JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **37**, No. 1, pp. 59-87

doi: 10.4208/jpde.v37.n1.4
March 2024

On Linear Homogeneous Biwave Equations

BAI Yaqian*

Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, China.

Received 25 May 2022; Accepted 19 April 2023

Abstract. The biwave maps are a class of fourth order hyperbolic equations. In this paper, we are interested in the solution formulas of the linear homogeneous biwave equations. Based on the solution formulas and the weighted energy estimate, we can obtain the $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates, respectively. By our results, we find that the biwave maps enjoy some different properties compared with the standard wave equations.

AMS Subject Classifications: 35L15, 35L72, 35Q75, 35A01

Chinese Library Classifications: O175.2, O175.4

Key Words: Biwave maps; Duhamel's principle; Fourier transform; Cauchy peoblem; deacy estimate.

1 Introduction

Let $u:\mathbb{R}^{1+n}\to N$ is a smooth map from a Minkowski space $\mathbb{R}^{1+n}$ into a Riemannian manifold N, then the bi-energy functional is given by

$$\begin{aligned} E(u)=&\frac{1}{2}\int_{\mathbb{R}^{1+n}}||(d+d^*)^2u||^2\mathrm{d}t\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}^{1+n}}||d^*du||^2\mathrm{d}t\mathrm{d}x\\ =&\frac{1}{2}\int_{\mathbb{R}^{1+n}}||\tau_\Box(u)||^2\mathrm{d}t\mathrm{d}x. \end{aligned}\tag{1.1}$$

Here

$$\tau^\alpha_\Box(u)=\Box u^\alpha+\Gamma^\alpha_{\beta\gamma}(-u^\beta_t u^\gamma_t+\sum_{i=1}^n u^\beta_i u^\gamma_i),$$

where $\Box=\frac{\partial^2}{\partial t^2}-\triangle_x$ is the wave operator on $\mathbb{R}^{1+n}$ and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of N. Clearly, the map u is a wave map iff the wave field $\tau^\alpha_\Box(u)$ vanishes.

*Corresponding author. *Email addresses:* qianyabai@163.com (Y. Q. Bai)

The Euler-Lagrange equation of (1.1) is

$$\begin{aligned}(\tau_2)_\square(u)^\alpha =& J_u(\tau_\square u)^\alpha = \triangle\tau_\square(u)^\alpha + R^{N\alpha}(df,df)\tau_\square(u)\\ =&\square\tau_\square(u)^\alpha + \Gamma^\alpha_{\mu\gamma}\Big(-\tau_\square(u)^\mu_t\tau_\square(u)^\gamma_t + \sum_{i=1}^n \tau_\square(u)^\mu_i\tau_\square(u)^\gamma_i\Big)\\ &+R^\alpha_{\beta\gamma\mu}\Big(-u^\beta_t u^\gamma_t + \sum_{i=1}^n u^\beta_i u^\gamma_i\Big)\tau_\square(u)^\mu = 0,\end{aligned} \tag{1.2}$$

i.e., $\tau_\square(u)$ is a Jacobi field, where $R^\alpha_{\beta\gamma\mu}$ is the Riemannian curvature of N, see [1].

Biwave maps are biharmonic maps on Minkowski space, which generalize wave maps, and have been first studied by Chiang [2–4] in 2009. Biwave maps satisfy a fourth order hyperbolic system of partial differential equations, which are different from biharmonic maps. Recently, Chiang and Wei [1] studied the long time behavior of the biwave maps by Klainerman's method of vector fields when the initial data are small.

In this paper, for simplicity, in order to study the well posedness of biwave maps, we first study the linear case. If we assume that the manifold is flat, which means $\Gamma^\alpha_{\mu\gamma}=0$ and $R^\alpha_{\beta\gamma\mu}=0$, we will obtain $\square^2 u=0$.

Then the Cauchy problem for the n-dimensional biwave equation satisfies the following system

$$\begin{cases}\square^2 u = 0,\\ u(x,0)=f_1(x),\\ u_t(x,0)=f_2(x),\\ u_{tt}(x,0)=f_3(x),\\ u_{ttt}(x,0)=f_4(x).\end{cases} \tag{1.3}$$

Here $u=u(x,t)$ is the unknown, $t>0$ and $x\in\mathbb{R}^n$, $u_t=\frac{\partial u}{\partial t}$, $f_i(x)\in C_0^\infty(\mathbb{R}^n)$, $i=1,\cdots,4$. Occasionally it is convenient to write $t=x^0$, in which case $\partial_0=\partial_t$.

The biwave equations are also related to the mathematical theory of elasticity [5], thus it is of great physical significance to study it. However, there are not many mathematical results on biwave equations, because it gets more difficult when studying the high-order PDEs. Feng and Neilan [6,7] develop the finite element methods for the approximations of biwave equations. Fushchych, Roman and Zhdanov [8] consider the symmetry analysis of biwave equations $\square^2 u = F(u)$ and obtain the exact solution by Ansätze invariants under the non-conjugate subalgebras of the extended Poincaré algebra and the conformal algebra. The existence and uniqueness of the solution to initial-value problem and boundary-value problem for the fourth-order linear PDE of hyperbolic and composite types are given by Korzyuk, Cheb and Konopel'ko [9,10], respectively. By the techniques of Fourier analysis, Korzyuk, Vinh and Minh [5] also get the solution formulas for the Cauchy problem of the $n-$dimensional biwave maps $(\frac{\partial^2}{\partial t^2}-a^2\triangle)(\frac{\partial^2}{\partial t^2}-b^2\triangle)u=0$, with $a^2\neq b^2$.

