h(x)}{r_p^2}\phi_{q,\varepsilon}^2 \mathrm{d}v_g = o(\varepsilon^4).$$

Hence, we obtain

$$I\left((1-\varepsilon^2)\phi_{p,\varepsilon}+\varepsilon^2\phi_{q,\varepsilon})\right) \tag{4.19}$$
$$=(1-\varepsilon^2)^2 \int_M \left(|\nabla_g\phi_{p,\varepsilon}|^2 - \frac{h}{r_p^2}\phi_{p,\varepsilon}^2 \right) \mathrm{d}v_g + \varepsilon^4 \int_{\mathbb{R}^n} |\nabla U|^2 \mathrm{d}x + o(\varepsilon^4).$$

On the other hand, since the functions $\phi_{p,\varepsilon}$ and $\phi_{q,\varepsilon}$ are of disjoint supports, we have

$$\left(\int_M f|(1-\varepsilon^2)\phi_{p,\varepsilon}+\varepsilon^2\phi_{q,\varepsilon}|^{2^*} \mathrm{d}v_g \right)^{-\frac{2}{2^*}}$$
$$= \left((1-\varepsilon^2)^{2^*} \int_M f|\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g + \varepsilon^{2\times2^*} \int_M f|\phi_{q,\varepsilon}|^{2^*} \mathrm{d}v_g \right)^{-\frac{2}{2^*}}.$$

Here again, by considering a normal geodesic coordinate system around q, we get (see [15, Lemma 4])

$$\int_M f|\phi_{q,\varepsilon}|^{2^*}\,\mathrm{d}v_g = f(q)\int_{\mathbb{R}^n}|U(x)|^{2^*}\,\mathrm{d}x + C(n,a)^{2^*}w_{n-1}\left(\frac{1}{2n}(\Delta_g f)(q) - \frac{1}{6n}f(q)Scal_g(q)\right)$$
$$\cdot I_n^{an+1}\varepsilon^2 + o(\varepsilon^2). \tag{4.20}$$

Then, since $2^* > 2$, we get that

$$\varepsilon^{2\times 2^*}\int_M f|\phi_{q,\varepsilon}|^{2^*}\,\mathrm{d}v_g = o(\varepsilon^4).$$

We get then

$$\left(\int_M f|(1-\varepsilon^2)\phi_{p,\varepsilon} + \varepsilon^2\phi_{q,\varepsilon}|^{2^*}\,\mathrm{d}v_g\right)^{-\frac{2}{2^*}} = (1-\varepsilon^2)^{-2}\left(\int_M f|\phi_{p,\varepsilon}|^{2^*}\,\mathrm{d}v_g\right)^{-\frac{2}{2^*}} + o(\varepsilon^4). \tag{4.21}$$

Thus, by (4.18), (4.19) and (4.21), we get

$$\left(nJ_{f,h}(\mathcal{I}_\varepsilon)\right)^{\frac{2}{n}} = \frac{\int_M\left(|\nabla_g\phi_{p,\varepsilon}|^2 - \frac{h(x)}{(\rho_p(x))^2}\phi_{p,\varepsilon}^2\right)\mathrm{d}v_g}{\left(\int_M f|\phi_{p,\varepsilon}|^{2^*}\,\mathrm{d}v_g\right)^{\frac{2}{2^*}}} + \frac{\int_{\mathbb{R}^n}|\nabla U|^2\mathrm{d}x}{\left(f(q)\int_{\mathbb{R}^n}|U|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}}\varepsilon^4 + o(\varepsilon^4).$$

Hence, by (4.12), (4.13) and (4.14), we get

$$\left(nJ_{f,h}(\mathcal{I}_\varepsilon)\right)^{\frac{2}{n}} = (nD^*)^{\frac{2}{n}} - \mathcal{K}(\varepsilon) + \left(AB + \frac{\int_{\mathbb{R}^n}|\nabla U|^2\mathrm{d}x}{\left(f(q)\int_{\mathbb{R}^n}|U|^{2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}}\right)\varepsilon^4 + o(\varepsilon^4),$$

with $\mathcal{K}(\varepsilon)$ is defined by (4.15). Hence, we conclude that $\mathcal{I}_\varepsilon \in \Sigma_\varepsilon$ as in Lemma 4.1. □

