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Quenching Phenomenon of Solutions for Parabolic Equations with Singular Absorption on Graphs

ZHU Liping*, ZHOU Yaxin and HUANG Lin

Faculty of Science, Xi'an University of Architecture and Technology, Xi'an 710055, China.

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Abstract. In this paper, we study the quenching phenomenon of solutions for parabolic equations with singular absorption under the mixed boundary conditions on graphs. Firstly, we prove the local existence of solutions via Schauder fixed point theorem. Then, under certain conditions we obtain the estimates of quenching time and quenching rate. Finally, numerical experiments are shown to explain the theoretical results.

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1 Introduction

In this article, we mainly discuss the quenching phenomenon for the problem

$$\begin{cases} u_t(x,t)=\Delta_\omega u(x,t)+\beta u^{-q}(x,t)-\lambda u^{-p}(x,t), & (x,t)\in S\times(0,\infty), \\ B[u]=0, & (x,t)\in\partial S\times[0,\infty), \\ u(x,0)=u_0(x)>0, & x\in\bar{S}, \end{cases} \tag{1.1}$$

where $\beta>0$, $\lambda>0$, $p,q>0$, and u_0 satisfies the compatibility conditions. Δ_ω is the discrete Laplacian operator defined as

$$\Delta_\omega u(x,t)=\sum_{y\in\bar{S}}\omega(x,y)[u(y,t)-u(x,t)], \quad \forall x\in S,$$

*Corresponding author. *Email addresses:* 78184385@qq.com (L. P. Zhu), 1749611315@qq.com (Y. X. Zhou), 791558012@qq.com (L. Huang)

and weighted function $\omega(x,y):\bar{S}\times\bar{S}\to[0,+\infty)$ satisfies

$$\omega(x,x)=0,\ \forall x\in\bar{S},\quad \omega(x,y)=\omega(y,x),\quad \forall x,y\in\bar{S},\quad \omega(x,y)>0\Leftrightarrow x\sim y.$$

$B[u]=0$ represents the mixed boundary condition

$$\mu(x)\frac{\partial u}{\partial n}(x,t)-\sigma(x)u(x,t)=0.$$

Here $\mu(x)$ is a function greater than 0, and $\sigma(x)$ is nonnegative function defined on ∂S, and for all $x\in\partial S$ there is $\mu(x)-\sigma(x)>0$. $\partial u/\partial n$ denotes the discrete normal derivative as

$$\frac{\partial u}{\partial n}(x,t)=\sum_{y\in S}\omega(x,y)[u(x,t)-u(y,t)],\quad \forall x\in\partial S.$$

Eq. (1.1) models a polarization phenomenon in ionic conductor [1], and it can also be regarded as a limit case of chemical catalyst kinetic model or enzyme kinetic model [2-5]. Moreover, the Robin boundary condition means that the system has energy exchange with the outside medium, and it includes various boundary conditions.

Over the past few decades, the study of the internal structure of graphs has attracted the attention of many researchers in various fields [6,7]. In particular, the properties of discrete Laplace operator on graphs and the solutions of various boundary value problems have been studied by many authors because of its wide range of applications, ranging from solving diffusion equations on networks [8-10], to energy flow modeling through networks or molecular vibration[11,12]. In recent years, some scholars have begun to pay attention to the behavior study of the solution of evolution equations defined on the network structure. Many objects and their relationship are often represented by networks. In mathematics, the weighted graph is another name for a network. Vertices represent objects, while edges represent connections between objects. They are widely used to analyze discrete objects. Chung [11] first introduced some concepts on the graph, such as integral, directional derivative and gradient etc., and proved the uniqueness of global solutions of the inverse problem and the solvability of the first and second boundary value problems under appropriate monotonic conditions, which provide a theoretical basis for the partial differential equations on graphs.

In [13], Chung et al. studied the homogeneous Dirichlet boundary value problem for the ω-heat equation with absorption on a network

$$\begin{cases} u_t(x,t)=\Delta_\omega u(x,t)-u^q, & (x,t)\in G\times(0,+\infty),\\ u(x,t)=0, & x\in\partial G, t>0,\\ u(x,0)=u_0(x), & x\in G.\end{cases}$$

The absorption term denotes that the heat passing through networks is affected by the reactive forces proportional to the power of their potentials. In addition, the behavior of the solution depends entirely on the sign of $q-1$. The authors proved that if $0<q<1$, the nontrivial solution becomes extinct in finite time, but if $q\ge 1$, it remains strictly positive. Furthermore, the extinction and positivity for the p,ω-heat equation with absorption was also studied in [14,15].

In [16], the authors considered the quenching phenomenon of nonlocal diffusion equation with singular absorption term. Liu [17] studied the asymptotic behavior of the solution of the ω-heat equation with reaction term and absorption term on the graph. In addition, Xin [18] studied the quenching problem of the solution of the discrete p,ω-Laplacian parabolic equation with singular absorption term on the graph

$$\begin{cases} u_t=\Delta_{p,\omega}u-\lambda u^{-q}, & (x,t)\in S\times(0,T),\\ u(x,t)=1, & (x,t)\in \partial S\times(0,T),\\ u(x,0)=u_0(x)\ge 0, & x\in S, \end{cases}$$

where $p\ge 2$, $q>0$. The authors proved the local existence and uniqueness of the solution by Banach fixed point theorem. And then, on some suitable conditions, they proved that the solution quenches in finite time by comparison principle. Moreover, the upper bound of quenching time to solution was also obtained.

In [19], the author studied the blow-up phenomenon of solutions for discrete p,ω-Laplacian parabolic equation under mixed boundary conditions

$$\begin{cases} u_t(x,t)=\Delta_{p,\omega}u(x,t)+f(u(x,t)), & (x,t)\in S\times(0,\infty),\\ \mu(z)\dfrac{\partial u}{\partial_p n}(z,t)+\sigma(z)|u(z,t)|^{p-2}u(z,t)=0, & (z,t)\in\partial S\times[0,\infty),\\ u(x,0)=u_0(x)\ge 0, & x\in\bar{S}, \end{cases}$$

where $p>1$, f is a nonnegative locally Lipschitz continuous function on R, $\mu,\sigma:\partial S\to[0,\infty)$ are functions such that $\mu(z)+\sigma(z)>0, z\in\partial S$. They proved that if f satisfies the following conditions:

$$\alpha\int_0^u f(s)\mathrm{d}s\le uf(u)+\beta u^p+\gamma,\quad u>0,$$

for some $\alpha>2$, $\gamma>0$ and $0\le\beta\le\frac{(\alpha-p)\lambda_{p,0}}{p}$. At the same time, when the initial value u_0 satisfies certain conditions, the solution blows up in finite time.