In this paper, we use the standard Duhamel's principle, spherical means formulas and the method of Fourier analysis respectively to obtain the exact solution formulas of (1.3). Furthermore, we also study the $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates.

The main results of this paper can be stated as follows.

Theorem 1.1 (Solution formula of biwave equation with $n=1$)**.** *Assume $f_1\in C^4(\mathbb{R}^n)$, $f_2\in C^3(\mathbb{R}^n)$, $f_3\in C^2(\mathbb{R}^n)$ and $f_4\in C^1(\mathbb{R}^n)$, then the solution formula of* (1.3) *is*

$$u(x,t)=\frac{1}{2}[f_1(x+t)+f_2(x-t)]+\frac{1}{2}\int_{x-t}^{x+t} f_2(y)\mathrm{d}y+\frac{1}{2}\int_0^t\int_{x-t'}^{x+t'} v(y,s)\mathrm{d}y\mathrm{d}s, \tag{1.4}$$

where

$$\begin{aligned}v(y,s)=&\frac{1}{2}[f_3(y+s)+f_3(y-s)]-\frac{1}{2}[\triangle f_1(y+s)+\triangle f_1(y-s)]\\&+\frac{1}{2}\int_{y-s}^{y+s} f_4(\omega)\mathrm{d}\omega-\frac{1}{2}\int_{y-s}^{y+s}\triangle f_2(\omega)\mathrm{d}\omega,\end{aligned}$$

and $t'=t-s\geq 0$.

Theorem 1.2 (Solution formula of biwave equation in odd dimensions)**.** *Assume n is an odd integer, $n\geq 3$, and suppose also $f_1\in C^{m+3}(\mathbb{R}^n)$, $f_2\in C^{m+2}(\mathbb{R}^n)$, $f_3\in C^{m+1}(\mathbb{R}^n)$ and $f_4\in C^m(\mathbb{R}^n)$ for $m=\frac{n+1}{2}$. Then the solution formula of* (1.3) *is*

$$\begin{aligned}u(x,t)=&\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_1(x)\Big)+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_2(x)\Big)\\&+\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-3}{2}}\Big((t')^{n-2}A_{t'}v(x,s)\Big)\mathrm{d}s\Big],\end{aligned} \tag{1.5}$$

where

$$v(x,s)=\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_sg_1(x)\Big)+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_sg_2(x)\Big)\Big],$$

and

$$A_tf(x)=\frac{1}{\omega_{n-1}}\int_{S^{n-1}}f(x+ty)\mathrm{d}\sigma(y),\quad \omega_{n-1}$$

denotes the area of the unit sphere $S^{n-1}\subset\mathbb{R}^n$, $g_1(x)=f_3(x)-\triangle f_1(x)$, $g_2(x)=f_4(x)-\triangle f_2(x)$.

Theorem 1.3 (Solution formula of biwave equation in even dimensions)**.** *Assume n is an even integer, $n\geq 2$, and suppose also $f_1\in C^{m+3}(\mathbb{R}^n)$, $f_2\in C^{m+2}(\mathbb{R}^n)$, $f_3\in C^{m+1}(\mathbb{R}^n)$ and $f_4\in C^m(\mathbb{R}^n)$ for $m=\frac{n+2}{2}$. Then the solution formula of* (1.3) *is*

$$u(x,t)=\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f_1(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y$$

$$+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f_2(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y$$
$$+\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-2}{2}}(t')^{n-1}\int_{|y|<1}\frac{v(x+t'y,s)}{\sqrt{1-|y|^2}}\mathrm{d}y\mathrm{d}s\Big], \tag{1.6}$$

where

$$v(x,s)=\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{g_1(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y$$
$$+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big].$$

Remark 1.1. In [5], the authors get the solution formulas for $(\frac{\partial^2}{\partial t^2}-a^2\triangle)(\frac{\partial^2}{\partial t^2}-b^2\triangle)u=0$ by the method of Fourier analysis. In their paper, $a^2\neq b^2$ is necessary. Here we can also get the solution formula by the method of Fourier analysis when $a^2=b^2$.

Remark 1.2. Comparing (1.5) and (1.6), we observe that if n is odd and $n\geq 3$, the data $f_i(i=1,\cdots,4)$ at a given point $x\in\mathbb{R}^n$ affect the solution u only on the boundary $\{(y,t)|t>0,|x-y|=t\}$ of the cone $C=\{(y,t)|t>0,|x-y|<t\}$. On the other hand, if n is even and $n\geq 2$, the data $f_i(i=1,\cdots,4)$ affect u within all of C. In other words, a "disturbance" originating at x propagates along a sharp wavefront in odd dimensions, but in even dimensions continues to have effects even after the leading edge of the wavefront passes. Therefore, the Huygen's principle holds as well for biwave maps.

Besides, similar to the standard wave equation, we also obtain the $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ $(N\geq[\frac{3n}{2}]+3)$ estimates, where $[a]$ denotes the largest integer less than a.