4.2 The map $\beta:\Sigma_\varepsilon \to M_{R_M}$.

In this subsection, we construct a map $\beta:\Sigma_\varepsilon \to M_{R_M}$. For this aim, we introduce the barycenter function $\beta:\mathcal{N}_{f,h} \to \mathbb{R}^n$ defined by

$$\beta(u) = \frac{\int_M (x+q-p)f|u|^{2^*}\,\mathrm{d}v_g}{\int_M f|u|^{2^*}\,\mathrm{d}v_g}.$$

The function β is well defined as $u \neq 0$ for all $u \in \mathcal{N}_{f,h}$ and the manifold M is embedded in some Euclidean space $\mathbb{R}^N$. We prove some properties of the function β through a series of lemmas:

Lemma 4.4. *We have*

$$\lim_{\varepsilon \to 0} \beta(\mathcal{I}_\varepsilon(q)) = q.$$

Proof. We begin with case $q = p$. By homogeneity of the function β, we have

$$\beta(\mathcal{I}_\varepsilon(p)) = \beta(\phi_{p,\varepsilon}) = \frac{\int_M x f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g}{\int_M f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g}.$$

Then

$$|\beta(\mathcal{I}_\varepsilon(p)) - p| = \left| \frac{\int_M x f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g}{\int_M f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g} - \frac{\int_M p f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g}{\int_M f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g} \right| \leq \frac{\int_M |x - p| f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g}{\int_M f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g}.$$

For the numerator, we have

$$\int_M |x - p| f(x) |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g = C(n,a) \int_M \varphi(r_p(x)) r_p(x) f(x) \left(\frac{\varepsilon^a r_p(x)^{a-1}}{\varepsilon^{2a} + r_p(x)^{2a}} \right)^{\frac{n}{2}-1} \mathrm{d}v_g.$$

We repeat the same calculation as in Lemma 4.3, we get

$$\int_M |x - p| f(x) |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g = \varepsilon f(p) \int_{\mathbb{R}^n} |U|^{2^*} \mathrm{d}x + \left(\frac{1}{2n} (\Delta_g f)(p) - \frac{1}{6n} f(p) Scal_g(p) \right)$$
$$\cdot (C(n,a))^{2^*} w_{n-1} I_n^{an+1} \varepsilon^3 + o(\varepsilon^3). \tag{4.22}$$

For the dominator, we have already

$$\int_M f |\phi_{p,\varepsilon}|^{2^*} \mathrm{d}v_g = f(p) \int_{\mathbb{R}^n} |U|^{2^*} \mathrm{d}x + \left(\frac{1}{2n} (\Delta_g f)(p) - \frac{1}{6n} f(p) Scal_g(p) \right)$$
$$\cdot (C(n,a))^{2^*} w_{n-1} I_n^{an+1} \varepsilon^2 + o(\varepsilon^2).$$

By letting $\varepsilon \to 0$, we get that $\lim_{\varepsilon \to 0} \beta(\mathcal{I}_\varepsilon(p)) = p$.

Now, for $q \neq p$, we choose δ small enough so that $B(q, 2\delta) \cap B(p, 2\delta) = \varnothing$ in such way that the functions $\phi_{p,\varepsilon}$ and $\phi_{q,\varepsilon}$ have disjoint supports. Then, similarly as above, we have

$$|\beta(\mathcal{I}_\varepsilon(q)) - q| \leq \frac{\int_M |x - p| f(x) |(1 - \varepsilon^2) \phi_{p,\varepsilon} + \varepsilon^2 \phi_{q,\varepsilon}|^{2^*} \mathrm{d}v_g}{\int_M f(x) |(1 - \varepsilon^2) \phi_{p,\varepsilon} + \varepsilon^2 \phi_{q,\varepsilon}|^{2^*} \mathrm{d}v_g}.$$

