JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **37**, No. 1, pp. 25-46

doi: 10.4208/jpde.v37.n1.2
March 2024

Robustness of Pullback Attractors for 2D Incompressible Navier-Stokes Equations with Delay

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Received 25 May 2022; Accepted 19 April 2023

Abstract. This paper is concerned with the pullback dynamics and robustness for the 2D incompressible Navier-Stokes equations with delay on the convective term in bounded domain. Under appropriate assumption on the delay term, we establish the existence of pullback attractors for the fluid flow model, which is dependent on the past state. Inspired by the idea in Zelati and Gal's paper (JMFM, 2015), the robustness of pullback attractors has been proved via upper semi-continuity in last section.

AMS Subject Classifications: 35B40, 35B41, 35Q30, 76D03, 76D05

Chinese Library Classifications: O175.29

Key Words: Navier-Stokes equations; pullback attractors; delay; upper semi-continuity.

1 Introduction

This presented is concerned with the pullback dynamics and robustness for the 2D Navier-Stokes equations with delay, which can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u(t-\rho(t))\cdot\nabla)u + \nabla p = f(t), & (t,x)\in(\tau,+\infty)\times\Omega, \\ \mathrm{div} u = 0, & (t,x)\in(\tau,+\infty)\times\Omega, \\ u(t,x) = 0, & (t,x)\in(\tau,+\infty)\times\partial\Omega, \\ u(\tau+\theta,x) = \phi(\theta), & (\theta,x)\in[-h,0]\times\Omega, \end{cases} \tag{1.1}$$

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where the kinematic viscosity $\nu > 0$, p is the unknown pressure, the external force term is $f(t)$, the delay function $\rho(t) \in C^1(\mathbb{R};[0,h])$ with

$$0 < \rho'(t) \le M < 1 \text{ for all } t \in \mathbb{R},$$

$h > 0$ is a constant, $u_t(s) = u(t+s)$, $s \in [-h,0]$, $\phi(\theta)$ is the initial datum in $[-h,0]$, and $u(\tau,x) = \phi(0)$.

The research on dynamic systems for the two dimensional incompressible Navier-Stokes equations has attracted mathematician's attention in 1980s, which contains the existence of attractors and its geometric structure on different domains, see [1–3] and the literatures therein. Delay effect can be found in many aspects, such as biology, economic and so on, which can lead to the instability of system, even if the delay is very small. Since the motion of fluid flow is not only dependent on current state, but also the past history such as delay and memory, which leads to the research on incompressible functional Navier-Stokes equations and some extended models. In 1963, Krasovskii [4] first noticed the system with delay, constructed the Navier-Stokes equations with delay and obtained the well-posedness of system. In past decades, there were many literatures about the hydrodynamic system with delay, especially the Navier-Stokes equations with constant, variable and distributed delays, which can be referred to, Taniguchi [5], Hale [6], Caraballo and Real [7], Garcín-Luengo, Marín-Rubio and Planas [8–10] and more literatures therein about the fluid flow with delays.

The hydrodynamic system with perturbation is a key research point in last thirty decades, which includes the convergence of attractors as perturbation vanishes, i.e., the robustness of attractors via upper and lower semi-continuity, see the theory and application in Chapter III of Carvalho, Langa and Robinson [11]. However, the lower semi-continuity is very difficult to verify since the lack of good regularity, which leads to the validity of upper semi-continuity for attractors as a tool to understand turbulence, see [11–13]. In 2009, another interesting method was given by Wang [14] to obtain the upper semi-continuity of random attractors, and Wang [15] for the pullback attractors. For more relating results to the convergence of attractors and solutions, we can refer to [16–18] and so on.

To our best knowledge, there are fruitful results on the dynamics for Navier-Stokes equations and related models with delay, which illustrated the complexity of fluid flow. However, the research on robustness of attractors for incompressible Navier-Stokes equations with delay on convective term, i.e., the convergence of pullback attractors as delay vanishes is still open, which is our main goal in this paper. The main features of this paper can be summarized as follows:

(1) In Section 2, some functional spaces and related conclusions on pullback $\mathcal{D}$-attractors are given. The definition of upper semicontinuity of attractors for system with delay term is presented, which is inspired by [17];

(2) In Section 3, we use the standard Galerkin method and conclusions on compactness to derive the wellposedness of solutions, and determine a continuous semi-flow

in space $M_H = H\times(C_H\cap L^2_V)$. In addition, based on the basic conclusions, the pullback $\mathcal{D}$-attractor $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ is derived after the existence of pullback $\mathcal{D}$-absorbing set $\mathcal{B}=\{B(t)\}_{t\in\mathbb{R}}$ for the semi-flow in Section 5;

(3) In Section 6, we present two lemmas to show that the pullback attractors $\mathcal{A}^h = \{A^h(t)\}_{t\in\mathbb{R}}$ in M_H and $\mathcal{A}^0=\{A^0(t)\}_{t\in\mathbb{R}}$ in H have the property of upper semi-continuity as $h\to 0$, i.e., the robustness of attractors has been shown.

2 Some preliminaries

2.1 Some functional settings

Let be $E:=\{u;\ u\in(C_0^\infty(\Omega))^2,\ \mathrm{div}u=0\}$, H is the closure of E in $(L^2(\Omega))^2$ topology, with norm $\|\cdot\|_H=|\cdot|$ and inner product $(\cdot,\cdot)$, where

$$(u,v)=\sum_{j=1}^{2}\int_\Omega u_j(x)v_j(x)\mathrm{d}x,\quad \forall\, u,v\in(L^2(\Omega))^2.$$

V is the closure of E in $(H^1(\Omega))^2$ topology, with norm $\|\cdot\|_V=\|\cdot\|$ and inner product $((\cdot,\cdot))$, where

$$((u,v))=\sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\mathrm{d}x,\quad \forall\, u,v\in(H_0^1(\Omega))^2,$$

and its dual space is written as V' with norm $\|\cdot\|_*$. Also, $V\hookrightarrow\hookrightarrow H\hookrightarrow V'$, and there hold

$$<u,v>_{V'\times V}=(u,v)_H,\quad \forall\, u\in H\subset V',\ v\in V.$$

$A:=-P_L\Delta$ is the Stokes operator with the domain $(H^2(\Omega))^2\cap V$, P_L is the Helmholtz-Leray projection in $(L^2(\Omega))^2$, and

$$P_L:\ H\oplus G(\Omega)\to H,$$

where

$$G(\Omega)=H^\perp:=\{u\in(L^2(\Omega))^2;\ \exists\,\varsigma\in(L^2_{loc}(\Omega))^2:\ u=\nabla\varsigma\}.$$

Especially, $<Au,v>=((u,v))$. The normalized eigenfunctions $\{\omega_j\}_{j=1}^\infty$ of A are defined on the Hilbert basis of H, which possesses eigenvalues as $\{\lambda_j\}_{j=1}^\infty(0<\lambda_1\le\cdots\le\lambda_j\le\cdots,\ \lambda_j\to+\infty\ as\ j\to\infty)$. The fractional power operator A^s is defined as

$$A^s u=\sum_j \lambda_j^s a_j\omega_j,\quad s\in\mathbb{R},\ j\in\mathbb{Z}^+,\qquad u=\sum_j a_j\omega_j,\ a_j=(u,\omega_j),$$

with the domain $D(A^s)=\{u;\ A^s u\in H\}$, and we still use $D(A^s)$ to denote $\overline{E}^{D(A^s)}$ whose norm is written as

$$\|u\|_{2s}^2=|A^s u|^2=\sum_j \lambda_j^{2s}|a_j|^2.$$

Also,

$$V=D(A^{\frac{1}{2}})=\{u;\ A^{\frac{1}{2}}u\in H\}=\{u=\sum_j a_j\omega_j;\ \sum_j \lambda_j|a_j|^2<+\infty\},$$