Theorem 1.4. *Through above solution formulas, when $t>1$, we have the following results*
(1) When $n=1$

$$||u||_{L^\infty(\mathbb{R})}\leq C\big(||f_1||_{L^\infty(\mathbb{R})}+||f_2||_{L^1(\mathbb{R})}\big)+C||f_4||_{L^1(\mathbb{R})}t^2$$
$$+C\big(||f_2||_{L^\infty(\mathbb{R})}+||f_1||_{W^{2,1}(\mathbb{R})}+||f_3||_{L^1(\mathbb{R})}\big)t.$$

(2) When $n=2$

$$||u||_{L^\infty(\mathbb{R}^2)}\leq C\big(||f_1||_{W^{6,1}(\mathbb{R}^2)}+||f_2||_{W^{5,1}(\mathbb{R}^2)}\big)t^{\frac{3}{2}}+C\big(||f_3||_{W^{2,1}(\mathbb{R}^2)}+||f_4||_{W^{2,1}(\mathbb{R}^2)}\big)t^2.$$

(3) When $n=3$

$$||u||_{L^\infty(\mathbb{R}^3)}\leq C\big(||f_1||_{W^{7,1}(\mathbb{R}^3)}+||f_2||_{W^{6,1}(\mathbb{R}^3)}\big)t\ln t+C\big(||f_3||_{W^{3,1}(\mathbb{R}^3)}+||f_4||_{W^{3,1}(\mathbb{R}^3)}\big)t^2\ln t.$$

(4) When $n>3$

$$||u||_{L^\infty(\mathbb{R}^n)}\leq C\big(||f_1||_{W^{[3n/2]+3,1}(\mathbb{R}^n)}+||f_2||_{W^{[3n/2]+2,1}(\mathbb{R}^n)}\big)t$$
$$+C\big(||f_3||_{W^{[n],1}(\mathbb{R}^n)}+||f_4||_{W^{[n],1}(\mathbb{R}^n)}\big)t^{\frac{n+1}{2}}.$$

Following the idea of [1], we can get the $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates, which shows that the solution will decay when $n>7$.

Theorem 1.5. *Assume that u is a solution to* (1.3), *then we get*

$$\begin{aligned}||u||_{L^\infty(\mathbb{R}^n)}\le & C\Big(||f_1||_{W_\Lambda^{[n/2]+2,2}(\mathbb{R}^n)}+||f_2||_{W_\Lambda^{[n/2]+1,2}(\mathbb{R}^n)}\Big)(1+t+|x|)^{\frac{3-n}{2}}(1+|t-|x||)^{-\frac{1}{2}}\\ &+C\Big(||f_1||_{W_\Lambda^{[n/2]+4,2}(\mathbb{R}^n)}+||f_2||_{W_\Lambda^{[n/2]+3,2}(\mathbb{R}^n)}+||f_3||_{W_\Lambda^{[n/2]+2,2}(\mathbb{R}^n)}\\ &+||f_4||_{W_\Lambda^{[n/2]+1,1}(\mathbb{R}^n)}\Big)(1+t+|x|)^{\frac{7-n}{2}}(1+|t-|x||)^{-\frac{1}{2}},\end{aligned}$$

where

$$||f||_{W_\Lambda^{N,2}(\mathbb{R}^n)}=\sum_{|a|\le N}||\Lambda^a f||_{L^2(\mathbb{R}^n)}=\sum_{|a|\le N}\Big(\int_{\mathbb{R}^n}|\Lambda^a f|^2\mathrm{d}x\Big)^{\frac{1}{2}},\quad \Lambda=(\partial,r\partial_r,x_i\partial_j-x_j\partial_i,x_i\partial_t)$$

denotes the Klainerman's vector field on $\{t=0\}$.

Remark 1.3. For the linear wave equation

$$\begin{cases}\Box u=0, & t>0,x\in\mathbb{R}^n,\\ u(x,0)=f(x), & u_t(x,0)=g(x).\end{cases}$$

We know that if u belongs to $C_0^\infty(\mathbb{R}^n)$

$$||u||_{L^\infty(\mathbb{R}^n)}\le C\big(||f||_{W^{n,1}(\mathbb{R}^n)}+||g||_{W^{n-1,1}(\mathbb{R}^n)}\big)(1+t)^{-\frac{n-1}{2}}.$$

If $u\in C^\infty(\mathbb{R}^n)$ vanishs when $|x|$ is large. Then, if $t>0$

$$||u||_{L^\infty(\mathbb{R}^n)}\le\begin{cases}C(||f||_{W^{[n/2]+2,2}(\mathbb{R}^n)}+||g||_{W^{[n/2]+1,2}(\mathbb{R}^n)})(1+t)^{-\frac{n-1}{2}}, & n \text{ is odd},\\ C(||f||_{W^{[n/2]+2,2}(\mathbb{R}^n)}+||g||_{W^{[n/2]+1,2}(\mathbb{R}^n)})(1+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{\frac{1}{2}}, & n \text{ is even}.\end{cases}$$

By Theorems 1.4 and 1.5, we see that the decay properties of biwave maps are different from the wave equation.

The paper is organized as follows: in Section 2, we give some preparations on wave equations and Fourier transformation; in Section 3, we obtain the solution formulas for (1.3) by two methods; we study the $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ estimates based on the solution formulas, and the $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates based on the weighted energy estimate in Section 4.

2 Preliminaries

In this section, we first introduce the notations and some important lemmas.