In addition, Chung [20] discussed the Fujita's blow-up problem of solutions for the discrete p,ω-Laplacian parabolic equation on the mixed boundary

$$\begin{cases} u_t(x,t)=\Delta_{p,\omega}u(x,t)+\psi(t)|u(x,t)|^{q-1}u(x,t), & (x,t)\in S\times(0,t^*),\\ \mu(z)\dfrac{\partial u}{\partial_p n}(z,t)+\sigma(z)|u(z,t)|^{p-2}u(z,t)=0, & (x,t)\in\partial S\times[0,t^*),\\ u(x,0)=u_0(x)\ge 0, & x\in\bar{S}, \end{cases}$$

where $p\geq 2, q>0$, ψ is a positive continuous function, t^* is the maximal existence time of the solution u. $\mu,\sigma:\partial S\to[0,\infty)$ are functions such that $\mu(z)+\sigma(z)>0, z\in\partial S$. In order to discuss Fujita's blow-up, they introduced a critical set

$$\Lambda_{p,\psi}=\left\{q>1\left|\int_0^\infty \psi(s)e^{-(q-p+1)\lambda_{p,0}s}\mathrm{d}s=\infty\right.\right\},$$

where $\lambda_{p,0}$ is the first eigenvalue of $\triangle_{p,\omega}$ on a network. In the case of $\sigma=0$, they proved that when $q>1$, $q\in\Lambda_{p,\psi}$, the solution blows up in finite time, and when $q>1$, $q\notin\Lambda_{p,\psi}$, the solution blows up in finite time when the initial data u_0 is sufficiently large. In the case of $\sigma\neq 0$, they obtained that blow-up phenomena can occur for $1\leq p-1\leq q$ with $q>1$. Moreover, if $1<p-1<q$ and $q\in\Lambda_{p,\psi}$, they obtained Fujita's blow-up solutions.

By far, although there are some works about quenching phenomenon for the discrete p,ω-Laplacian parabolic equation [13, 18], all the problems therein are considered with Dirichlet boundary conditions and there is no work about quenching problem under the mixed boundary conditions as far as we known. So based on the above works, the purpose of this paper is to study the quenching phenomenon of solutions for parabolic equation with singular absorption under mixed boundary conditions. In Section 2, we discuss the local existence of solutions through the Schauder fixed point theorem, and study the discrete version of the comparison principle. In Section 3, we prove that the solution quenches in finite time and obtain the quenching rate. Finally, in Section 4, we give some numerical experiments to explain our theoretical results.

2 Local existence of solutions

In this section, we devote to discussing the local existence of solutions for problem (1.1) via the Schauder fixed point theorem. To do this, we need to give the following related knowledge.

Lemma 2.1. ([21]) *Let H be a compact subset of $\mathbb{R}$ and $\bar{S}$ be a network. Consider a Banach space $D(\bar{S}\times H)$ and the maximum norm $||u||_{\bar{S},H}=\max_{(x,t)\in S\times H}|u(x,t)|$. Then a subset Y of $D(\bar{S}\times H)$ is relatively compact if Y is uniformly bounded on $\bar{S}\times H$ and equicontinuous on H for each $x\in\bar{S}$.*

Next we first define a function $\phi:\mathbb{R}\to\mathbb{R}$ by

$$\phi(\beta)=\sum_{x\in S}[\beta-u(x,t)]c(x)-d\beta,$$

here $c(x)>0$ for $x\in S$, and $d\geq 0$ with $c(x)-d>0$ for some $x\in S$. By a simple calculation, we can conclude that ϕ is a strictly increasing bijective continuous function on $\mathbb{R}$, which means that function $\phi(\beta)$ has only one zero point $\rho\in\mathbb{R}$.

Theorem 2.1. (Local existence) *Let $u_0\in C(\bar{S})$ and $u_0>0$, then there exists a finite time $T^*>0$ such that problem* (1.1) *has at least one solution $u\in C^1(0,T^*)\times C(\bar{S})$.*

Proof. Firstly, we define a Banach space and its maximum norm:

$$\begin{aligned}&X(S\times[0,T^*])=\{u:S\times[0,T^*]\to\mathbb{R}|u(x,\cdot)\in C[0,T^*],\forall x\in S\},\\&\|u\|_{S,T^*}=\max_{(x,t)\in S\times[0,T^*]}|u(x,t)|,\end{aligned}$$

and consider a subspace of Banach space $X(S\times[0,T^*])$,

$$A_{T^*}=\left\{u\in X(S\times[0,T^*])\,|\,\frac{1}{2}\|u_0\|_{S,T^*}\le\|u\|_{S,T^*}\le\frac{3}{2}\|u_0\|_{S,T^*}\right\}.$$

Then it is obvious that A_{T^*} is convex and closed.

Next we define an operator $Y:A_{T^*}\to A_{T^*}$ by

$$Y[u](x,t)=u_0(x)+\int_0^t\Delta_\omega u(x,s)+\beta u^{-q}(x,s)-\lambda u^{-p}(x,s)\mathrm{d}s,\quad (x,t)\in S\times[0,T^*].$$

Let

$$T^*=\frac{1}{6\omega_0+\beta\left[\frac{1}{2}\|u_0\|_{S,T^*}\right]^{-q-1}+\lambda\left[\frac{1}{2}\|u_0\|_{S,T^*}\right]^{-p-1}},$$

where $\omega_0=\max_{x\in S}d_\omega x$ and $d_\omega x=\sum_{y\in\bar{S}}\omega(x,y)$. Then it can be proved that the operator Y is well-defined on A_{T^*}. For $\forall u,v\in A_{T^*}$ and all $(x,t)\in S\times[0,T^*]$,

$$\begin{aligned}&|Y[u(x,t)]-Y[v(x,t)]|\\&\le\left|\int_0^t\Delta_\omega(u-v)\mathrm{d}s\right|+\beta\left|\int_0^t(u^{-q}-v^{-q})\mathrm{d}s\right|+\lambda\left|\int_0^t(u^{-p}-v^{-p})\mathrm{d}s\right|\\&\le T^*\left[2\omega_0\|u-v\|_{S,T^*}+\beta q\|u-v\|_{S,T^*}\|\xi_1\|_{S,T^*}^{-q-1}+\lambda p\|u-v\|_{S,T^*}\|\xi_2\|_{S,T^*}^{-p-1}\right]\\&\le\|u-v\|_{S,T^*}T^*\left\{2\omega_0+\beta q\left[\frac{1}{2}\|u_0\|_{S,T^*}\right]^{-q-1}+\lambda p\left[\frac{1}{2}\|u_0\|_{S,T^*}\right]^{-p-1}\right\},\end{aligned}$$

where ξ_1,ξ_2 are between $u(x,t)$ and $v(x,t)$. So by the definition of T^* we know that the operator Y is continuous.