and

$$\|u\|_1=|A^{\frac{1}{2}}u|=\|u\|,\quad \text{for any } u\in V.$$

Given a Banach space X, let $C_X=C([-h,0];X)$ with norm

$$\|u(\theta)\|_{C_X}=\sup_{\theta\in[-h,0]}\|u(\theta)\|_X,$$

L_X^2 is Lebesgue space in the delayed interval $[-h,0]$, and the product space $M_H=H\times(C_H\cap L_V^2)$ with norm

$$\|(u(t),u_t)\|_{M_H}^2=|u(t)|^2+\|u_t\|_{C_H}^2+\|u_t\|_{L_V^2}^2.$$

Let P_1 and P_2 be projections from M_H to H and $C_H\cap L_V^2$, that is

$$P_1M_H=H,\quad P_2M_H=C_H\cap L_V^2.$$

And we define the bilinear and trilinear operators $B(\cdot,\cdot)$, $b(\cdot,\cdot,\cdot)$ given by

$$B(u,v):=P_L((u\cdot\nabla)v),\quad \forall\ u,v\in E,$$
$$b(u,v,w)=\sum_{i,j=1}^2\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j\mathrm{d}x=(B(u,v),w)$$

respectively, where $B(u,v)$ is a linear continuous operator from V to V', and for all $u,v,w\in V$ there hold

$$\begin{cases} b(u,v,v)=0,\\ b(u,v,w)=-b(u,w,v),\\ |b(u,v,w)|\le C|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\|\|w\|^{\frac{1}{2}}|w|^{\frac{1}{2}}. \end{cases}\tag{2.1}$$

Lemma 2.1. (The Lions-Aubin Lemma [19]) *Let X,Y and Z be three Banach spaces, X and Y are reflexive, $X\hookrightarrow\hookrightarrow Z\hookrightarrow Y$, $1<p_0,p_1<\infty$, and $T>\tau$ is a fixed constant. If a bounded sequence $\{u_n\}$ satisfies that $u_n\in L^{p_0}(\tau,T;X)$ and $\frac{\partial u_n}{\partial t}\in L^{p_1}(\tau,T;Y)$, then the sequence $\{u_n\}$ is precompact in $L^{p_0}(\tau,T;Z)$.*

Lemma 2.2. (The Lions-Magenes Theorem [19]) *For $1<p<+\infty$, $1/p+1/p'=1$, $u\in L^p(\tau,T;V)$, and $\frac{\partial}{\partial t}u\in L^{p'}(\tau,T;V')$, then $u\in C([\tau,T];H)$ (if necessary, some function values could be changed in some set of zero measure in $[\tau,T]$).*

2.2 Some conclusions about tempered pullback dynamic system

Before analyzing our model, we recall briefly some interesting results on pullback $\mathcal{D}$-attractors, which can be found in [11, 15, 20].

Definition 2.1. *Let $\mathcal{P}(X)$ be the family of all nonempty subsets in X, and $\mathcal{D}$ is a nonempty class of families $\{D(t)\}_{t\in\mathbb{R}}\subset\mathcal{P}(X)$. Then $\mathcal{D}$ is called a universe in $\mathcal{P}(X)$.*

Definition 2.2. *Let be a metric space (X,d), $\mathbb{R}^2_d=\{(t,\tau)\in\mathbb{R}^2|\tau\le t\}$, and a family of two-parameter mappings $\{U(\cdot,\cdot)\}$: $\mathbb{R}^2_d\times X\to X$ is called process in X if*
(i) $U(\tau,\tau)=Id$;
(ii) $U(t,l)U(l,\tau)=U(t,\tau)$, $\forall\tau\le l\le t$.

Definition 2.3. *Let X be a Banach space, the process $\{U(t,\tau)|t\ge\tau\}$ is said to be continuous in X if the mapping $U(t,\tau):X\to X$ is continuous.*

Definition 2.4. *The family of subsets $\mathcal{D}_0=\{D_0(t)\}_{t\in\mathbb{R}}$ is said to be pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}$ in X if for any $\{D(t)\}_{t\in\mathbb{R}}\in\mathcal{D}$ and any $t\in\mathbb{R}$, there exists $\tau(t,D)\le t$ such that $U(t,\tau)D(\tau)\subset D_0(t)$ for all $\tau\le\tau(t,D)$.*

Definition 2.5. *A process $\{U(t,\tau)\}$ in X is said to be pullback $\mathcal{D}$-asymptotically compact if for any $t\in\mathbb{R}$, any sequence $\{\tau_n\}\subset(-\infty,t]$ with $\tau_n\to-\infty$ as $n\to+\infty$, and $\{u_n\}\subset D(\tau_n)$, there holds that $\{U(t,\tau_n)u_n\}$ is relatively compact in X.*

Definition 2.6. *A family of compact subsets $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ is called the pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}$ in X, if for any $\tau\le t\in\mathbb{R}$,*
(i) $\mathcal{A}$ is invariant ($U(t,\tau)A(\tau)=A(t)$);
(ii) $\mathcal{A}$ is pullback $\mathcal{D}$-attracting, i.e.,

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),A(t))=0.$$

Theorem 2.1. (See [8]) *Let $\{U(t,\tau)\}$ be a continuous process in a Banach space X, and $\mathcal{D}$ is a universe in $\mathcal{P}(X)$. A family of subsets $\mathcal{D}_0=\{D_0(t)|t\in\mathbb{R}\}$ is pullback $\mathcal{D}$-absorbing for $\{U(t,\tau)\}$, and $\{U(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in X. Then the process $\{U(t,\tau)\}$ possesses the pullback $\mathcal{D}$-attractors $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ in X, where*

$$\mathcal{A}=\Lambda(\mathcal{D},t)=\bigcap_{s\le t}\left(\overline{\bigcup_{\tau\le s}U(t,\tau)D(\tau)}\right).$$

Remark. For a family of closed sets $\mathcal{C}=\{C(t)\}_{t\in\mathbb{R}}$ which satisfies that

$$\lim_{\tau\to-\infty}\mathrm{dist}_X(U(t,\tau)D(\tau),C(t))=0,$$

if there always holds that $\mathcal{A}\subset C(t)$, then it is said that the pullback $\mathcal{D}$-attractors $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ is minimal.

If $\mathcal{A}\subset\mathcal{D}$, then it is said that the pullback $\mathcal{D}$-attractors $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ is unique. And the sufficient condition for $\mathcal{A}\subset\mathcal{D}$ is that $\mathcal{D}_0\in\mathcal{D}$, $D(t)$ is closed for any $t\in\mathbb{R}$, and $\mathcal{D}$ is inclusion-closed.

To study the relationship among the pullback attractors, we provide the following definition of upper semi-continuity and related conclusion, see [11].

Definition 2.7. *Suppose that X and Λ are metric spaces, and $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ is a family of subsets of X. Then it is said that the family of subsets $\{\mathcal{A}_\lambda\}$ has the property of upper semicontinuity as $\lambda\to\lambda_0$ in X if*

$$\lim_{\lambda\to\lambda_0}\mathrm{dist}_X(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_0})=0.$$

Theorem 2.2. *Suppose that, $\lambda\in[\lambda_0,\lambda_0+1)$, $\{S_\lambda(\cdot,\cdot)\}$ is a family of processes such that*
(i) $S_\lambda(\cdot,\cdot)$ has a pullback attractor $\mathcal{A}_\lambda(\cdot)$ for all $\lambda\in[\lambda_0,\lambda_0+1)$;
(ii) For any $t\in\mathbb{R}$, any $T\geq 0$, and any bounded set $D\subset X$,

$$\sup_{s\in[0,T],\ u_0\in D} d(S_\lambda(t+s,t)u_0,S_{\lambda_0}(t+s,t)u_0)\to 0 \text{ as } \lambda\to\lambda_0;$$

(iii) There exist $\delta>0$ and $t_0\in\mathbb{R}$ such that

$$\bigcup_{\lambda\in(\lambda_0,\lambda_0+\delta)}\bigcup_{s\leq t_0}\mathcal{A}_\lambda(s)$$

is bounded. Then the pullback attractors have the property of upper semicontinuity as $\lambda\to\lambda_0$: for each $t\in\mathbb{R}$,

$$\lim_{\lambda\to\lambda_0}\mathrm{dist}(\mathcal{A}_\lambda,\mathcal{A}_{\lambda_0})=0.$$

Remark. Considering the incompressible Navier-Stokes equations, the nonlinearity of convective term leads to that we can not use the above method to study the upper semi-continuity of pullback attractors directly, and thus the technique in Zelati and Gal's paper [18] has been introduced and applied to achieve our goal.