Since the functions $\phi_{p,\varepsilon}$ and $\phi_{q,\varepsilon}$ have disjoint supports, we have

$$\int_M |x - p| f(x) |(1 - \varepsilon^2) \phi_{p,\varepsilon} + \varepsilon^2 \phi_{q,\varepsilon}|^{2^*} \mathrm{d}v_g$$

$$=(1-\varepsilon^2)^{2^*}\int_M |x-p|f(x)|\phi_{p,\varepsilon}^{2^*}|\,dv_g+\varepsilon^{2\times2^*}\int_M |x-p|f(x)|\phi_{q,\varepsilon}|^{2^*}\,dv_g$$

$$\leq(1-\varepsilon^2)^{2^*}\int_M |x-p|f(x)|\phi_{p,\varepsilon}^{2^*}|\,dv_g+\varepsilon^{2\times2^*}\int_M |x-q|f(x)|\phi_{q,\varepsilon}|^{2^*}\,dv_g$$

$$+\varepsilon^{2\times2^*}|q-p|\int_M f(x)|\phi_{q,\varepsilon}|^{2^*}\,dv_g. \tag{4.23}$$

By (4.20), (4.22) and the expansion

$$\int_M |x-q|f(x)|\phi_{q,\varepsilon}|^{2^*}\,dv_g=\varepsilon f(q)\int_{\mathbb{R}^n}|U|^{2^*}\,dx+\left(\frac{1}{2n}(\Delta_g f)(q)-\frac{1}{6n}f(p)Scal_g(q)\right)$$
$$\cdot(C(n,a))^{2^*}w_{n-1}I_n^{an+1}\varepsilon^3+o(\varepsilon^3),$$

the three terms of right-hand side of inequality (4.23) tend to zero as ε goes to zero.

Besides, by (4.10), (4.11) and (4.21), we have

$$\int_M f(x)|(1-\varepsilon^2)\phi_{p,\varepsilon}+\varepsilon^2\phi_{q,\varepsilon}|^{2^*}\,dv_g\to f(p)\int_{\mathbb{R}^n}|U|^{2^*}\,dx \text{ as } \varepsilon\to0.$$

Therefore, we get

$$|\beta(\mathcal{I}_\varepsilon(q))-q|\to0 \text{ as } \varepsilon\to0.$$

$\square$

Lemma 4.5. *For any $\gamma\in(0,1)$ and for every $u_\varepsilon\in\Sigma_\varepsilon$, we have*

$$\int_{B(p,\frac{r_M}{2})}f|u_\varepsilon|^{2^*}\,dv_g>(1-\gamma)(f(p))^{\frac{2-n}{2}}(S_{h(p)})^{\frac{n}{2}},$$

where $S_{h(p)}$ is defined by (2.9) with $\lambda=h(p)$.

Proof. Suppose by contradiction that there exist $\gamma_o\in(0,1)$, a sequence $\varepsilon_m\to0$ as $m\to\infty$ and a sequence $u_m=u_{\varepsilon_m}\in\Sigma_{\varepsilon_m}$ such that

$$\int_{B(p,\frac{r_M}{2})}f|u_m|^{2^*}\,dv_g\leq(1-\gamma_o)(f(p))^{\frac{2-n}{2}}(S_{h(p)})^{\frac{n}{2}}. \tag{4.24}$$

By proceeding as in [13, Lemma 5.4], we can assume that $D_{\mathcal{N}_{f,h}}J_{f,h}(u_m)\to0$ as $m\to\infty$. Since $D^*-g(\varepsilon_m)<J_{f,h}(u_m)<D^*$, for some $g(\varepsilon_m)>0$ and $g(\varepsilon_m)\to0$ as $m\to\infty$ and since the manifold $\mathcal{N}_{f,h}$ defines a natural constraint for the functional $J_{f,h}$ (see [10]), we can assume that u_m is a P-S sequence of $J_{f,h}$ at level D^*. Thus by Corollary 3.2, up to a subsequence, either u_m converges strongly in $H_1^2(M)$ to nontrivial critical point u of $J_{f,h}$, or there exists a sequence of reals $R_m\to0$ as $m\to\infty$ and a sequence $w_m\in H_1^2(M)$ that converges strongly to 0 in $H_1^2(M)$ such that