Finally, because of the fact $Y(A_{T^*})\subset A_{T^*}$, we know that $Y(A_{T^*})$ is uniformly bounded. Meanwhile, for any $(x,t_1),(x,t_2)\in\bar{S}\times[0,T^*]$ and all $u\in A_{T*}$, we have

$$\begin{aligned}&|Y[u(x,t_1)]-Y[u(x,t_2)]|\\&\le\left|\int_{t_1}^{t_2}\Delta_\omega u\mathrm{d}s\right|+\beta\left|\int_{t_1}^{t_2}u^{-q}\mathrm{d}s\right|+\lambda\left|\int_{t_1}^{t_2}u^{-p}\mathrm{d}s\right|\\&\le|t_2-t_1|\left[2\omega_0\|u\|_{S,T^*}+\beta\|u\|_{S,T^*}^{-q}+\lambda\|u\|_{S,T^*}^{-p}\right]\\&\le|t_2-t_1|\left\{3\omega_0\|u_0\|_{S,T^*}+\beta\left[\frac{1}{2}\|u_0\|_{S,T^*}\right]^{-q}+\lambda\left[\frac{1}{2}\|u_0\|_{S,T^*}\right]^{-p}\right\},\end{aligned}$$

which means that $Y(A_{T^*})$ is equicontinuous on $[0,T^*]$. So $Y(A_{T^*})$ is relatively compact by Lemma 2.1. Therefore, by the Schauder fixed point theorem, there exists a function $u\in A_{T^*}$ satisfying Eq. (1.1).

Moreover, for all $(z,t)\in\partial S\times[0,T^*]$, from the definition of ϕ we know that the value of $u(z,t)$ can be uniquely determined by the boundary condition

$$\mu(z)\frac{\partial u}{\partial n}(z,t)-\sigma(z)u(z,t)=0,$$

and there is $|u(z,t)|\geq||u||_{S,T^*}$. Also, it is not hard to get that u is bounded. In addition, by the definition of Y and the boundary condition $B[u]=0$, we get that for every $x\in\bar{S}$, $u(x,\cdot)$ is continuous on $[0,T^*]$ and differentiable in $(0,T^*)$. $\square$

Definition 2.1. *Suppose that non-negative function $U(x,t)\in C(\bar{S}\times[0,T))$ is differentiable in $(0,T)$ and satisfies*

$$\begin{cases} U_t(x,t)\geq\Delta_w U(x,t)+\beta U^{-q}(x,t)-\lambda U^{-p}(x,t), & x\in S\times(0,T),\\ \mu(x)\dfrac{\partial U}{\partial n}(x,t)-\sigma(x)U(x,t)\leq 0, & x\in\partial S\times(0,T),\\ U(x,0)\geq U_0(x), & x\in\bar{S}, \end{cases} \tag{2.1}$$

then we call $U(x,t)$ a supersolution of (1.1). *On the contrary, if $V(x,t)\in C(\bar{S}\times[0,T))$ is differentiable in* $(0,T)$ *and satisfies the reverse inequality in* (2.1), *we call $V(x,t)$ a subsolution of problem* (1.1).

Theorem 2.2. (Comparison Principle) *Let $U(x,t)$ and $V(x,t)$ be the supersolution and subsolution of problem* (1.1) *respectively, then $U(x,t)\geq V(x,t)$ for all $(x,t)\in S\times[0,T)$ and $U(x,t)\leq V(x,t)$ for all $(x,t)\in\partial S\times[0,T)$.*

Proof. Combining the definition of supersolution and subsolution and mean value theorem, we have

$$\begin{aligned} &U_t(x,t)-V_t(x,t)-\Delta_\omega[U(x,t)-V(x,t)]-\beta(U^{-q}(x,t)-V^{-q}(x,t))\\ &\qquad+\lambda(U^{-p}(x,t)-V^{-p}(x,t))\\ =&U_t(x,t)-V_t(x,t)-\Delta_\omega[U(x,t)-V(x,t)]+\beta q{\zeta_1}^{-q-1}(U(x,t)-V(x,t))\\ &\qquad-\lambda p{\zeta_2}^{-p-1}(U(x,t)-V(x,t))\geq 0, \end{aligned} \tag{2.2}$$

where $\zeta_i(i=1,2)$ are between $u(x,t)$ and $v(x,t)$. For any given $\widehat{T}\in(0,T)$ and $(x,t)\in\bar{S}\times[0,\widehat{T}]$, let

$$\widetilde{U}(x,t)=e^{-2\lambda pmt}U(x,t),\quad \widetilde{V}(x,t)=e^{-2\lambda pmt}V(x,t),$$

where $m:=[\min_{(x,t)\in\bar{S}\times[0,\widehat{T}]}\{|U(x,t)|,|V(x,t)|\}]^{-p-1}$. Then (2.2) can be written as

$$\widetilde{U_t}(x,t)-\widetilde{V}_t(x,t)-\Delta_\omega[\widetilde{U}(x,t)-\widetilde{V}(x,t)]$$

$$+\Big\{\lambda p[2m-{\zeta_2}^{-p-1}(x,t)]+\beta q\zeta_1^{-q-1}(x,t)\Big\}[\widetilde{U}(x,t)-\widetilde{V}(x,t)]\geq 0. \tag{2.3}$$

Since $\widetilde{U}$ and $\widetilde{V}$ are continuous on $[0,\widehat{T}]$ and $\bar{S}$ is finite, there exists $(x_0,t_0)\in\bar{S}\times[0,\widehat{T}]$ such that

$$(\widetilde{U}-\widetilde{V})(x_0,t_0)=\min_{(x,t)\in\bar{S}\times[0,\widehat{T}]}(\widetilde{U}-\widetilde{V})(x,t),$$

that is to say,

$$\widetilde{V}(x,t_0)-\widetilde{V}(x_0,t_0)\leq\widetilde{U}(x,t_0)-\widetilde{U}(x_0,t_0),\quad \forall x\in\bar{S}. \tag{2.4}$$

Next for the conclusion, we use reductio to prove $(\widetilde{U}-\widetilde{V})(x_0,t_0)\geq 0$ when $x_0\in S$, so we suppose that $(\widetilde{U}-\widetilde{V})(x_0,t_0)<0$. Besides, we also need to prove $(\widetilde{U}-\widetilde{V})(x_0,t_0)\leq 0$ when $x_0\in\partial S$.