3 Wellposedness

3.1 Abstract form and weak solution

The problems (1.1) can be written as the abstract equivalent form

$$\begin{cases}\dfrac{\partial}{\partial t}u+\nu Au+B(u(t-\rho(t)),u)=P_L f(t),\\ u(\tau+\theta,x)=\phi(\theta),\qquad(\theta,x)\in[-h,0]\times\Omega.\end{cases}\tag{3.1}$$

Definition 3.1. *A solution $u\colon [\tau,\infty)\to H$ is called the weak solution to* (3.1) *if there hold*
(i) $u\in C([\tau-h,T];H)\cap L^2(\tau,T;V)$, $\frac{\partial u}{\partial t}\in L^2(\tau,T;V')$;
(ii) $u(\tau+\theta,x)=\phi(\theta)$, $\theta\in[-h,0]$;
(iii) for any $v\in V$ and $s,t\in[\tau,T)$ with $s\le t$, there holds

$$\begin{aligned}&(u(t),v)+\nu\int_s^t <Au(l),v>\mathrm{d}l+\int_s^t <B(u(l-\rho(l),u(l)),v>\mathrm{d}l\\ =&(u(s),v)+\int_s^t (P_Lf,v)\mathrm{d}l;\end{aligned}\tag{3.2}$$

(iv) for any $T\ge\tau$, the energy inequality holds in the distribution sense that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|^2+\nu\|u(t)\|^2\le(f,u(t)).\tag{3.3}$$

Also, the inequality (3.3) can be understood in the way that there holds for any positive test function $\psi\in C_0^\infty[\tau,T]$ that

$$-\frac{1}{2}\int_\tau^T |u(t)|^2\psi'(t)\mathrm{d}t+\nu\int_\tau^T \|u(t)\|^2\psi(t)\mathrm{d}t\le\int_\tau^T (f,u(t))\psi(t)\mathrm{d}t.\tag{3.4}$$

3.2 Existence, uniqueness and continuity of solutions

To study the well-posedness of solutions, we first assume that

$$\int_{-\infty}^t e^{rs}|f|^2\mathrm{d}s,\quad 0<r<\nu.$$

Theorem 3.1. *Let $(u(\tau),\phi)\in M_H$, then there exists a unique weak solution $u(t;\tau,\phi(\theta))$ to system* (3.1) *on $[\tau-h,T]$.*

Proof. We first use the standard Faedo-Galerkin method to establish the existence of solution to (3.1).

Step 1: Approximation solution. Consider an orthonormal basis $\{e_j\}_{j=1}^\infty$ of $(L^2(\Omega))^2$ which is dense in space V. Fix $m\ge1$, write $X_m=\mathrm{span}\{e_1,\cdots,e_m\}$, and define

$$P_mu=\sum_{j=1}^m(u,e_j)e_j,\ \forall\ u\in(L^2(\Omega))^2.$$

We also assume that $u_m(t)=\sum_{j=1}^m a_{mj}(t)e_j$ satisfies for a.e. $t>\tau$ that

$$\begin{cases}(u_m(t),e_j)+\nu\int_\tau^t <Au_m,e_j>\mathrm{d}s+\int_\tau^t b(u_m(s-\rho(s)),u_m,e_j)\mathrm{d}s\\ \qquad=(u_m(\tau),e_j)+\int_\tau^t (P_mP_Lf(s),e_j)\mathrm{d}s,\\ u_m(\tau+\theta,x)=\phi_m(\theta),\quad(\theta,x)\in[-h,0]\times\Omega,\end{cases}\tag{3.5}$$

where

$$u_m(\tau+\theta)=\phi_m(\theta)=\sum_{k=1}^{m} b_{mj}e_j \to \phi(\theta) \text{ in } H,$$

and

$$P_m P_L f(t) \to P_L f(t) \text{ in } H \text{ as } m \to +\infty.$$

From the basic conclusion of ordinary differential equations [21] we know that the system (3.5) has a unique solution on some local interval $[\tau-h,t_m)$, where $t_m \in (\tau,\infty)$.

Step 2: Priori estimate. Letting $\tau<T<t_m$, from (3.5) we obtain that

$$\frac{\partial}{\partial t}|u_m|^2+2\nu\|u_m\|^2 \leq 2|u_m||f| \leq \nu\lambda_1|u_m|^2+\frac{1}{\nu\lambda_1}|f|^2, \quad \text{a.e. } t\in[\tau,T],$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u_m|^2+\nu\|u_m\|^2 \leq \frac{1}{\nu\lambda_1}|f|^2. \tag{3.6}$$

Using Gronwall's inequality, we obtain

$$|u_m(t)|^2 \leq e^{-\nu\lambda_1(t-\tau)}|u(\tau)|^2+\frac{1}{\nu^2\lambda_1^2}|f|^2, \tag{3.7}$$

$$\nu\int_s^t \|u_m(k)\|\mathrm{d}k \leq |u_m(s)|^2+\frac{1}{\nu\lambda_1}\int_s^t |f|^2\mathrm{d}k, \tag{3.8}$$

it follows from (3.7) and (3.8) that $t_m=\infty$ for all m, and

$$u_m \in L^\infty(\tau-h,T;H)\cap L^2(\tau-h,T;V).$$

Also, if we integrate (3.6) from t to $t+\theta$, $\theta\in[-h,0]$, from the fact that $\phi\in C_H$ we have

$$\|(u_m)_t\|_{C_H}^2 \leq \|\phi(\theta)\|_{C_H}^2+\frac{1}{\nu\lambda_1}\int_\tau^t |f|^2\mathrm{d}s.$$

For any $v\in V$, there holds

$$\begin{aligned}&|<B(u_m(t-\rho(t)),u_m),v>|\\ =&\int_\Omega u_m(t-\rho(t))\nabla v u_m \mathrm{d}x \leq C\|u_m(t-\rho(t))\|_{L^\infty}|\nabla v||u_m|,\end{aligned}$$

then

$$\|B(u_m(t-\rho(t)),u_m(t)\|_{V'} \leq C\|u_m(t-\rho(t))\|_{L^\infty}|u_m| \leq C|u_m|,$$

it follows that

$$B(u_m(t-\rho(t)),u_m)\in L^2(\tau,T;V').$$

Since $Au_m \in L^2(\tau-h,T;V')$, and

$$\frac{\partial}{\partial t}u_m = -\nu Au_m - B(u_m(t-\rho(t)),u_m) + f(t),$$

we have $\frac{\partial}{\partial t}u_m \in L^2(\tau,T;V')$.