$$u_m=(f(p))^{\frac{2-n}{4}}\phi_{p,R_m}+w_m. \tag{4.25}$$

Suppose that u_m converges strongly in $H_1^2(M)$, up to a subsequence, to a nontrivial critical point $u \in H_1^2(M)$ of $J_{f,h}$. Then, u satisfies $\int_M f|u|^{2^*} dv_g = nD^*$ and

$$\int_{B(p,\frac{r_M}{2})} f|u|^{2^*} dv_g \le (1-\gamma_0)(f(p))^{\frac{2-n}{2}}(S_{h(p)})^{\frac{n}{2}}. \tag{4.26}$$

Let $\delta > 0$ be a constant and define functions $w_m: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$v_m(x) = (\varepsilon_m)^{\frac{n-2}{2}} \eta_\delta(\varepsilon_m x) u(\exp_p(\varepsilon_m x)),$$

where η_δ is defined in (3.2).

It is known, see [1, Theorem 1.53] (see also the proof of [1, Lemma 2.24]), that in the normal geodesic coordinates $(B(p,\delta), \exp_p^{-1})$, for every $\varepsilon \in (0,1)$, we can have

$$(1-\varepsilon)^n dx \le dv_g \le (1+\varepsilon)^n dx.$$

We proceed as in the proof of [11, Lemma 3.3] (see also the proof of [13, Lemma 5.7]) to show that $(DJ_\infty)(v_m)\varphi \to 0$, $\forall \varphi \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ as $m \to \infty$. Then,

$$J_\infty(v_m) = \frac{f(p)}{n} \int_{\mathbb{R}^n} |v_m|^{2^*} dx + o(1) \ge D^* = \frac{(f(p))^{\frac{2-n}{2}}(S_{h(p)})^{\frac{n}{2}}}{n}, \tag{4.27}$$

where J_∞ is defined by (3.1).

On the other hand, by continuity of f on p and by (4.26), for m large, we have

$$\begin{aligned}
\int_{B(0,\delta)} f(p)|v_m|^{2^*} dx &= \int_{B(0,\varepsilon_m\delta)} f(p)|\eta_\delta(x)u(\exp_p(x))|^{2^*} dx \\
&\le \frac{1}{(1-\varepsilon)^n} \int_{B(p,\varepsilon_m\delta)} f(x)|u|^{2^*} dv_g + \frac{\varepsilon}{(1-\varepsilon)^n} \int_{B(p,\varepsilon_m\delta)} |u|^{2^*} dv_g \\
&\le \frac{(1-\gamma_o)nD^*}{(1-\varepsilon)^n} + \frac{\varepsilon}{(1-\varepsilon)^n} \int_{B(p,\varepsilon_m\delta)} |u|^{2^*} dv_g.
\end{aligned}$$

Then, we can easily see that for ε small, there exists a small positive constant η_ε such that $\eta_\varepsilon < \gamma_o$ and

$$\int_{B(0,\delta)} f(p)|v_m|^{2^*} dx \le (1-(\gamma_o-\eta_\varepsilon))nD^* + o(1),$$

so that, by (4.27), we get the contradiction.