Case 1: $x_0\in S$.

Since $\widetilde{U}(x,0)-\widetilde{V}(x,0)\geq 0$ on S, we have $(x_0,t_0)\in S\times(0,T']$. Then we get from (2.4) that

$$\Delta_\omega\widetilde{U}(x_0,t_0)-\Delta_\omega\widetilde{V}(x_0,t_0)\geq 0. \tag{2.5}$$

Also, from the differentiability of $(\widetilde{U}-\widetilde{V})(x,t)$ in $(0,\widehat{T}]$ for each $x\in S$, we see

$$(\widetilde{U}_t-\widetilde{V}_t)(x_0,t_0)=0. \tag{2.6}$$

Applying (2.5) and (2.6) to (2.3), we get

$$\begin{aligned}&\widetilde{U}_t(x_0,t_0)-\widetilde{V}_t(x_0,t_0)-[\Delta_\omega\widetilde{U}(x_0,t_0)-\Delta_\omega\widetilde{V}(x_0,t_0)]\\&\quad+\Big\{\lambda p[2m-{\xi_2}^{-p-1}(x_0,t_0)]+\beta q\xi_1^{-q-1}(x_0,t_0)\Big\}[\widetilde{U}(x_0,t_0)-\widetilde{V}(x_0,t_0)]<0,\end{aligned}$$

which contradicts with (2.3). Then we get $U(x,t)\geq V(x,t)$ for all $(x,t)\in S\times[0,T)$ for the arbitrariness of $\widehat{T}$.

Case 2: $x_0\in\partial S$.

By Definition 2.1 and the boundary condition, we can easily get $(\widetilde{U}-\widetilde{V})(x_0,t_0)\leq 0$. Then we have $U(x,t)\leq V(x,t)$ for all $(x,t)\in\partial S\times[0,T)$ for the arbitrariness of $\widehat{T}$. □

Remark 2.1. In fact, by Theorem 2.2 and a similar proof in [19], if there exists x^* such that $u_0(x^*)>v_0(x^*)$, we have $U(x,t)>V(x,t)$ on $S\cup\{x\in\partial S|\mu(x)>0\}\times(0,T)$.

3 Quenching phenomena and quenching rate

In this section, we only consider the case of $p\geq q>0$. First of all, we discuss the case $\sigma=0$ (Neumann boundary condition)

Theorem 3.1. *When $\sigma=0$, the solution u of problem* (1.1) *satisfies*

$$\sum_{x\in S}u_t(x,t)=\beta\sum_{x\in S}u^{-q}(x,t)-\lambda\sum_{x\in S}u^{-p}(x,t).$$

Moreover, if p,q,λ,β and u_0 satisfies one of the following conditions

(i) $p>q$, $\Delta_\omega u_0+\beta u_0^{-q}-\lambda u_0^{-p}<0$ and $\max_{x\in S}u_0(x)<(\frac{\beta}{\lambda})^{\frac{1}{-p+q}}$,

(ii) $p=q$ and $\lambda>\beta$,

then u quenches in the finite time T.

Proof. Summing Eq. (1.1) over S, we have

$$\begin{aligned}\sum_{x\in S}u_t(x,t)&=\sum_{x\in S}\Delta_\omega u(x,t)+\beta\sum_{x\in S}u^{-q}(x,t)-\lambda\sum_{x\in S}u^{-p}(x,t)\\&=\sum_{x\in \bar{S}}\Delta_\omega u(x,t)-\sum_{x\in\partial S}\Delta_\omega u(x,t)+\beta\sum_{x\in S}u^{-q}(x,t)-\lambda\sum_{x\in S}u^{-p}(x,t)\\&=\beta\sum_{x\in S}u^{-q}(x,t)-\lambda\sum_{x\in S}u^{-p}(x,t).\end{aligned}\tag{3.1}$$

(i) By the conditions of u_0 and the maximum principle, we know that $u_t<0$, which leads to the fact that $u(x,t)$ decreases about t for any $x\in S$. Then from (3.1) we have

$$\begin{aligned}\sum_{x\in S}u_t(x,t)&=\sum_{x\in S}u^{-q}(x,t)[\beta-\lambda u^{-p+q}(x,t)]\\&\le\sum_{x\in S}u^{-q}(x,t)\{\beta-\lambda[\max_{x\in S}u(x,t)]^{-p+q}\}\\&\le -C_0\frac{1}{\left[\sum_{x\in S}u(x,t)\right]^q},\end{aligned}$$

where $C_0=-\beta+\lambda[\max_{x\in S}u_0(x)]^{-p+q}$. Solving the above differential inequality, we obtain

$$\sum_{x\in S}u(x,t)\le\left\{\left[\sum_{x\in S}u_0(x)\right]^{q+1}-C_0t(q+1)\right\}^{\frac{1}{q+1}},\tag{3.2}$$

which implies that u quenches in the finite time

$$T=\frac{\left[\sum_{x\in S}u_0(x)\right]^{q+1}}{(q+1)C_0}.$$

(ii) Since $p=q$, then from (3.1) we have

$$\sum_{x\in S}u_t(x,t)=-(\lambda-\beta)\sum_{x\in S}u^{-q}(x,t)\le-(\lambda-\beta)\frac{1}{\left[\sum_{x\in S}u(x,t)\right]^q}.$$

Therefore, we can easily get that

$$\sum_{x\in S} u(x,t) \le \left\{ \left[\sum_{x\in S} u_0(x) \right]^{q+1} - (\lambda-\beta)(q+1)t \right\}^{\frac{1}{q+1}}$$

which shows that the solution u to problem (1.1) quenches in finite time

$$T = \frac{\left[\sum_{x\in S} u_0(x) \right]^{q+1}}{(\lambda-\beta)(q+1)}. \qquad \square$$

Now let λ_1 be the eigenvalue of the eigenvalue problem

$$\begin{cases} -\Delta_\omega u(x) = \lambda u(x), & x\in S, \\ \mu(x)\dfrac{\partial u}{\partial n}(x) - \sigma(x)u(x) = 0, & x\in \partial S, \end{cases} \tag{3.3}$$

and ϕ_1 be the eigenfunction corresponding to λ_1. Then from [13] we know that $\lambda_1 > 0$. Next, we set

$$M = \max_{x\in S} \phi_1(x) > 0, \quad m = \min_{x\in S} \phi_1(x) > 0.$$

Theorem 3.2. *Suppose that $\sigma \neq 0$, $p = q$, then*

(i) when $\lambda > \beta$, the solution u of problem (1.1) *quenches in finite time T.*