Step 3: Limit procedure. From the Lions-Aubin lemma we obtain that there exist functions $u \in L^\infty(\tau-h,T;H)\cap L^2(\tau-h,T;V)$, $u_1 \in L^2(\tau,T;V')$, and $u_2 \in L^2(\tau,T;V')$ such that

$$\begin{cases} u_m \rightharpoonup u, & \text{weakly star in } L^\infty(\tau-h,T;H); \\ u_m \rightharpoonup u, & \text{weakly in } L^2(\tau-h,T;V); \\ Au_m \rightharpoonup u_1, & \text{weakly in } L^2(\tau-h,T;V'); \\ \frac{\partial}{\partial t}u_m \rightharpoonup u_2, & \text{weakly in } L^2(\tau,T;V'); \\ u_m \to u, & \text{strongly in } L^2(\tau,T;H); \\ u_m(s) \to u(s) & \text{in } \Omega_T \text{ (a.e.)}, \end{cases} \tag{3.9}$$

and from the compactness results we can obtain $u_1 = Au$, $u_2 = \frac{\partial}{\partial t}u$. Also, the Lions-Magenes Theorem can lead to that $u_m \in C([\tau,T];H)$.

Also, from (3.9) and the Sobolev interpolation inequality we can obtain that

$$\|u_m(t)\|_{L^4(\Omega)} \le C\|u_m(t)\|^{\frac{1}{2}} \cdot |u_m(t)|^{\frac{1}{2}} \le C\|u_m(t)\|^{\frac{1}{2}},$$

and $u_m(t) \in L^4(\tau-h,T;L^4(\Omega))$, it follows that $u_m(t-\rho(t))u_m(t)$ is bounded in $L^2(\tau,T;H)$, which means there exists a subsequence (also written $\{u_m\}$) such that

$$u_m(t-\rho(t))u_m(t) \rightharpoonup u(t-\rho(t))u(t) \text{ weakly in } L^2(\tau,T;H).$$

In addition, $u_m(t-\rho(t))u_m(t) \rightharpoonup u(t-\rho(t))u(t)$ weakly in $L^2(\Omega_T)$, and for any $v \in V$ there holds

$$\begin{aligned} &\int_\tau^T b(u_m(t-\rho(t)),u_m,v)\mathrm{d}t = \int_\tau^T \sum_{i,j=1}^2 \int_\Omega u_{mi}(t-\rho(t))D_i u_{mj} v_j \mathrm{d}x\mathrm{d}t \\ =& -\sum_{i,j=1}^2 \int_\tau^T \int_\Omega u_{mi}(t-\rho(t))D_i v_j u_{mj} \mathrm{d}x\mathrm{d}t \to -\sum_{i,j=1}^2 \int_\tau^T \int_\Omega u_i(t-\rho(t))D_i v_j u_j \mathrm{d}x\mathrm{d}t \\ =& \int_\tau^T b(u_(t-\rho(t)),u,v)\mathrm{d}t. \end{aligned} \tag{3.10}$$

From (3.9), we can also obtain that for any $v \in V$,

$$(u_m(t),v) + \nu\int_\tau^T < Au_m(t),v > \mathrm{d}t \to (u(t),v) + \nu\int_\tau^T < Au(t),v > \mathrm{d}t, \tag{3.11}$$

and passing to the limit in (3.5)can lead to that u satisfies (3.2) and (3.3).

Step 4: Continuous dependence on initial data. Suppose that u and v are two solutions to (3.1) with initial data ϕ_1 and ϕ_2 in $[-h,0]\times\Omega$ respectively, we denote $w(t)=u(t)-v(t)$, and there holds

$$\frac{\partial w}{\partial t}+\nu Aw+B(u(t-\rho(t)),u)-B(v(t-\rho(t)),v)=0.$$

Noting that

$$B(u(t-\rho(t)),u)-B(v(t-\rho(t)),v)=B(w(t-\rho(t)),u)+B(v(t-\rho(t)),w),$$

we get

$$\frac{\partial w}{\partial t}+\nu Aw+B(w(t-\rho(t)),u)+B(v(t-\rho(t)),w)=0. \tag{3.12}$$

Multiplying (3.12) by w, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|w|^2+2\nu\|w\|^2\le 2|b(w(t-\rho(t)),u,w)|\\
\le& C|w(t-\rho(t))|^{\frac{1}{2}}\|w(t-\rho(t))\|^{\frac{1}{2}}\|u\|\|w\|^{\frac{1}{2}}|w|^{\frac{1}{2}}\\
\le& C\operatorname*{esssup}_{r\in[t-h,t]}|w(r)|\|u\|\|w(t-\rho(t))\|^{\frac{1}{2}}\|w(t)\|^{\frac{1}{2}}\\
\le& \frac{C}{\varepsilon}\operatorname*{esssup}_{r\in[t-h,t]}|w(r)|^2\|u\|^2+\varepsilon\|w(t-\rho(t))\|\|w(t)\|\\
\le& \frac{C}{\varepsilon}\operatorname*{esssup}_{r\in[t-h,t]}|w(r)|^2\|u\|^2+\frac{\varepsilon^2}{2\nu}\|w(t-\rho(t))\|^2+\frac{\nu}{2}\|w(t)\|^2,
\end{aligned}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2+\frac{3\nu}{2}\|w\|^2\le\frac{C}{\varepsilon}\operatorname*{esssup}_{r\in[t-h,t]}|w(r)|^2\|u\|^2+\frac{\varepsilon^2}{2\nu}\|w(t-\rho(t))\|^2. \tag{3.13}$$

Choosing $\varepsilon=\nu\sqrt{1-M}$ and integrating (3.13) over $[\tau,t]$, we get

$$\begin{aligned}
&|w(t)|^2-|w(\tau)|^2+\frac{3\nu}{2}\int_\tau^t\|w\|^2\mathrm{d}s\\
\le&\frac{C}{\varepsilon}\int_\tau^t\operatorname*{esssup}_{r\in[s-h,s]}|w(r)|^2\|u(s)\|^2\mathrm{d}s+\frac{\varepsilon^2}{2\nu}\int_\tau^t\|w(s-\rho(s))\|^2\mathrm{d}s\\
\le&\frac{C}{\varepsilon}\int_\tau^t\operatorname*{esssup}_{r\in[s-h,s]}|w(r)|^2\|u(s)\|^2\mathrm{d}s+\frac{\varepsilon^2}{2\nu(1-M)}\int_{\tau-h}^t\|w(s)\|^2\mathrm{d}s
\end{aligned}$$

$$\leq \frac{C}{\nu\sqrt{1-M}}\int_\tau^t \operatorname*{esssup}_{r\in[s-h,s]}|w(r)|^2\|u(s)\|^2\mathrm{d}s+\frac{\nu}{2}\int_\tau^t\|w(s)\|^2\mathrm{d}s+\frac{\nu}{2}\int_{\tau-h}^{\tau}\|w(s)\|^2\mathrm{d}s \tag{3.14}$$

i.e.,

$$\begin{aligned}&|w(t)|^2+\nu\int_\tau^t\|w\|^2\mathrm{d}s \\ \leq&|w(\tau)|^2+\frac{\nu}{2}\int_{\tau-h}^{\tau}\|w(s)\|^2\mathrm{d}s+\frac{C}{\nu\sqrt{1-M}}\int_\tau^t\operatorname*{esssup}_{r\in[s-h,s]}|w(r)|^2(1+\|u(s)\|^2)\mathrm{d}s,\end{aligned} \tag{3.15}$$

and

$$\operatorname*{esssup}_{r\in[t-h,t]}|w(r)|^2\leq\|w_\tau\|^2_{C_H}+\frac{\nu}{2}\|w_\tau\|^2_{L^2_V}+C\int_\tau^t\operatorname*{esssup}_{r\in[s-h,s]}|w(r)|^2(1+\|u(s)\|^2)\mathrm{d}s.$$

By the Gronwall inequality, we conclude that

$$\operatorname*{esssup}_{r\in[t-h,t]}|w(r)|^2\leq(\|\phi_1-\phi_2\|^2_{C_H}+\frac{\nu}{2}\|\phi_1-\phi_2\|^2_{L^2_V})\cdot\exp(C\int_\tau^t(1+\|u(s)\|^2)\mathrm{d}s),$$

from (3.15) we can also have

$$\begin{aligned}\nu\int_\tau^t\|w\|^2\mathrm{d}s\leq&(\|\phi_1-\phi_2\|^2_{C_H}+\frac{\nu}{2}\|\phi_1-\phi_2\|^2_{L^2_V})\\ &\cdot(1+C\exp(C\int_\tau^t(1+\|u(s)\|^2)\mathrm{d}s)\int_\tau^t(1+\|u(s)\|^2)\mathrm{d}s),\end{aligned}$$

which leads to the continuous dependence on initial data, and the uniqueness of solutions holds naturally. □

4 Regularity of solution

In [22], the regularity of solution to the 2D non-homogeneous Navier-Stokes equations in non-smooth domain was derived, and in the same way we can study the system (3.1) and obtain the conclusion as follows.