2.1 Notations

The scaling operator is defined by

$$S=t\partial_t+r\partial_r=t\partial_t+\sum_{i=1}^{n}x^i\partial_i,$$

and the angular-momentum operators are defined by

$$\Omega_{ij}=x_i\partial_j-x_j\partial_i,\quad (i,j=1,\cdots,n),$$

and we also have the Lorentz boost

$$L_{0i}=t\partial_i+x_i\partial_t,\quad (i=1,\cdots,n).$$

Combining with the original derivative $\partial=(\partial_t,\partial_1,\cdots,\partial_n)$, we get the Klainerman's vector field

$$\Gamma=(\partial,S,\Omega_{ij},L_{0i}).$$

For a function $v(x,t)$, we denote

$$\Gamma^a v=\partial^{a_1}S^{a_2}\Omega^{a_3}L^{a_4}v,\quad |a|=|a_1|+|a_2|+|a_3|+|a_4|.$$

We define

$$\begin{aligned}
&||v(\cdot,t)||_{L^2(\mathbb{R}^n)}=\Big(\int_{\mathbb{R}^n}|v(x,t)|^2\mathrm{d}x\Big)^{\frac{1}{2}},\quad ||v(\cdot,t)||_{L^\infty(\mathbb{R}^n)}=\sup_{x\in\mathbb{R}^n}|v(x,t)|.\\
&||v(\cdot,t)||_{W^{N,1}(\mathbb{R}^n)}=\sum_{|a|\le N}\int_{\mathbb{R}^n}|\partial^a v(x,t)|\mathrm{d}x,\\
&||v(\cdot,t)||_{W^{N,2}_\Gamma(\mathbb{R}^n)}=\sum_{|a|\le N}\Big(\int_{\mathbb{R}^n}|\Gamma^a v(x,t)|^2\mathrm{d}x\Big)^{\frac{1}{2}}.
\end{aligned}$$

2.2 Some important lemmas

In order to study the biwave equations, we first introduce some basic theories of linear wave equations, and these details can be found in Evans [11].

For the Cauchy problem of the linear wave equation, which takes the form

$$\begin{cases}\Box u=0,\\ u(x,0)=f(x),\quad \partial_t u(x,0)=g(x),\end{cases}\tag{2.1}$$

where $u=u(x,t)$, $x\in\mathbb{R}^n$, $f\in C^{m+1}(\mathbb{R}^n)$, $g\in C^m(\mathbb{R}^n)$, m depends on the dimensions.

Lemma 2.1 (Some useful identities)**.** *Let $\phi:\mathbb{R}\to\mathbb{R}$ be C^{k+1}, then for $k\geq 1$:*

(i) $\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1}\left(r^{2k-1}\phi(r)\right)=\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k}\left(r^{2k}\frac{\mathrm{d}\phi}{\mathrm{d}r}(r)\right)$,

(ii) $\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^{k-1}\left(r^{2k-1}\phi(r)\right)=\sum\limits_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{\mathrm{d}^j\phi}{\mathrm{d}r^j}(r)$,

where the constants $\beta_j^k(j=0,\cdots,k-1)$ are independent of ϕ, and $\beta_0^k=1\cdot 3\cdot 5\cdots(2k-1)$.

Proof. (i) Since

$$\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)=\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\right)\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}=-\frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2},$$

so

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}=r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}=r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^2+\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}.$$

When $k=1$, we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\phi(r))=&r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^2(r\phi)+\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)(r\phi)\\=&r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\phi)\right]+\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)(r\phi)\\=&\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left[r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)(r\phi)\right]-\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)(r\phi)\\=&\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r\phi+r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)-\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)(r\phi)\\=&\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right).\end{aligned}$$

When $k=2$, we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)(r^3\phi(r))=&3\frac{\mathrm{d}^2}{\mathrm{d}r^2}(r\phi)+\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\\=&3\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)+r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)+\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\\=&4\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)+\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left[r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\right]-2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\\=&\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left[r^2\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\right]+2r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big]\\=&\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left[\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(r^2\cdot r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)\right]\\=&\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)^2\left(r^4\frac{\mathrm{d}\phi}{\mathrm{d}r}\right).\end{aligned}$$

Assume $k=b$, (i) is true. i.e.,

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big)\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b-1}\Big(r^{2k-1}\phi(r)\Big)=\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}(r)\Big),$$

then when $k=b+1$, we have

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(r^{2b+1}\phi(r)\Big)=\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b-1}\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Big(r^{2b+1}\phi(r)\Big)\\
=&(2b+1)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b-1}\Big(r^{2b-1}\phi(r)\Big)+\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b-1}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)\\
=&(2b+1)\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)+r^2\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{2}\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b-1}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)+\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)\\
=&2b\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)+\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Big[r^2\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b-1}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)\Big]-2\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)\\
=&(2b-2)\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)+\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{2}\Big[r^2\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b-1}\Big(r^{2b}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)\Big]\\
=&\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b}\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)\Big(r^{2b+2}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big)\\
=&\Big(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\Big)^{b+1}\Big(r^{2b+2}\frac{\mathrm{d}\phi}{\mathrm{d}r}\Big).
\end{aligned}$$

Thus, the proof of (i) is completed, the proof of (ii) can be found in [12]. □

Lemma 2.2 (The one-dimensional wave equation). *For $n=1$, $m=1$, the solution can be obtained by D'Alembert's formula:*

$$u(x,t)=\frac{1}{2}[f(x+t)+f(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}g(y)\mathrm{d}y, \tag{2.2}$$

where (2.2) *can also be written as*

$$u(x,t)=\frac{\partial}{\partial t}(tA_tf)+tA_tg, \tag{2.3}$$

where $A_tf(x)=\frac{1}{2t}\int_{x-t}^{x+t}f(\xi)\mathrm{d}\xi$ is the integral average of the function f on the interval $[x-t,x+t]$.