(ii) when $\lambda \le \beta$, the solution u of problem (1.1) *exists globally.*

Proof. (i) From Theorem 2.1 we know that problem (1.1) has at least one solution $u \in C^1(0,T^*)\times C(\bar{S})$. Let T^* be small enough such that u doesn't quench before T^*, and $U(x,t) = f(t)\phi_1(x)$, where $f(t)$ satisfies

$$\begin{cases} f'(t) + (\lambda-\beta)M^{-p-1} f^{-p}(t) = 0, & t > T^*, \\ f(T^*) = \dfrac{\max_{x\in\bar{S}} u(x,T^*)}{m}. \end{cases} \tag{3.4}$$

By Remark 2.1 we know that $f(T^*) > 0$, and we can obtain that the solution of problem (3.4)

$$f(t) = \left[f^{p+1}(T^*) - (p+1)(\lambda-\beta)(t-T^*)M^{-p-1} \right]^{\frac{1}{p+1}}, \quad t > T^*,$$

which implies that $\lim_{t\to T} f(t) = 0$ with

$$T = T^* + \frac{1}{(p+1)(\lambda-\beta)M^{-p-1}} f^{p+1}(T^*).$$

Then for all $x \in \bar{S}$, we have

$$U(x,T^*) = f(T^*)\phi_1(x) = \frac{\max_{x\in\bar{S}} u(x,T^*)}{m}\phi_1(x) \ge \max_{x\in\bar{S}} u(x,T^*) \ge u(x,T^*)$$

for all $x\in\bar{S}$ and

$$\mu(x)\frac{\partial U}{\partial n}(x,t)-\sigma(x)U(x,t)=f(t)\left[\mu(x)\frac{\partial\phi_1}{\partial n}(x)-\sigma(x)\phi_1(x)\right]=0$$

for all $(x,t)\in\partial S\times[T^*,T)$. Moreover, we have

$$\begin{aligned}&U_t(x,t)-\Delta_\omega U(x,t)+(\lambda-\beta)U^{-p}\\=&\phi_1(x)f'(t)+\lambda_1\phi_1(x)f(t)+(\lambda-\beta){\phi_1}^{-p}(x)f^{-p}(t)\\\geq&\phi_1(x)\left[f'(t)+(\lambda-\beta)M^{-p-1}f^{-p}(t)\right]=0,\end{aligned}$$

for all $(x,t)\in S\times[t_0,T^*)$. So $U(x,t)$ is a supersolution of problem (1.1). Then due to the fact that $\lim_{t\to T}U(x,t)=\phi_1(x)\lim_{t\to T}f(t)=0$, the solution $u(x,t)$ of problem (1.1) quenches in finite time T.

(ii) When $\lambda\leq\beta$, let $V(x,t)=f(t)\phi_1(x)$, where $f(t)$ satisfies

$$\begin{cases}f'(t)=-\lambda_1 f(t)+(\beta-\lambda)M^{-p-1}f^{-p}(t),\\f(0)=\dfrac{\min_{x\in S\cup\{x\in\partial S|\mu(x)>0\}}u_0(x)}{M}.\end{cases}\tag{3.5}$$

By solving problem (3.5) we get

$$f(t)=\left\{e^{-(p+1)\lambda_1 t}\left[f^{p+1}(0)-\frac{(\beta-\lambda)M^{-p-1}}{\lambda_1}\right]+\frac{(\beta-\lambda)M^{-p-1}}{\lambda_1}\right\}^{\frac{1}{p+1}},$$

which means that f exists globally. Then for all $x\in\bar{S}$,

$$\begin{aligned}V(x,0)=f(0)\phi_1(x)=&\frac{\min_{x\in S\cup\{x\in\partial S|\mu(x)>0\}}u(x,0)}{M}\phi_1(x)\\\leq&\min_{x\in S\cup\{x\in\partial S|\mu(x)>0\}}u(x,0)\leq u(x,0),\end{aligned}$$

and for all $(x,t)\in\partial S\times[0,+\infty)$,

$$\mu(x)\frac{\partial V}{\partial n}(x,t)-\sigma(x)V(x,t)=f(t)\left[\mu(x)\frac{\partial\phi_1}{\partial n}(x)-\sigma(x)\phi_1(x)\right]=0.$$

Furthermore, we have

$$\begin{aligned}&V_t(x,t)-\Delta_\omega V(x,t)-(\beta-\lambda)V^{-p}\\=&\phi_1(x)f'(t)+\lambda_1\phi_1(x)f(t)-(\beta-\lambda){\phi_1}^{-p}(x)f^{-p}(t)\\\leq&\phi_1(x)\left[f'(t)+\lambda_1 f(t)-(\beta-\lambda)M^{-p-1}f^{-p}(t)\right]=0,\end{aligned}$$

for all $(x,t)\in S\times[0,+\infty)$. So $V(x,t)$ is a subsolution of problem (1.1) and the solution $u(x,t)$ of problem (1.1) exists globally consequently. □

Theorem 3.3. *Suppose that $\sigma\neq 0$, $p>q>0$, $1<\lambda<\beta+1$ and $d_\omega x=\sum_{y\in\bar{S}}\omega(x,y)>1$ for all $x\in S$, and let constant*

$$K=\min\left\{\left(\frac{\lambda-1}{\beta}\right)^{\frac{1}{p-q}},\left(\frac{1}{\max_{x\in S}d_\omega x}\right)^{\frac{1}{p}}\right\}.$$

Then under the conditions of

(i) $\sigma(x)>\mu(x)d_\omega x(1-K)$, $\forall x\in\partial S$,

(ii) $\max_{x\in S}u_0(x)\leq K<1$,

the solution u of problem (1.1) *quenches in finite time.*

Proof. Define

$$U(x,t)=\begin{cases}K, & x\in S,\\ 1, & x\in\partial S.\end{cases}$$

Then combining Definition 2.1 and condition (i), it is easy to prove that $U(x,t)$ is a supersolution of problem (1.1). From boundary condition we have

$$\mu(x)\sum_{y\in S}\omega(x,y)[u(x,t)-u(y,t)]=\sigma(x)u(x,t),\quad \forall x\in\partial S.$$