Theorem 4.1. *Let $(u(\tau),\phi)\in M_H$, and $u(t)$ is the solution to system* (3.1) *as in Theorem* 3.1, *then $u(t)\in L^\infty(\tau,T;D(A^{\frac{1}{4}}))$.*

Proof. We also use the Faedo-Galerkin method to consider the regularity of solution.

Step 1: Priori Estimate. From (3.5) we get for a.e. $t>\tau$

$$\frac{\mathrm{d}}{\mathrm{d}t}<u_m,A^{\frac{1}{2}}u_m>+\nu<Au_m,A^{\frac{1}{2}}u_m>\mathrm{d}s+b(u_m(t-\rho(t)),u_m,A^{\frac{1}{2}}u_m)$$

$$=< f(t), A^{\frac{1}{2}}u_m >, \tag{4.1}$$

and from the Sobolev embedding inequality, Hölder inequality and Young inequality, we can also derive

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}|A^{\frac{1}{4}}u_m(t)|^2+2\nu|A^{\frac{3}{4}}u_m|^2\\
\leq& 2\int_\Omega u_m(t-\rho(t))\nabla u_m A^{\frac{1}{2}}u_m\mathrm{d}x+2|f||A^{\frac{1}{2}}u_m|\\
\leq& C\|u_m(t-\rho(t))\|_{L^4}\|u_m\|\|A^{\frac{1}{2}}u_m\|_{L^4}+2|f||A^{\frac{1}{2}}u_m|\\
\leq& C|A^{\frac{1}{4}}u_m(t-\rho(t))|\|u_m\||A^{\frac{3}{4}}u_m|+C|f|\cdot|A^{\frac{3}{4}}u_m|\\
\leq& \nu|A^{\frac{3}{4}}u_m|^2+\frac{C}{\nu}|A^{\frac{1}{4}}u_m(t-\rho(t))|^2\|u_m\|^2+\frac{C}{\nu\lambda_1^{1/2}}|f|^2,
\end{aligned} \tag{4.2}$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}|A^{\frac{1}{4}}u_m|^2+\nu|A^{\frac{3}{4}}u_m|^2\leq\frac{C}{\nu}\|u_m\|^2|A^{\frac{1}{4}}u_m(t-\rho(t))|^2+\frac{C}{\nu\lambda_1^{1/2}}|f|^2. \tag{4.3}$$

Integrating (4.3) over $[s,t]$, we obtain that

$$\begin{aligned}
&|A^{\frac{1}{4}}u_m(t)|^2+\nu\int_s^t|A^{\frac{3}{4}}u_m|^2\mathrm{d}k\\
\leq& |A^{\frac{1}{4}}u_m(s)|^2+\frac{C}{\nu}\int_s^t|A^{\frac{1}{4}}u_m(k-\rho(k))|^2\|u_m(k)\|^2\mathrm{d}k+\frac{C}{\nu\lambda_1^{1/2}}|f|^2(t-s)\\
\leq& |A^{\frac{1}{4}}u_m(s)|^2+\frac{C}{\nu}\int_s^t\underset{k_1\in[k-h,k]}{\mathrm{esssup}}|A^{\frac{1}{4}}u_m(k_1)|^2\|u_m(k)\|^2\mathrm{d}k+\frac{C}{\nu\lambda_1^{1/2}}|f|^2(t-s),
\end{aligned} \tag{4.4}$$

which means

$$\begin{aligned}
&\underset{k\in[t-h,t]}{\mathrm{esssup}}|A^{\frac{1}{4}}u_m(k)|^2\\
\leq&\frac{C}{\nu}\int_s^t\underset{k_1\in[k-h,k]}{\mathrm{esssup}}|A^{\frac{1}{4}}u_m(k_1)|^2\|u_m(k)\|^2\mathrm{d}k+|A^{\frac{1}{4}}u_m(s)|^2+\frac{C}{\nu\lambda_1^{1/2}}|f|^2(t-s),
\end{aligned} \tag{4.5}$$

and applying the Gronwall lemma lead to that

$$\begin{aligned}
&\underset{k\in[t-h,t]}{\mathrm{esssup}}|A^{\frac{1}{4}}u_m(k)|^2\\
\leq&\left[|A^{\frac{1}{4}}u_m(s)|^2+\frac{C}{\nu\lambda_1^{1/2}}|f|^2(t-s)\right]\cdot\exp\left(\frac{C}{\nu}\int_s^t\|u_m(k)\|^2\mathrm{d}k\right).
\end{aligned} \tag{4.6}$$

Also, Combing (3.7) and (3.8), we have

$$\nu\int_s^t\|u_m(k)\|\mathrm{d}k\leq|u(\tau)|^2+\frac{1}{\nu\lambda_1}|f|^2\left(t-s+\frac{1}{\nu\lambda_1}\right). \tag{4.7}$$

Let $t-s=\frac{1}{\nu\lambda_1}$, we use the technique as shown as in Theorem 4.11 in [22] and show that, in any interval of length $\frac{1}{\nu\lambda_1}$, there exists k such that

$$\|u_m(k)\|^2 \leq 2\lambda_1|u(\tau)|^2 + \frac{2|f|^2}{\nu^2\lambda_1}, \tag{4.8}$$

which, with (4.6) and (4.7), can lead to that

$$\sup_{t>1/\nu\lambda_1} |A^{\frac{1}{4}}u_m(t)|^2 \leq \left(2\lambda_1^{\frac{1}{2}}|u(\tau)|^2 + \frac{C|f|^2}{\nu^2\lambda_1^{3/2}}\right)\cdot \exp\left[\frac{C}{\nu^2}\left(|u(\tau)|^2 + \frac{|f|^2}{\nu^2\lambda_1^2}\right)\right]. \tag{4.9}$$

Step 2: Limit procedure. From (3.9) we know

$$u_m \rightharpoonup u, \text{ weakly in } L^2(\tau-h,T;V),$$

the fact that $V \hookrightarrow D(A^{1/4})$ is compact leads to there is subsequence (still written $\{u_m\}$) satisfying

$$u_m \to u, \text{ strongly in } L^2(\tau,T;D(A^{\frac{1}{4}})).$$

It thus follows that there exists another subsequence $\{u_{m_j}\}$ such that

$$u_{m_j} \to u \text{ in } D(A^{\frac{1}{4}}) \text{ for a.e. } t,$$

which leads to fact that

$$\sup_{t>1/\nu\lambda_1} |A^{\frac{1}{4}}u(t)| \leq C,$$

and the conclusion holds naturally. □

5 Existence of pullback $\mathcal{D}$-attractors $\mathcal{A}$

From Theorem 4.1 we know that the system (3.1) generates a continuous semi-flow $S(t,\tau)=(U^0(t,\tau),U^h(t,\tau))$ in M_H by

$$S(t,\tau)(u(\tau),\phi)=(u(t),u_t).$$

To obtain the pullback $\mathcal{D}$-attractors, we must establish the existences of pullback $\mathcal{D}$-absorbing set and the pullback $\mathcal{D}$-asymptotic compactness of semi-flow. We first provide $\mathcal{D}$ to denote a class of all families $\{D(t)\}_{t\in\mathbb{R}} \subset \mathcal{P}(M_H)$ satisfying

$$\lim_{\tau\to-\infty} e^{r\tau} \sup_{(u(\tau),\phi)\in D(\tau)} |u(\tau)|^2 = 0, \quad 0<r<\nu.$$