Lemma 2.3 (Solution formula of wave equation in odd dimensions). *For $n\geq 3$, $m=\frac{n+1}{2}$, we have the representation formula:*

$$u(x,t)=\frac{1}{\gamma_n}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}⨍_{\partial B(x,t)}f\mathrm{d}S\Big)+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}⨍_{\partial B(x,t)}g\mathrm{d}S\Big)\Big]. \tag{2.4}$$

We can also rewrite (2.4) *as*

$$u(x,t)=\frac{1}{\gamma_n}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf(x)\Big)+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tg(x)\Big)\Big], \tag{2.5}$$

where $\gamma_n=1\cdot 3\cdot 5\cdots(n-2)$.

Lemma 2.4. (Solution formula of wave equation in even dimensions) *For $n\geq 2$, $m=\frac{n+2}{2}$, we have the following formula:*

$$u(x,t)=\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y + \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{g(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big]. \tag{2.6}$$

For the inhomogeneous wave equation with zero initial data,

$$\begin{cases} u_{tt}-\triangle u=h, & \text{in } \mathbb{R}^n\times(0,\infty),\\ u=0,\quad u_t=0, & \text{on } \mathbb{R}^n\times(t=0). \end{cases} \tag{2.7}$$

Lemma 2.5. *Motivated by Duhamel's principle, assume $u=u(x,t;s)$ is the solution to*

$$\begin{cases} u_{tt}(\cdot;s)-\triangle u(\cdot;s)=0, & \text{in } \mathbb{R}^n\times(s,\infty),\\ u(\cdot;s)=0,\quad u_t(\cdot;s)=h(\cdot,s), & \text{on } \mathbb{R}^n\times(t=s), \end{cases} \tag{2.8}$$

then the solution of (2.7) *is given by*

$$u(x,t)=\int_0^t u(x,t;s)\mathrm{d}s \quad (x\in\mathbb{R}^n,t\geq 0).$$

Obviously, combining Lemmas 2.2-2.5, we have

Lemma 2.6. *If $n=1$, we get the solution of* (2.7) *is*

$$u(x,t)=\frac{1}{2}\int_0^t\int_{x-t'}^{x+t'}h(y,s)\mathrm{d}y\mathrm{d}s=\int_0^t t'A_{t'}h(x,s)\mathrm{d}s; \tag{2.9}$$

if n is an odd integer and $n\geq 3$, we can obtain the solution formula of (2.7) *is*

$$u(x,t)=\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-3}{2}}\Big((t')^{n-2}A_{t'}h(x,s)\Big)\mathrm{d}s; \tag{2.10}$$

if n is an even integer and $n\geq 2$, the solution formula of (2.7) *is*

$$u(x,t)=\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-2}{2}}(t')^{n-1}\int_{|y|<1}\frac{h(x+t'y,s)}{\sqrt{1-|y|^2}}\mathrm{d}y\mathrm{d}s, \tag{2.11}$$

where $t'=t-s$.

The next lemma, which can be found in [12], plays an important role in getting the $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ estimates of the biwave equations.

Lemma 2.7. *Let $f\in C_0^\infty(\mathbb{R}^n)$, $g\in C_0^\infty(\mathbb{R}^n)$, $u(x,t)$ is the solution of* (2.1)*. We have*

$$|u(x,t)|\leq C\big(||f||_{W^{n,1}(\mathbb{R}^n)}+||g||_{W^{n-1,1}(\mathbb{R}^n)}\big)(1+t)^{-\frac{(n-1)}{2}},$$

if $t>1$,

$$|u(x,t)|\leq C\big(||f||_{W^{[n/2]+1,1}(\mathbb{R}^n)}+||g||_{W^{[n/2],1}(\mathbb{R}^n)}\big)t^{-\frac{(n-1)}{2}}.$$

In the meantime, in order to apply the method of Fourier transformation, we need to introduce some theories of Fourier analysis, see [5] and references therein.

Lemma 2.8. *We have*

$$\begin{aligned}
&\cos(|\xi|t)=\frac{e^{i|\xi|t}+e^{-i|\xi|t}}{2},\quad \sin(|\xi|t)=\frac{e^{i|\xi|t}-e^{-i|\xi|t}}{2i},\\
&\frac{\sin(|\xi|t)}{|\xi|}=\frac{e^{i|\xi|t}-e^{-i|\xi|t}}{2i|\xi|}=\frac{1}{2}\int_{-t}^{t}e^{i|\xi|\theta}\mathrm{d}\theta,\\
&\frac{\cos(|\xi|t)}{|\xi|^2}=\frac{e^{i|\xi|t}+e^{-i|\xi|t}}{2|\xi|^2}=-\frac{1}{2}\int_0^t\int_0^y e^{i|\xi|u}\mathrm{d}u\mathrm{d}y-\frac{1}{2}\int_0^t\int_0^y e^{-i|\xi|u}\mathrm{d}u\mathrm{d}y+\frac{1}{|\xi|^2},\\
&\frac{\sin(|\xi|t)}{|\xi|^3}=\frac{e^{i|\xi|t}-e^{-i|\xi|t}}{2i|\xi|^3}=-\frac{1}{2}\int_{-t}^{t}\int_0^y\int_0^\tau e^{i|\xi|u}\mathrm{d}u\mathrm{d}\tau\mathrm{d}y+\frac{t}{|\xi|^2}.
\end{aligned}$$

Furthermore,

$$\widehat{\delta}(x-\alpha t)=\int_{-\infty}^{+\infty}e^{-i|\xi|x}\delta(x-\alpha t)\mathrm{d}x=e^{-i\alpha|\xi|t},$$

where $\delta(x)$ is the Dirac delta function.