Now, suppose that up to a subsequence, u_m is such that (4.25) is satisfied. By using the inequality

$$(a+b)^{2^*} \ge a^{2^*} + b^{2^*} + 2^* a^{2^*-1} b + 2^* a b^{2^*-1}, \qquad a \ge 0, \, b \ge 0,$$

and by using the fact that $w_m \to 0$ strongly in $H_1^2(M)$ together with (4.24), we obtain

$$\int_{B(p,\frac{r_M}{2})} f|\phi_{p,R_m}|^{2^*} dv_g \le (1-\gamma_o)f(p)(S_{h(p)})^{\frac{n}{2}} + o(1). \tag{4.28}$$

Put $\varepsilon_m^* = R_m \to 0$ as $m \to \infty$. Thus, by using the expansion (4.11), we have

$$\int_M f|\phi_{p,\varepsilon_m^*}|^{2^*}\,dv_g = f(p)\int_{\mathbb{R}^n}|U|^{2^*}\,dx - (C(n,a))^{2^*}w_{n-1}I_n^{an+1}$$
$$\cdot\left(\frac{1}{2n}(\Delta_g f)(p) - \frac{1}{6n}f(p)Scal_g(p)\right)(\varepsilon_m^*)^2 + o((\varepsilon_m^*)^2).$$

As the function U is a positive solution of (2.6) with $\lambda = h(p)$, we get

$$\int_M f|\phi_{p,\varepsilon_m^*}|^{2^*}\,dv_g = f(p)(S_{h(p)})^{\frac{n}{2}} + o(\varepsilon_m).$$

Recall that the function ϕ_{p,ε_m^*} is supported in $B(p,2\delta)$, then by choosing δ small (so that $\delta < \frac{r_M}{2}$), we obtain by (4.28)

$$f(p)(S_{h(p)})^{\frac{n}{2}} + o(\varepsilon_m) \le (1-\gamma_0)f(p)(S_{h(p)})^{\frac{n}{2}} + o(1).$$

Hence, by letting $m \to \infty$, we get the contradiction

$$f(p)(S_{h(p)})^{\frac{n}{2}} \le (1-\gamma_0)f(p)(S_{h(p)})^{\frac{n}{2}}. \qquad \square$$

Lemma 4.6. *For ϵ small, $\beta(u_\varepsilon) \in M_{r_M}$ for all $u_\varepsilon \in \Sigma_\varepsilon$.*

Proof. Let $u_\varepsilon \in \Sigma_\varepsilon$, by Lemma 4.5, we get that for any $\gamma \in (0,1)$

$$\frac{\displaystyle\int_{B(p,\frac{r_M}{2})} f|u_\varepsilon|^{2^*}\,dv_g}{\displaystyle\int_M f|u_\varepsilon|^{2^*}\,dv_g} > \frac{(1-\gamma)(f(p))^{\frac{2-n}{2}}(S_{h(p)})^{\frac{n}{2}}}{nD^*} = (1-\gamma). \qquad (4.29)$$

Recall that

$$M_{r_M} = \{x \in \mathbb{R}^n : d(x,M) < r_M\}.$$

By (4.29), we obtain

$$|\beta(u) - p| = \left|\frac{\displaystyle\int_M (x-p)f|u_\varepsilon|^{2^*}\,dv_g}{\displaystyle\int_M f|u_\varepsilon|^{2^*}\,dv_g}\right|$$

$$\le \frac{\displaystyle\int_{B(p,\frac{r_M}{2})}|x-p|f|u_\varepsilon|^{2^*}\,dv_g}{\displaystyle\int_M f|u_\varepsilon|^{2^*}\,dv_g} + \frac{\displaystyle\int_{M\setminus B(p,\frac{r_M}{2})}|x-p|f|u_\varepsilon|^{2^*}\,dv_g}{\displaystyle\int_M f|u_\varepsilon|^{2^*}\,dv_g}$$

$$\le \frac{r_M}{2} + \mathrm{Diam}(M)\left(1 - \frac{\displaystyle\int_{B(p,\frac{r_M}{2})} f|u_\varepsilon|^{2^*}\,dv_g}{\displaystyle\int_M f|u_\varepsilon|^{2^*}\,dv_g}\right)$$

$$\leq \frac{r_M}{2}+\mathrm{Diam}(M)\gamma,$$

where $\mathrm{Diam}(M)$ is the diameter of M. Thus, by choosing γ small, we get the conclusion.