5.1 Pullback $\mathcal{D}$-absorbing sets in M_H

Theorem 5.1. *Assume $(u(\tau),\phi)\in M_H$, the semi-flow $\{S(t,\tau)\}$ to system* (3.1) *possesses a pullback $\mathcal{D}$-absorbing set $\mathcal{B}=\{B(t)\}_{t\in\mathbb{R}}$ in M_H, where*

$$B(t)=\bar{B}_H(0,\rho_H(t))\times(\bar{B}_{C_H}(0,\rho_H(t))\cap\bar{B}_{L^2_V}(0,\rho_{L^2_V}(t))),$$

in which

$$\rho_H^2(t)=1+\frac{1}{\nu\lambda_1}e^{-rt}e^{rh}\int_{-\infty}^t e^{rs}|f|^2\mathrm{d}s,\qquad \rho_{L^2_V}^2(t)=\frac{1}{\nu}\rho_H^2(t)+\frac{1}{\nu^2\lambda_1}\int_{t-h}^t|f|^2\mathrm{d}s.$$

Proof. Multiplying (3.1) by u, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+2\nu\|u\|^2\le 2|u||f|\le\nu\lambda_1|u|^2+\frac{1}{\nu\lambda_1}|f|^2,$$

i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2+\nu\|u\|^2\le\frac{1}{\nu\lambda_1}|f|^2, \tag{5.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{rt}|u|^2+(\nu-r)e^{rt}\|u\|^2\le\frac{1}{\nu\lambda_1}e^{rt}|f|^2. \tag{5.2}$$

Integrating (5.2) on $[\tau,t]$, we have

$$|u(t)|^2\le e^{r(\tau-t)}|u(\tau)|^2+\frac{e^{-rt}}{\nu\lambda_1}\int_\tau^t e^{rs}|f|^2\mathrm{d}s,$$

which means there exists $\tau(\mathcal{D},t)<t-h$ such that, for $\tau<\tau(\mathcal{D},t)$,

$$|u(t)|^2<1+\frac{1}{\nu\lambda_1}e^{-rt}e^{rh}\int_{-\infty}^t e^{rs}|f|^2\mathrm{d}s=\rho_H^2(t).$$

Also, we can derive that

$$\|u_t\|_{C_H}^2\le\rho_H^2(t),\quad \tau<\tau(\mathcal{D},t).$$

From (5.1) we obtain

$$\nu\int_{t-h}^t\|u(s)\|^2\mathrm{d}s\le|u(t-h)|^2+\frac{1}{\nu\lambda_1}\int_{t-h}^t|f|^2\mathrm{d}s, \tag{5.3}$$

and there holds

$$\|u_t\|_{L^2_V}^2\le\frac{1}{\nu}\rho_H^2(t)+\frac{1}{\nu^2\lambda_1}\int_{t-h}^t|f|^2\mathrm{d}s=\rho_{L^2_V}^2(t),\quad\tau<\tau(\mathcal{D},t),$$

and the conclusion holds. □

Remark 5.1. For the process $U^0(t,\tau)$, it has the pullback $\mathcal{D}$-absorbing set $\mathcal{B}^0=\{B^0(t)\}_{t\in\mathbb{R}}$ in H, where

$$B^0(t)=\left\{u(t)\in H;\ |u(t)|^2\le\rho_0^2(t)=1+\frac{e^{-rt}}{\nu\lambda_1}\int_{-\infty}^{t}e^{rs}|f|^2\mathrm{d}s\right\}. \tag{5.4}$$

For the process $U^h(t,\tau)$, it has the pullback $\mathcal{D}$-absorbing set $\mathcal{B}^h=\{B^h(t)\}_{t\in\mathbb{R}}$ in C_H, where

$$B^h(t)=\left\{u_t\in C_H;\ \|u_t\|^2\le\rho_H^2(t)=1+\frac{e^{-rt}e^{rh}}{\nu\lambda_1}\int_{-\infty}^{t}e^{rs}|f|^2\mathrm{d}s\right\}. \tag{5.5}$$

And there holds

$$\limsup_{h\to 0}\rho_H^2(t)=\rho_0^2(t). \tag{5.6}$$

5.2 Pullback $\mathcal{D}$-asymptotic compactness of $S(t,\tau)$ in M_H

Theorem 5.2. *Let $(u(\tau),\phi)\in M_H$, then the semi-flow $S(t,\tau):\ M_H\to M_H$ generated by the system* (3.1) *is pullback $\mathcal{D}$-asymptotically compact.*

Proof. For any family $\hat{D}=\{D(t)\}_{t\in\mathbb{R}}$ in $\mathcal{D}$, the sequence $\{\tau_n\}\subset(-\infty,t]$ satisfies $\tau_n\to-\infty$ as $n\to\infty$, $\{(u(\tau_n),\phi^n)\}\subset D(\tau_n)$, and we write $u^n(\cdot)=u(\cdot;\tau_n,u(\tau_n),\phi^n)$.

Step 1. We establish the weak convergence of $\{u^n(t,x)\}$ in $[t-h,t]$ and in H for any arbitrary fixed $t\ge\tau$ respectively.

From the conclusions and techniques in Theorems 4.1 and 5.1, we know there exists $\tau(\hat{D},t)\le t-3h-1$ such that, for $\tau\le\tau(\hat{D},t)$,

$$\{u^n\}\subset L^\infty(t-3h-1,t;H)\cap L^2(t-2h-1,t;V),\quad \left\{\frac{\partial}{\partial t}(u^n)\right\}\subset L^2(t-h-1,t;V').$$

Which together with the compactness Lemma and the diagonal procedure can lead to that there exists a subsequence (written also as $\{u^n\}$) satisfying

$$\begin{cases} u^n\rightharpoonup u \text{ weakly star in } L^\infty(t-3h-1,t;H);\\ u^n\rightharpoonup u \text{ weakly in } L^2(t-2h-1,t;V);\\ \dfrac{\partial}{\partial t}u^n\rightharpoonup\dfrac{\partial}{\partial t}u \text{ weakly in } L^2(t-h-1,t;V');\\ u^n\to u \text{ strongly in } L^2(t-h-1,t;H);\\ u^n\to u \text{ in } H,\text{ a.e. } s\in(t-h-1,t). \end{cases} \tag{5.7}$$

Thus, from (5.7) we can conclude that $u\in C([t-h-1,t];H)$ is a weak solution for system (3.1) with the initial datum $u(t-h-1)$.

Since $\{u^n\}\subset L^\infty(t-h-1,t;D(A^{1/4}))$ and $\{\frac{\partial}{\partial t}(u^n)\}\subset L^2(t-h-1,t;V')$, the Aubin-Lions-Simon Lemma leads to that

$$u^n \to u \text{ strongly in } C([t-h-1,t];H). \tag{5.8}$$

It thus follows for any $\{s_n\}\subset[t-h-1,t]$ with $s_n\to s\in[t-h-1,t]$ that

$$u^n(s_n)\rightharpoonup u(s) \text{ weakly in } H, \tag{5.9}$$

and there holds that

$$\liminf_{n\to\infty}|u^n(s_n)|\geq|u(s)|. \tag{5.10}$$

Step 2. We establish the strong convergence of sequence $\{u^n(s_n)\}$ in $C([t-h,t];H)$ for any sequence $\{s_n\}\subset[t-h,t]$ with $s_n\to s$ as $n\to+\infty$.