Lemma 2.9. *Assume that n is an odd integer and $n\geq 3$, then*

$$\frac{\sin(|\xi|t)}{|\xi|}=\frac{1}{2}\int_{-t}^{t}e^{is|\xi|}\mathrm{d}s=c_n\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(0,t)}e^{-ix\cdot\xi}\mathrm{d}\sigma(x)\Big),\tag{2.12}$$

where $t>0$, ρ_n is the surface measure of the unit ball in $\mathbb{R}^n$ and $c_n^{-1}=(n-2)(n-4)\cdots 1$.

On the other hand, if n is an even integer and $n\geq 2$, then

$$\frac{\sin(|\xi|t)}{|\xi|}=\frac{1}{2}\int_{-t}^{t}e^{is|\xi|}\mathrm{d}s=d_n\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(\frac{1}{v_n}\int_{B(0,t)}\frac{1}{(t^2-|x|^2)^{1/2}}e^{-ix\cdot\xi}\mathrm{d}x\Big),\tag{2.13}$$

where $t>0$, v_n is the volume of the unit ball in $\mathbb{R}^n$, $d_n^{-1}=n(n-2)(n-4)\cdots 2$ and $\rho_n=nv_n$.

Remark 2.1. For every $\varphi\in S(\mathbb{R}^n)$, we notice that

$$\begin{aligned}
\frac{1}{\rho_n R^{n-1}}\int_{\partial B(x,R)}\varphi(y)\mathrm{d}\sigma(y)&=⨍_{\partial B(x,R)}\varphi(y)\mathrm{d}\sigma(y)=⨍_{\partial B(0,1)}\varphi(x+Ry)\mathrm{d}\sigma(y)\\
&=\frac{1}{\omega_{n-1}}\int_{S^{n-1}}\varphi(x+Ry)\mathrm{d}\sigma(y)=A_R\varphi(x).
\end{aligned}$$

In order to get the $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates of the biwave equations, we require the following Klainerman Sobolev inequality and Hardy inequality, which can be found in [1].

Lemma 2.10. *Suppose that $u=u(x,t)$ is a function with compact support in the variable x for any fixed $t\geq 0$. Then, for any integer $M\geq 0$, we have*

$$\sum_{|\alpha|\leq M}|\Gamma^\alpha u(\cdot,t)|_{L^\infty(\mathbb{R}^n)}\lesssim(1+t+|x|)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}}\sum_{|\alpha|\leq M+[\frac{n}{2}]+1}||\Gamma^\alpha u(\cdot,t)||_{L^2(\mathbb{R}^n)}.$$

Lemma 2.11. *If $n\geq 1$, we assume that $u(x,t)=0$ on $|x|\geq 1+t$, then*

$$||u(\cdot,t)||_{L^2(\mathbb{R}^n)}\lesssim||\nabla u(\cdot,t)||_{L^2(\mathbb{R}^n)}(1+t).$$

3 Proof of Theorems 1.1-1.3

In this section, we give the solution formulas for the biwave equations (1.3) by exploiting the Duhamel's principle and the Fourier analysis, respectively.

3.1 The method of Duhamel's principle

Introducing a new variable $v=\Box u$, then (1.3) is equivalent to

$$\begin{cases}\Box u=v,\\ u(x,0)=f_1(x),\\ u_t(x,0)=f_2(x),\end{cases}\tag{3.1}$$

$$\begin{cases}\Box v=0,\\ v(x,0)=f_3(x)-\triangle f_1(x):=g_1(x),\\ v_t(x,0)=f_4(x)-\triangle f_2(x):=g_2(x).\end{cases}\tag{3.2}$$

By Duhamel's principle, we have $u=\overline{u}+\widetilde{u}$, where

$$\begin{cases}\Box\overline{u}=0,\\ \overline{u}(x,0)=f_1(x),\\ \overline{u}_t(x,0)=f_2(x),\end{cases}\tag{3.3}$$

$$\begin{cases}\Box\widetilde{u}=v,\\ \widetilde{u}(x,0)=0,\\ \widetilde{u}_t(x,0)=0.\end{cases}\tag{3.4}$$

We discuss the solution formulas of different dimensions in different cases.