$\square$

5 Proof of the main result

Proof. By Lemmas 4.3 and 4.6 the maps $\mathcal{I}_\varepsilon: M \to \Sigma_\varepsilon$ and $\beta: \sigma_\varepsilon \to M_{r_M}$ are well defined. Moreover, by Lemma 4.4 the composition $\beta \circ \mathcal{I}_\varepsilon: M \to M_{r_M}$ is well defined and is homotopic to the identity. Thus, by the properties of Lusternik-Schnirelmann category, $Cat(\Sigma_\varepsilon) \geq Cat(M)$. Since the Palais-Smale conditions are satisfied in the set Σ_ε, by Theorem 4.1 there are at least $cat(M)$ critical points of the functional $J_{f,h}$.

It remains, to achieve the proof of the theorem, to prove that there exists another critical point u with $J_{f,h}(u) > D^*$. For this task, following [13], we construct a set P_ε which is contractible in $\mathcal{N}_{f,h} \cap J_{f,h}^c$.

Put $\varphi_\varepsilon = (1-\varepsilon^2)\phi_{p,\varepsilon} + \varepsilon^2\phi_{q,\varepsilon}$ and define the set

$$\Omega_{\varepsilon,t} = \left\{ (1-t)\varphi_\varepsilon + t\varepsilon^6\phi_{p,\frac{1}{\varepsilon}}, \quad t \in [0,1] \right\}.$$

Consider $P_{\varepsilon,t}$, the projection of Ω_ε on the Nehari manifold $\mathcal{N}_{f,h}$

$$P_{\varepsilon,t} = \left\{ \Phi(\omega_{\varepsilon,t}), \quad \omega_{\varepsilon,t} \in \Omega_\varepsilon \right\}.$$

We notice immediately that $\mathcal{I}_\varepsilon(M) \subset P_{\varepsilon,t}$ and $P_{\varepsilon,t}$ is contractible in $H_1^2(M)$. Put

$$c_\varepsilon = \sup_{t \in [0,1]} J_{f,h}(\Phi(\omega_{\varepsilon,t})).$$

We show that $0 < c_\varepsilon < D^*$. We have already, by Lemma 4.3, that $c_\varepsilon \geq J_{f,h}(\Phi(\varphi_{\varepsilon,t})) > 0$. Let $\omega_{\varepsilon,t} \in \Omega_\varepsilon$. Then, we have

$$\int_M |\nabla_g \omega_{\varepsilon,t}|^2 dv_g = (1-t)^2 \int_M |\nabla_g \varphi_\varepsilon|^2 dv_g + t^2\varepsilon^{12} \int_M |\nabla_g \phi_{p,\frac{1}{\varepsilon}}|^2 dv_g$$
$$+ 2t(1-t)\varepsilon^6 \int_M g(\nabla_g \varphi_\varepsilon, \nabla_g \phi_{p,\frac{1}{\varepsilon}}) dv_g.$$

By (4.6), we have

$$\int_M |\nabla_g \phi_{p,\frac{1}{\varepsilon}}|^2 dv_g = \int_{\mathcal{R}^n} |\nabla U|^2 dx - \frac{1}{6n} Scal_g(p) C_1(n,a) I_n^{an+1} \left(\frac{1}{\varepsilon^2} \right) + o\left(\frac{1}{\varepsilon^2} \right).$$

Then,

$$\varepsilon^{12} \int_M |\nabla_g \phi_{p,\frac{1}{\varepsilon}}|^2 dv_g = -\frac{1}{6n} Scal_g(p) C_1(n,a) I_n^{an+1} \varepsilon^{10} + o(\varepsilon^{10}) = o(\varepsilon^8),$$

and

$$\varepsilon^6 \left| \int_M g(\nabla_g \varphi_\varepsilon, \nabla_g \phi_{\frac{1}{\varepsilon}}) dv_g \right| \le \int_M |\nabla_g \varphi_\varepsilon| \cdot |\nabla_g \phi_{\frac{1}{\varepsilon}}| dv_g$$

$$\le \left(\int_M |\nabla_g \varphi_\varepsilon|^2 dv_g \right)^{\frac{1}{2}} \left(\varepsilon^{12} \int_M |\nabla_g \phi_{\frac{1}{\varepsilon}}|^2 dv_g \right)^{\frac{1}{2}} = o(\varepsilon^4).$$