According to the energy equality for u^n and u, we define the following two functionals in $[t-h-1,t]$

$$J_n(s)=\frac{1}{2}|u^n(s)|^2-\int_{t-h-1}^s (f,u^n(r))\mathrm{d}r, \tag{5.11}$$

$$J(s)=\frac{1}{2}|u(s)|^2-\int_{t-h-1}^s (f,u(r))\mathrm{d}r, \tag{5.12}$$

which are continuous and non-increasing in $[t-h-1,t]$. From (5.7) we know as $n\to+\infty$ that

$$\int_{t-h-1}^s (f,u^n(r))\mathrm{d}r\to\int_{t-h-1}^s (f,u(r))\mathrm{d}r, \tag{5.13}$$

it follows that

$$J_n(s)\to J(s) \text{ a.e. } s\in(t-h-1,t), \tag{5.14}$$

which means that, for any $\varepsilon>0$, there exists a constant $\hat{N}$ such that, for any $n\geq\hat{N}$ and any sequence $\{s_n\}\subset[t-h-1,t]$,

$$|J_n(s_n)-J(s_n)|\leq\frac{\varepsilon}{2}. \tag{5.15}$$

Also, the uniform continuity of $J(s)$ with respect to s can lead to that for any $\varepsilon>0$, there exist a constant $\tilde{N}$ such that, for any $n\geq\tilde{N}$ and any sequence $\{s_n\}\subset[t-h-1,t]$ satisfying $s_n\to s$,

$$|J(s_n)-J(s)|\leq\frac{\varepsilon}{2}. \tag{5.16}$$

Letting $N=\max\{\hat{N},\tilde{N}\}$ and using (5.15) and (5.16), we can obtain for $n\geq N$ that

$$|J_n(s_n)-J(s)|\leq|J_n(s_n)-J(s_n)|+|J(s_n)-J(s)|\leq\varepsilon. \tag{5.17}$$

It thus follows for any $\{s_n\}\subset[t-h-1,t]$ that

$$\limsup_{n\to\infty}J_n(s_n)\leq J(s), \tag{5.18}$$

which can lead to the strong convergence of $\{u^n(s_n)\}$ in $C([t-h,t];H)$ by using (5.10).

Step 3. We establish the strong convergence of $\{u^n(s_n)\}$ in $L^2(t-h,t;V)$ for any sequence $\{s_n\}\subset[t-h,t]$ with $s_n\to s$ as $n\to+\infty$. For u^n and u, there hold

$$|u^n(t)|^2+2\nu\int_{t-h}^t\|u^n(r)\|^2\mathrm{d}r=|u^n(t-h)|^2+2\int_{t-h}^t(f,u^n(r))\mathrm{d}r, \tag{5.19}$$

$$|u(t)|^2+2\nu\int_{t-h}^t\|u(r)\|^2\mathrm{d}r=|u(t-h)|^2+2\int_{t-h}^t(f,u(r))\mathrm{d}r. \tag{5.20}$$

Combining (5.7), (5.19) and (5.20), we can derive that as $n\to+\infty$

$$2\nu\int_{t-h}^t\|u^n(r)\|^2\mathrm{d}r\to2\nu\int_{t-h}^t\|u(r)\|^2\mathrm{d}r, \tag{5.21}$$

and the strong convergence of $\{u^n(s_n)\}$ in $L^2(t-h,t;V)$ can be derived by using (5.7) and (5.21).

To sum up, from the upward three steps we can obtain that the semi-flow $\{S(t,\tau)\}:M_H\to M_H$ generated by the system (3.1) is pullback $\mathcal{D}$-asymptotically compact. □

5.3 Existence of pullback $\mathcal{D}$-attractors

In Theorem 4.1 the continuity of semi-flow $\{S(t,\tau)\}$ generated by system (3.1) is derived, the pullback $\mathcal{D}$-absorbing sets are established in Theorem 5.1, and we show that $\{S(t,\tau)\}$ is pullback $\mathcal{D}$-asymptotically compact in M_H. According to Theorem 2.1 or the technique as in [17], we obtain the conclusion as following.

Theorem 5.3. *Assume $(u(\tau),\phi)\in M_H$, the semi-flow $S(t,\tau):\ M_H\to M_H$ generated by the system* (3.1) *possesses the pullback $\mathcal{D}$-attractor $\mathcal{A}=\{A(t)\}_{t\in\mathbb{R}}$ in M_H, and the pullback attractors corresponding to the processes $U^0(t,\tau)$ and $U^h(t,\tau)$ exist respectively, where*

$$\mathcal{A}^0=P_1\mathcal{A},\qquad\mathcal{A}^h=P_2\mathcal{A}.$$

6 Upper semi-continuity of pullback attractors

When $h=0$, the system (3.1) can be reduced to the following system

$$\begin{cases} \dfrac{\partial}{\partial t}u+\nu Au+B(u,u)=f(t), \\ u(\tau,x)=\phi(0), \quad x\in\Omega. \end{cases} \tag{6.1}$$

In the following way, we intend to establish some results on the convergence of pullback attractors $\mathcal{A}^h$ to system (3.1) and $\mathcal{A}^0$ to system (6.1) as $h\to 0$. In view of the complexity of problem in the case of strong topology in H, we use the concept of weak topology of L^2 to consider the convergence problem, denote the metric by d_H° in H in which the weak topology on bounded sets is metrizable, and give the corresponding definition of upper semi-continuity as follows.

Definition 6.1. *It is said that a family of subsets $\{A^h(t)\}_{t\in\mathbb{R}}$ in H has the property of upper semi-continuity if*

$$\lim_{h\to 0}\sup_{-h\le\theta\le 0}\mathrm{dist}_H^\circ(A^h(t),A^0(t))=0, \quad \textit{for any } t\in\mathbb{R},$$

where

$$\mathrm{dist}_H^\circ(E,S)=\sup_{\phi\in E}\inf_{\varphi\in S}||\phi(\theta)-\varphi||_H, \quad \forall\, E\subset H,\, S\subset H.$$

Lemma 6.1. *Let $\{h_n\}\subset(0,h]$ be any sequence such that $h_n\to 0$ as $n\to+\infty$, $(u_{\tau,n},\phi_n)\in M_H$, and $U_{h_n}u_{\tau,n}$ is the solution to the system* (3.1) *with $h=h_n$. Assume that $u:[\tau,+\infty)\to H$ is the solution to system* (6.1)*, then there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to+\infty} d_H^\circ(U^{h_{n_k}}u_{\tau,n},u(t))=0, \quad \forall t\ge\tau.$$

Proof. Similar to the proof of Theorem 4.1, we can show that, for every $T>\tau$, the sequence of solution $\{u_n(t)=U^{h_n}u_{\tau,n}\}$ satisfies

$$\{u_n\}\subset L^\infty(\tau-h,T;H)\cap L^2(\tau-h,T;V), \quad \left\{\frac{\partial}{\partial t}u_n\right\}\subset L^2(\tau,T;V').$$

And there exists a function $u:[\tau,+\infty)\to H$ satisfying

$$\{u(t)\}\subset L^\infty(\tau-h,T;H)\cap L^2(\tau-h,T;V), \quad \left\{\frac{\partial}{\partial t}u(t)\right\}\subset L^2(\tau,T;V'),$$

with

$$\begin{cases} u_n\to u, \text{ strongly in } L^2(\tau,T;H); \\ u_n\rightharpoonup u, \text{ weakly in } L^2(\tau-h,T;V). \end{cases} \tag{6.2}$$

For each $n\in\mathbb{N}$ and any $s,t\in[\tau,T]$, we have

$$\begin{aligned}&(u_n(t),v)+\nu\int_s^t(A^{\frac{1}{2}}u_n(l),A^{\frac{1}{2}}v)\mathrm{d}l+\int_s^t<u_n(l-\rho(l)\cdot\nabla u_n(l),v>\mathrm{d}l\\=&(u_n(s),v)+\int_s^t(f,v)\mathrm{d}l,\qquad\forall v\in V.\end{aligned}\tag{6.3}$$

From (6.2) we know that the pointwise convergence holds in H for any $t\in[\tau,T]$, which means

$$(u_n(t),v)\to(u(t),v),\text{ for any } v\in H \text{ as } n\to+\infty,$$

and

$$\nu\int_s^t(A^{\frac{1}{2}}u_n(l),A^{\frac{1}{2}}v)\mathrm{d}l\to\nu\int_s^t(A^{\frac{1}{2}}u(l),A^{\frac{1}{2}}v)\mathrm{d}l\text{ as } n\to+\infty,$$

it thus follows that

$$\begin{aligned}&(u(t),v)+\nu\int_s^t(A^{\frac{1}{2}}u(l),A^{\frac{1}{2}}v)\mathrm{d}l+\int_s^t<u(l)\cdot\nabla u(l),v>\mathrm{d}l\\=&(u(s),v)+\int_s^t(f,v)\mathrm{d}l,\qquad\forall v\in V,\end{aligned}\tag{6.4}$$

which means u is the weak solution to system (6.1).