Case I: $n=1$**.**

By Lemma 2.2, if $n=1$, the solution of (3.2) is

$$v(x,t)=\frac{1}{2}[g_1(x+t)+g_1(x-t)]+\frac{1}{2}\int_{x-t}^{x+t} g_2(y)\mathrm{d}y. \tag{3.5}$$

The solution of (3.3) is

$$\overline{u}(x,t)=\frac{1}{2}[f_1(x+t)+f_1(x-t)]+\frac{1}{2}\int_{x-t}^{x+t} f_2(y)\mathrm{d}y. \tag{3.6}$$

For (3.4), by Lemma 2.6, if $n=1$, we have

$$\widetilde{u}(x,t)=\frac{1}{2}\int_0^t\int_{x-t'}^{x+t'} v(y,s)\mathrm{d}y\mathrm{d}s. \tag{3.7}$$

Inserting (3.6)-(3.7) into (3.1), we get

$$\begin{aligned} u(x,t)=&\overline{u}(x,t)+\widetilde{u}(x,t)\\ =&\frac{1}{2}[f_1(x+t)+f_1(x-t)]+\frac{1}{2}\int_{x-t}^{x+t} f_2(y)\mathrm{d}y+\frac{1}{2}\int_0^t\int_{x-t'}^{x+t'} v(y,s)\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} v(y,s)=&\frac{1}{2}[f_3(y+s)+f_3(y-s)]-\frac{1}{2}[\triangle f_1(y+s)+\triangle f_1(y-s)]\\ &+\frac{1}{2}\int_{y-s}^{y+s} f_4(\omega)\mathrm{d}\omega-\frac{1}{2}\int_{y-s}^{y+s}\triangle f_2(\omega)\mathrm{d}\omega, \end{aligned}$$

and $t'=t-s\geq 0$.

Case II: $n\geq 3$ **and** n **is odd.**

If n is odd and $n\geq 3$, by Lemma 2.3, the solution of (3.2) is

$$\begin{aligned} v(x,t)=&\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_t g_1(x)\Big)\\ &+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_t g_2(x)\Big)\Big]. \end{aligned} \tag{3.9}$$

The solution of (3.3) is

$$\begin{aligned} \overline{u}(x,t)=&\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_t f_1(x)\Big)\\ &+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_t f_2(x)\Big)\Big]. \end{aligned} \tag{3.10}$$

For (3.4), by Lemma 2.6, if n is odd and $n\geq 3$, we have

$$\widetilde{u}(x,t)=\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-3}{2}}\Big((t')^{n-2}A_{t'}v(x,s)\Big)\mathrm{d}s. \tag{3.11}$$

Combining (3.10) and (3.11), when n is odd and $n\geq 3$, we get

$$\begin{aligned}u(x,t)=&\overline{u}(x,t)+\widetilde{u}(x,t)\\=&\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_1(x)\Big)+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_2(x)\Big)\\&+\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-3}{2}}\Big((t')^{n-2}A_{t'}v(x,s)\Big)\mathrm{d}s\Big],\end{aligned} \tag{3.12}$$

where

$$v(x,s)=\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_sg_1(x)\Big)+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_sg_2(x)\Big)\Big],$$

and $g_1=f_3-\triangle f_1$, $g_2=f_4-\triangle f_2$.

Case III: $n\geq 2$ **and** n **is even.**

If n is even and $n\geq 2$, by Lemma 2.4, the solution of (3.2) is

$$\begin{aligned}v(x,t)=&\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{g_1(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y\\&+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{g_2(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big].\end{aligned} \tag{3.13}$$

The solution of (3.3) is

$$\begin{aligned}\overline{u}(x,t)=&\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f_1(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y\\&+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f_2(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big].\end{aligned} \tag{3.14}$$

For (3.4), by Lemma 2.6, if n is even and $n\geq 2$, we have

$$\widetilde{u}(x,t)=\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-2}{2}}(t')^{n-1}\int_{|y|<1}\frac{v(x+t'y,s)}{\sqrt{1-|y|^2}}\mathrm{d}y\mathrm{d}s. \tag{3.15}$$

Therefore, when n is even and $n\geq 2$, combining (3.14) and (3.15), we get

$$u(x,t)=\overline{u}(x,t)+\widetilde{u}(x,t)$$

$$
\begin{aligned}
=&\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f_1(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y\\
&+\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}t^{n-1}\int_{|y|<1}\frac{f_2(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y\\
&+\int_0^t\Big(\frac{1}{t'}\frac{\partial}{\partial t'}\Big)^{\frac{n-2}{2}}(t')^{n-1}\int_{|y|<1}\frac{v(x+t'y,s)}{\sqrt{1-|y|^2}}\mathrm{d}y\mathrm{d}s\Big],
\end{aligned} \tag{3.16}
$$

where

$$
\begin{aligned}
v(x,s)=&\frac{2}{1\cdot 3\cdot 5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{g_1(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\\
&+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big].
\end{aligned}
$$

Thus, combining (3.8), (3.12) and (3.16), proof of the Theorems 1.1-1.3 is completed. □

3.2 The method of Fourier analysis

In this subsection, we restudy (1.3) by the method of Fourier analysis.

Definition 3.1. *The Fourier transform of Schwarz function ϕ is defined by*

$$
\mathcal{F}[\phi](\xi)=\widehat{\phi}(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\phi(x)\mathrm{d}x. \tag{3.17}
$$

The convolution of two integrable functions ϕ and ψ is written as $\phi*\psi$, i.e.,

$$
(\phi*\psi)(t)=\int_{\mathbb{R}^n}\phi(\tau)\psi(t-\tau)\mathrm{d}\tau. \tag{3.18}
$$

The Fourier's inversion formula for Schwarz function ϕ is defined by

$$
\mathcal{F}^{-1}[\phi](x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\widehat{\phi}(\xi)\mathrm{d}\xi. \tag{3.19}
$$

Eq. (1.3) can be rewritten as a fourth-order PDE, which has the following form

$$
\frac{\partial^4 u}{\partial t^4}-2\frac{\partial^2}{\partial t^2}\triangle u+\triangle^2 u=0. \tag{3.20}
$$