Because of the fact that $\mu(x)>0,\sigma(x)>0$ and $u_0(x)\geq 0$, there exists $y\in S$ such that $u(y,t)\leq u(x,t)$ for each $x\in\partial S$. So we assume $u(x,t)$ gets its minimum value at $x^*\in S$, and obtain

$$\begin{aligned}u_t(x^*,t)=&\Delta_\omega u(x^*,t)+\beta u^{-q}(x^*,t)-\lambda u^{-p}(x^*,t)\\ \leq&\sum_{y\in\bar{S}}[u(y,t)-u(x^*,t)]\omega(x^*,y)-u^{-p}(x^*,t)[\lambda-\beta K^{p-q}]\\ \leq& d_\omega x^*-d_\omega x^*u(x^*,t)-u^{-p}(x^*,t)\\ \leq&\max_{x\in S}d_\omega x-d_\omega x^*u(x^*,t)-K^{-p}\\ \leq&-d_\omega x^*u(x^*,t),\end{aligned}\tag{3.6}$$

by the definition of K. Integrating above inequality about t on $(0,t)$, we have

$$u(x^*,t)\leq u_0(x^*)e^{-d_\omega x^*t}\leq\max_{x\in S}u_0(x)e^{-d_\omega x^*t}\leq Ke^{-d_\omega x^*t}.\tag{3.7}$$

Hence, let $t_0=\frac{\ln(2K^p\max_{x\in S}d_\omega x)}{pd_\omega x^*}$, and then for any $t\geq t_0$, from inequalities (3.6) and (3.7), we have

$$\begin{aligned}u_t(x^*,t)\leq&\max_{x\in S}d_\omega x-\frac{1}{2}u^{-p}(x^*,t)-\frac{1}{2}u^{-p}(x^*,t)\\ \leq&\max_{x\in S}d_\omega x-\frac{1}{2}K^{-p}e^{pd_\omega x^*t}-\frac{1}{2}u^{-p}(x^*,t)\end{aligned}$$

$$\leq -\frac{1}{2}u^{-p}(x^*,t). \tag{3.8}$$

Integrating (3.8) on $[t_0,t]$, it is easy to obtain

$$u^{p+1}(x^*,t) \leq u^{p+1}(x^*,t_0) - \frac{1}{2}(p+1)(t-t_0).$$

From (3.7), the above inequality becomes

$$u^{p+1}(x^*,t) \leq u_0^{p+1}(x^*)e^{-(p+1)d_\omega x^* t_0} - \frac{1}{2}(p+1)(t-t_0).$$

So we get that $u(x,t)$ quenches in finite time T with

$$0 < T \leq t_0 + \frac{2}{p+1}u_0^{p+1}(x^*)e^{-d_\omega x^*(p+1)t_0}.$$

□

Theorem 3.4. *Suppose that $\sigma \neq 0$, $p = q > 0$, $\lambda > \beta$, and let $x^* \in S$ and $T > 0$ be the quenching point and the quenching time of u respectively, then we have*

$$\lim_{t\to T^-}(T-t)^{-\frac{1}{p+1}}u(x^*,t) = [(\lambda-\beta)(p+1)]^{\frac{1}{p+1}}.$$

Proof. Firstly, at the quenching point x^*, for any $t \in [0,T)$ we have

$$\begin{aligned} u_t(x^*,t) &= \Delta_\omega u(x^*,t) + \beta u^{-q}(x^*,t) - \lambda u^{-p}(x^*,t) \\ &\geq (\beta-\lambda)u^{-p}(x^*,t). \end{aligned} \tag{3.9}$$

Integrating above inequality with respect to t on (t,T), we obtain

$$u^{p+1}(x^*,t) \leq (\lambda-\beta)(p+1)(T-t). \tag{3.10}$$

Thus there is

$$\lim_{t\to T^-}(T-t)^{-\frac{1}{p+1}}u(x^*,t) \leq [(\lambda-\beta)(p+1)]^{\frac{1}{p+1}}. \tag{3.11}$$

On the other hand, from Definition 2.1 it can be shown that $U(x,t) = \left(\frac{\lambda-\beta}{\lambda-\beta+1}\right)^{\frac{1}{p}}$ is a super-solution to problem (1.1) for all $(x,t) \in \bar{S} \times [0,T)$. Then by (3.9) and (3.10) we get

$$\begin{aligned} u_t(x^*,t) &\leq d_\omega x^* \left(\frac{\lambda-\beta}{\lambda-\beta+1}\right)^{\frac{1}{p}} + (\beta-\lambda)u^{-p}(x^*,t) \\ &\leq u^{-p}(x^*,t)\{(\beta-\lambda) + C(T-t)^{\frac{p}{p+1}}\}, \end{aligned}$$

where

$$C = \max_{x\in S} d_\omega x \left(\frac{\lambda-\beta}{\lambda-\beta+1}\right)^{\frac{1}{p}} [(\lambda-\beta)(p+1)]^{\frac{p}{p+1}}.$$

Integrating above inequality with t on (t,T) again, we have

$$u(x^*,t) \geq \left\{ (\lambda-\beta)(p+1)(T-t) - C\frac{(p+1)^2}{2p+1}(T-t)^{\frac{2p+1}{p+1}} \right\}^{\frac{1}{p+1}}.$$

That is,

$$\begin{aligned}\lim_{t\to T^-}(T-t)^{-\frac{1}{p+1}}u(x^*,t) \geq & \lim_{t\to T^-}\left\{ (\lambda-\beta)(p+1) - C\frac{(p+1)^2}{2p+1}(T-t)^{\frac{p}{p+1}} \right\}^{\frac{1}{p+1}} \\ =&[(\lambda-\beta)(p+1)]^{\frac{1}{p+1}}. \end{aligned} \tag{3.12}$$

Then the conclusion is obtained from (3.11) and (3.12). □

Theorem 3.5. *Suppose that x^* is the quenching point and T is the quenching time of u, then under the conditions of Theorem* 3.3 *we have*

$$\lim_{t\to T^-}(T-t)^{-\frac{1}{p+1}}u(x^*,t) = [\lambda(p+1)]^{\frac{1}{p+1}}.$$

Proof. (i) Firstly, at the quenching point x^*, for any $t\in[0,T)$ we have

$$\begin{aligned} u_t(x^*,t) =& \Delta_\omega u(x^*,t) + \beta u^{-q}(x^*,t) - \lambda u^{-p}(x^*,t) \\ =& \sum_{y\in\bar{S}}[u(y,t)-u(x^*,t)]\omega(x^*,y) + \beta u^{-q}(x^*,t) - \lambda u^{-p}(x^*,t) \\ \geq & -\lambda u^{-p}(x^*,t). \end{aligned}$$