For any $n\in\mathbb{N}$, u_n is the weak solution to system (3.1) with $h=h_n$, then from the definition of weak solution we know

$$-\frac{1}{2}\int_\tau^T|u_n(t)|^2\psi'(t)\mathrm{d}t+\nu\int_\tau^T\|u_n(t)\|^2\psi(t)\mathrm{d}t=\int_\tau^T(u_n(t),f)\psi(t)\mathrm{d}t,\tag{6.5}$$

for any $T>\tau$ and any positive function $\psi(t)\in C_0^\infty(\tau,T)$. To gain our ends, we need to get the convergence of each term in (6.5).

Since $|u_n(t)|^2\to|u(t)|^2$ as $n\to+\infty$ for almost all $t\in[\tau,T]$, and the sequence $\{|u_n(t)|^2\cdot\psi'(t)\}\subset L^1(\tau,T)$, the Lebesgue dominated convergence theorem can lead to

$$\int_\tau^T|u_n(t)|^2\psi'(t)\mathrm{d}t\to\int_\tau^T|u(t)|^2\psi'(t)\mathrm{d}t,\quad\text{as } n\to+\infty.$$

We use the conclusion (6.2) again, and from the property of weak convergence we have

$$\nu\int_\tau^T\|u(t)\|^2\psi(t)\mathrm{d}t\le\liminf_{n\to+\infty}\nu\int_\tau^T\|u_n(t)\|^2\psi(t)\mathrm{d}t.$$

Hence, it yields

$$-\frac{1}{2}\int_\tau^T|u(t)|^2\psi'(t)\mathrm{d}t+\nu\int_\tau^T\|u(t)\|^2\psi(t)\mathrm{d}t=\int_\tau^T(u(t),f)\psi(t)\mathrm{d}t,\tag{6.6}$$

which is just the energy inequality for u, and the conclusion holds naturally. $\square$

Lemma 6.2. *Let $(u(\tau),\phi)\in M_H$, $\mathcal{B}^h=\{B^h(t)\}_{t\in\mathbb{R}}$ is the pullback absorbing set of $U^h(t,\tau)$ in M_H to system* (3.1)*, and $\mathcal{B}^0=\{B^0(t)\}_{t\in\mathbb{R}}$ is the correspondingly pullback absorbing set of $U^0(t,\tau)$ in H to system* (6.1)*, then*

$$\lim_{h\to 0}\text{dist}_H^\circ(U^h(t,\tau)\mathcal{B}^h,U^0(t,\tau)\mathcal{B}^0)=0.$$

Proof. In the following way, we use the proof by contradiction to obtain the conclusion. Assume that there are $t>\tau$, the sequence $\{h_n\}\subset(0,1)$, where $h_n\to 0$ as $n\to+\infty$, and $u_{\tau,n}\in B^{h_n}(\tau)\in\mathcal{B}^h$ such that for any $n\in\mathbb{N}$

$$\inf_{v\in U^0(t,\tau)\mathcal{B}^0}\text{dist}_H^\circ(U^{h_n}(t,\tau)u_{\tau,n},v)\geq\varepsilon.$$

Then, for any $v\in U^0(t,\tau)\mathcal{B}^0$, we obtain that for any $n\in\mathbb{N}$

$$\text{dist}_H^\circ(U^{h_n}(t,\tau)u_{\tau,n},v)\geq\varepsilon,$$

which is a contradiction with Lemma 6.1. □

Theorem 6.1. *Let $(u(\tau),\phi)\in M_H$, $\mathcal{A}^h=\{A^h(t)\}_{t\in\mathbb{R}}$ is the pullback attractor of $U^h(t,\tau)$ in $C_H\cap L^2_V$ to system* (3.1)*, and $\mathcal{A}^0=\{A^0(t)\}_{t\in\mathbb{R}}$ is the correspondingly pullback attractor of $U^0(t,\tau)$ in H to system* (6.1)*, then*

$$\lim_{h\to 0}\text{dist}_H^\circ(\mathcal{A}^h,\mathcal{A}^0)=0.$$

Proof. It is known that $\mathcal{A}^0=\{A^0(t)\}_{t\in\mathbb{R}}$ is the pullback attractor of $U^0(t,\tau)$ in H to system (6.1), from the definition of pullback attractor we know that, for any arbitrarily fixed constant $\varepsilon>0$, there exists $\tau_\varepsilon<0$ such that

$$\text{dist}_H^\circ(U^0(t,\tau_\varepsilon)\mathcal{B}^0,\mathcal{A}^0)\leq\frac{\varepsilon}{2}.$$

From the upper lemma we know that there exist $0<h_\varepsilon<1$ such that for $h<h_\varepsilon$

$$\text{dist}_H^\circ(U^h(t,\tau_\varepsilon)\mathcal{B}^h,U^0(t,\tau_\varepsilon)\mathcal{B}^0)\leq\frac{\varepsilon}{2}.$$

And the conclusion

$$U^h(t,\tau)A(\tau)=A(t),\quad \mathcal{A}^h\subset\mathcal{B}^h$$

leads to

$$\begin{aligned}\text{dist}_H^\circ(\mathcal{A}^h,U^0(t,\tau_\varepsilon)\mathcal{B}^0)&=\text{dist}_H^\circ(U^h(t,\tau_\varepsilon)\mathcal{A}^h,U^0(t,\tau_\varepsilon)\mathcal{B}^0)\\&\leq\text{dist}_H^\circ(U^h(t,\tau_\varepsilon)\mathcal{B}^h,U^0(t,\tau_\varepsilon)\mathcal{B}^0),\end{aligned}\tag{6.7}$$

which implies that

$$\text{dist}_H^\circ(\mathcal{A}^h,U^0(t,\tau_\varepsilon)\mathcal{B}^0)\leq\frac{\varepsilon}{2}.$$

In addition, the triangle inequality

$$\text{dist}_H^\circ(\mathcal{A}^h,\mathcal{A}^0)\leq\text{dist}_H^\circ(\mathcal{A}^h,U^0(t,\tau_\varepsilon)\mathcal{B}^0)+\text{dist}_H^\circ(U^h(t,\tau_\varepsilon)\mathcal{B}^h,U^0(t,\tau_\varepsilon)\mathcal{B}^0)$$

ensures that the conclusion holds. □

6.1 Further research

The convergence of pullback attractors $\mathcal{A}^h$ to system (3.1) as $h\to 0$ has been attained in weak topology, which implies the robustness of dynamic system. Whereas the topology here we considered is weak, what about the case of strong topology? Can we obtain the structure and stability as in [23], and pullback dynamics on Lipschitz-like domain as in [24]? This is the objective we want to be next.

Acknowledgement

Xin-Guang Yang was partially supported by the Fund of Young Backbone Teacher in Henan Province (No. 2018GGJS039), cultivation Fund of Henan Normal University (No. 2020PL17) and Henan Overseas Expertise Introduction Center for Discipline Innovation (No. CXJD2020003).

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