Then, we get

$$\int_M |\nabla_g \omega_{\varepsilon,t}|^2 dv_g = (1-t)^2 \int_M |\nabla_g \varphi_\varepsilon|^2 dv_g + o(\varepsilon^4). \tag{5.1}$$

Similarly, we get

$$\int_M \frac{h}{\rho_p^2} w_{\varepsilon,t}^2 dv_g = (1-t)^2 \int_M \frac{h}{\rho_p^2} \varphi_\varepsilon^2 dv_g + o(\varepsilon^4).$$

Besides,

$$\int_M f |\omega_{\varepsilon,t}|^{2^*} dv_g = \int_M f |(1-t)\varphi_\varepsilon + t\varepsilon^6 \phi_{p,\frac{1}{\varepsilon}}|^{2^*} dv_g$$

$$\ge (1-t)^{2^*} \int_M f |\varphi_\varepsilon|^{2^*} dv_g + t^{2^*} \varepsilon^{6 \times 2^*} \int_M f |\phi_{p,\frac{1}{\varepsilon}}|^{2^*} dv_g.$$

As before, by (4.10) we have

$$\int_M f |\phi_{p,\frac{1}{\varepsilon}}|^{2^*} dv_g = f(p) \int_{\mathbb{R}^n} |U(x)|^{2^*} dx$$

$$+ C(n,a)^{2^*} w_{n-1} \left(\frac{1}{2n} (\Delta_g f)(p) - \frac{1}{6n} f(p) Scal_g(p) \right) I_n^{an+1} \frac{1}{\varepsilon^2} + o\left(\frac{1}{\varepsilon^2} \right).$$

Then, we can easily see that

$$\varepsilon^{6 \times 2^*} \int_M f |\phi_{p,\frac{1}{\varepsilon}}|^{2^*} dv_g = o(\varepsilon^4).$$

Then

$$\int_M f |\omega_{\varepsilon,t}|^{2^*} dv_g = \int_M f |(1-t)\varphi_\varepsilon + t\varepsilon^6 \phi_{p,\frac{1}{\varepsilon}}|^{2^*} dv_g \ge (1-t)^{2^*} \int_M f |\varphi_\varepsilon|^{2^*} dv_g + o(\varepsilon^4).$$

Hence, we get

$$\left(nJ_{f,h}(\Phi(\omega_{\varepsilon,t})) \right)^{\frac{2}{n}} = \frac{\displaystyle \int_M |\nabla_g \omega_{\varepsilon,t}|^2 dv_g - \int_M \frac{h}{\rho_p^2} \omega_{\varepsilon,t}^2 dv_g}{\left(\displaystyle \int_M f |\omega_{\varepsilon,t}|^{2^*} dv_g \right)^{\frac{2^*}{2}}}$$

$$\leq \frac{\int_M |\nabla_g \varphi_\varepsilon|^2 \mathrm{d}v_g - \int_M \frac{h}{\rho_p^2} \varphi_\varepsilon^2 \mathrm{d}v_g + o(\varepsilon^4)}{\left(\int_M f |\varphi_\varepsilon|^{2^*} \mathrm{d}v_g \right)^{\frac{2^*}{2}} + o(\varepsilon^4)}.$$

Then, the conclusion $J_{f,h}(\Phi(\omega_{\varepsilon,t}) < D^*$ follows as in Lemma 4.3 and thus $0 < c_\varepsilon < D^*$.

Now, since $Cat(\Sigma_\varepsilon) \geq Cat(M) > 1$ and the (P-S) condition, with levels in the interval $]0, D^*[$, is satisfied (see Corollary 3.3), by Theorem 4.1 there exists another critical point u of $J_{f,h}$ with $D^* < J_{f,h}(u)$ and the proof is complete.   □

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