Integrating from t to T, we obtain

$$u^{p+1}(x^*,t) \leq [\lambda(p+1)(T-t)]. \tag{3.13}$$

(ii) Since of the fact that $U(x,t)$ is the super-solution of problem (1.1) and (3.13), where $U(x,t)$ is defined in Theorem 3.3, we get

$$\begin{aligned} u_t(x^*,t) \leq & d_\omega x^* + \beta u^{-q}(x^*,t) - \lambda u^{-p}(x^*,t) \\ \leq & u^{-p}(x^*,t)\left\{ -\lambda + \max_{x\in S} d_\omega x[\lambda(p+1)(T-t)]^{\frac{p}{p+1}} + \beta[\lambda(p+1)(T-t)]^{\frac{p-q}{p+1}} \right\}. \end{aligned}$$

Integrating above inequality over (t,T), we have

$$\begin{aligned} u^{p+1}(x^*,t) \geq & \lambda(p+1)(T-t) - \max_{x\in S} d_\omega x[\lambda(p+1)]^{\frac{p}{p+1}}\frac{(p+1)^2}{2p+1}(T-t)^{\frac{2p+1}{p+1}} \\ & - \beta[\lambda(p+1)]^{\frac{p-q}{p+1}}\frac{(p+1)^2}{2p-q+1}(T-t)^{\frac{2p-q+1}{p+1}}. \end{aligned}$$

Then combining above inequality with (3.13) and let $t\to T^-$, it is derived that

$$\lim_{t\to T^-}(T-t)^{-\frac{1}{p+1}}u(x^*,t) = [\lambda(p+1)]^{\frac{1}{p+1}}.$$

□

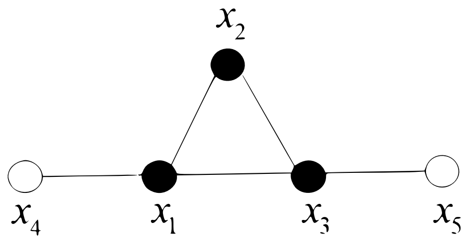

Figure 1: Graph G

4 Numerical experiments

In this section, we give some numerical experiments to illustrate our conclusions. Consider the graph G (as shown in Fig. 1) with interior nodes $S=\{x_1,x_2,x_3\}$ and boundary nodes $\partial S=\{x_4,x_5\}$, then the discrete form of problem (1.1) can be written as

$$\begin{aligned}
u_t(x_1,t)=&[u(x_2,t)-u(x_1,t)]\omega(x_1,x_2)+[u(x_3,t)-u(x_1,t)]\omega(x_1,x_3)\\
&+[u(x_4,t)-u(x_1,t)]\omega(x_1,x_4)+\beta u^{-q}(x_1,t)-\lambda u^{-p}(x_1,t),\\
u_t(x_2,t)=&[u(x_1,t)-u(x_2,t)]\omega(x_1,x_2)+[u(x_3,t)-u(x_2,t)]\omega(x_2,x_3)+\beta u^{-q}(x_2,t)\\
&-\lambda u^{-p}(x_2,t),\\
u_t(x_3,t)=&[u(x_1,t)-u(x_3,t)]\omega(x_1,x_3)+[u(x_2,t)-u(x_3,t)]\omega(x_2,x_3)\\
&+[u(x_5,t)-u(x_3,t)]\omega(x_3,x_5)+\beta u^{-q}(x_3,t)-\lambda u^{-p}(x_3,t).
\end{aligned}$$

Firstly, we consider the case $\sigma=0$ (Neumann boundary condition). By the boundary condition $B[u]=0$, we get

$$u(x_4,t)=u(x_1,t),\quad u(x_5,t)=u(x_3,t),\qquad t\geq 0.$$

Let $\omega(x_1,x_2)=0.25$, $\omega(x_1,x_3)=0.5$, $\omega(x_1,x_4)=0.35$, $\omega(x_2,x_3)=0.5$ and $\omega(x_3,x_5)=0.1$.

Example 4.1. ($\sigma=0$) Let $p=2,q=1,\lambda=\beta=1$ or $p=q=1,\lambda=4.247,\beta=1$, and the initial value $u_0(x_1)=u_0(x_4)=0.4167$, $u_0(x_2)=0.34$ and $u_0(x_3)=u_0(x_5)=0.5$. Figs. 2 and 3 show that the solutions of problem (1.1) quench in finite time, which is consistent with the conclusion of Theorem 3.1.

Next, we consider the case $\sigma\neq 0$, and we only discuss $\mu(x_4)=\mu(x_5)=1$, $\sigma(x_4)=\sigma(x_5)=0.31$.Taking $\omega(x_1,x_2)=0.6$, $\omega(x_1,x_3)=0.2$, $\omega(x_1,x_4)=1.1$, $\omega(x_2,x_3)=0.5$ and $\omega(x_3,x_5)=1.1$. Then from the boundary condition we have

$$u(x_4,t)=\frac{1.1}{0.79}u(x_1,t),\qquad u(x_5,t)=\frac{1.1}{0.79}u(x_3,t),\quad t\geq 0.$$

Example 4.2. ($\sigma\neq 0$) Let $p=q=2$, $\lambda=2.2$, $\beta=1$, and the initial value $u_0(x_1)=0.5$, $u_0(x_2)=0.4$, $u_0(x_3)=0.45$, $u_0(x_4)=\frac{1.1}{0.79}u_0(x_1)$ and $u_0(x_5)=\frac{1.1}{0.79}u_0(x_3)$. The numerical

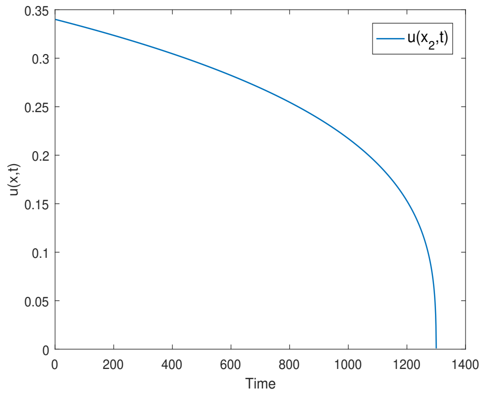

Figure 2: The quenching phenomenon when $p>q$

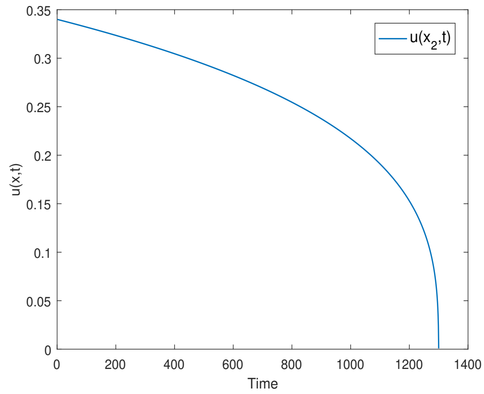

Figure 3: The quenching phenomenon when $p=q$

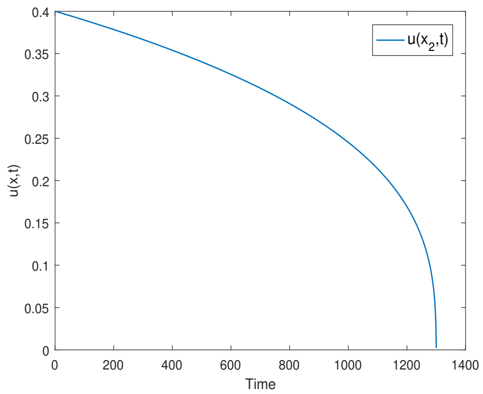

Figure 4: Quenching of $u(x_2,t)$ when $p=q$

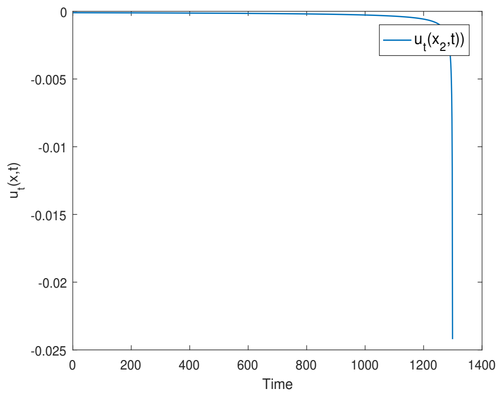

Figure 5: Blow-up of $u_t(x_2,t)$ when $p=q$

experiment results are shown in Figs. 4 and 5. From these numerical experiments, we know that the solution u quenches and u_t blows up in finite time, which illustrate the result of Theorem 3.2.

Example 4.3. ($\sigma\neq 0$) Let $p=2$, $q=1$, $\lambda=1.5$, $\beta=0.69$, the initial conditions $u_0(x_1)=0.45$, $u_0(x_2)=0.409$, $u_0(x_3)=0.5815$, $u_0(x_4)=\frac{1.1}{0.79}u_0(x_1)$ and $u_0(x_5)=\frac{1.1}{0.79}u_0(x_3)$. Figs. 6 and 7 illustrate the Theorem 3.3. That is to say, the solution u quenches and u_t blows up in finite time.

Example 4.4. ($\sigma\neq 0$) Let $p=2$, $q=1$, $\lambda=1.82968$, $\beta=1$, the initial conditions $u_0(x_1)=0.5$, $u_0(x_2)=0.43$, $u_0(x_3)=0.7$, $u_0(x_4)=\frac{1.1}{0.79}u_0(x_1)$ and $u_0(x_5)=\frac{1.1}{0.79}u_0(x_3)$. Figs. 8 and 9 show that the solution u quenches and u_t blows up in finite time, which exploit the result of Theorem 3.3.

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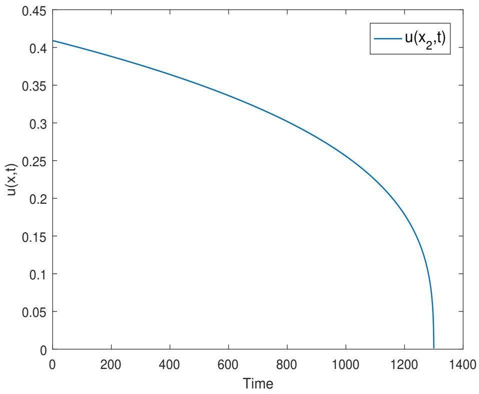

Figure 6: Quenching of $u(x_2,t)$ when $p>q$

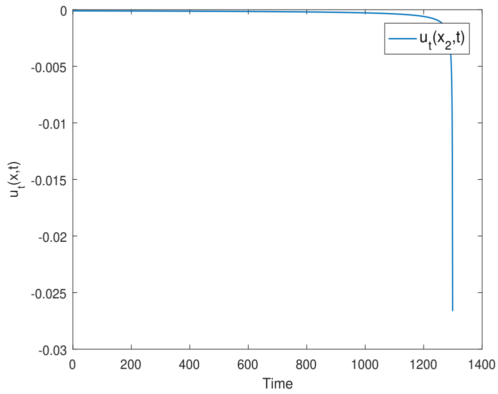

Figure 7: Blowup of $u_t(x_2,t)$ when $p>q$

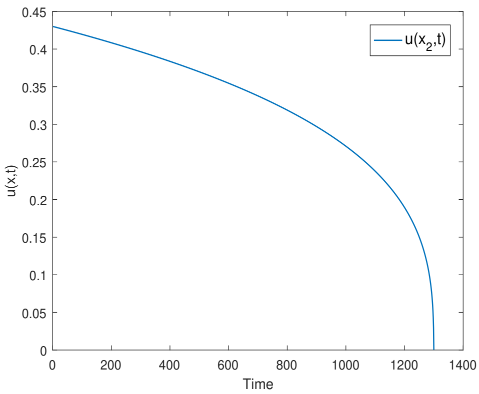

Figure 8: Quenching of $u(x_2,t)$ when when $p>q$

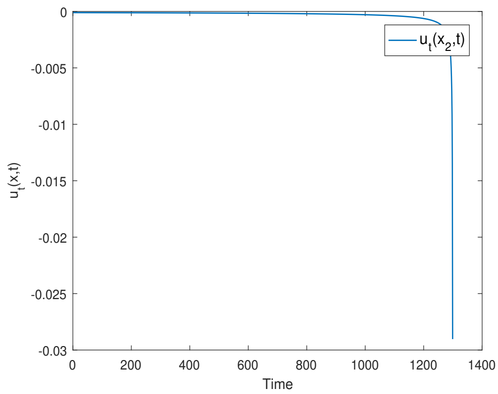

Figure 9: Blow-up of $u_t(x_2,t)$ when when $p>q$

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