Taking Fourier transform on both sides of Eq. (3.20) over $\mathbb{R}^n$, we obtain

$$
\frac{\partial^4}{\partial t^4}\widehat{u}(\xi,t)+2|\xi|^2\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t)+|\xi|^4\widehat{u}(\xi,t)=0. \tag{3.21}
$$

Solving (3.21), we have

$$\widehat{u}(\xi,t)=(C_1t+C_2)\cos(|\xi|t)+(D_1t+D_2)\sin(|\xi|t), \tag{3.22}$$

where C_1, C_2, D_1 *and* D_2 are determined by the initial conditions:

$$\begin{cases} \widehat{u}(\xi,t)|_{t=0}=C_2=\widehat{f}_1(\xi), \\ \dfrac{\partial\widehat{u}(\xi,t)}{\partial t}\Big|_{t=0}=C_1+D_2|\xi|=\widehat{f}_2(\xi), \\ \dfrac{\partial^2\widehat{u}(\xi,t)}{\partial t^2}\Big|_{t=0}=-C_2|\xi|^2+2D_1|\xi|=\widehat{f}_3(\xi), \\ \dfrac{\partial^3\widehat{u}(\xi,t)}{\partial t^3}\Big|_{t=0}=-3C_1|\xi|^2-D_2|\xi|^3=\widehat{f}_4(\xi). \end{cases}$$

Then we have

$$\begin{cases} C_1=-\dfrac{1}{2}\widehat{f}_2(\xi)-\dfrac{1}{2}\dfrac{\widehat{f}_4(\xi)}{|\xi|^2}, \\ C_2=\widehat{f}_1(\xi), \\ D_1=\dfrac{1}{2}|\xi|\widehat{f}_1(\xi)+\dfrac{1}{2}\dfrac{\widehat{f}_3(\xi)}{|\xi|}, \\ D_2=\dfrac{3}{2}\dfrac{\widehat{f}_2(\xi)}{|\xi|}+\dfrac{1}{2}\dfrac{\widehat{f}_4(\xi)}{|\xi|^3}. \end{cases}$$

Finally, we get the expression of $\widehat{u}(\xi,t)$

$$\begin{aligned}\widehat{u}(\xi,t)=&\Big(-\frac{1}{2}\widehat{f}_2(\xi)t-\frac{1}{2}\frac{\widehat{f}_4(\xi)}{|\xi|^2}t+\widehat{f}_1(\xi)\Big)\cos(|\xi|t)\\ &+\Big(\frac{1}{2}|\xi|\widehat{f}_1(\xi)t+\frac{1}{2}\frac{\widehat{f}_3(\xi)}{|\xi|}t+\frac{3}{2}\frac{\widehat{f}_2(\xi)}{|\xi|}+\frac{1}{2}\frac{\widehat{f}_4(\xi)}{|\xi|^3}\Big)\sin(|\xi|t). \end{aligned}\tag{3.23}$$

Next, we discuss the Fourier transform formulas of different dimensions in different cases.

Case I: $n=1$.

By Lemma 2.8, we know

$$\Big(\frac{e^{i|\xi|t}+e^{-i|\xi|t}}{2}\Big)\widehat{f}_1(\xi)=\frac{\mathcal{F}[\delta(x+t)*f_1(x)]+\mathcal{F}[\delta(x-t)*f_1(x)]}{2}, \tag{3.24}$$

$$\begin{aligned}|\xi|\widehat{f}_1(\xi)\frac{e^{i|\xi|t}-e^{-i|\xi|t}}{2i}&=|\xi|^2\widehat{f}_1(\xi)\frac{e^{i|\xi|t}-e^{-i|\xi|t}}{2i|\xi|}\\ &=|\xi|^2\widehat{f}_1(\xi)\Big(\frac{1}{2}\int_{-t}^{t}e^{i|\xi|\theta}\mathrm{d}\theta\Big)=\frac{1}{2}\int_{-t}^{t}\mathcal{F}[\delta(x+\theta)*\triangle f_1(x)]\mathrm{d}\theta, \end{aligned}\tag{3.25}$$

$$\widehat{f}_2(\xi)\Big(\frac{1}{2}\int_{-t}^{t}e^{i|\xi|\theta}\mathrm{d}\theta\Big)=\frac{1}{2}\int_{-t}^{t}\mathcal{F}[\delta(x+\theta)*f_2(x)]\mathrm{d}\theta, \tag{3.26}$$

$$\widehat{f}_4(\xi)\Big(-\frac{1}{2}\int_0^t\int_0^y e^{i|\xi|u}\mathrm{d}u\mathrm{d}y-\frac{1}{2}\int_0^t\int_0^y e^{-i|\xi|u}\mathrm{d}u\mathrm{d}y\Big)$$
$$=\Big(-\frac{1}{2}\int_0^t\int_0^y\mathcal{F}[\delta(x+u)*f_4(x)]\mathrm{d}u\mathrm{d}y-\frac{1}{2}\int_0^t\int_0^y\mathcal{F}[\delta(x-u)*f_4(x)]\mathrm{d}u\mathrm{d}y\Big), \tag{3.27}$$

$$\widehat{f}_4(\xi)\Big(-\frac{1}{2}\int_{-t}^t\int_0^y\int_0^\tau e^{i|\xi|u}\mathrm{d}u\mathrm{d}\tau\mathrm{d}y\Big)=-\frac{1}{2}\int_{-t}^t\int_0^y\int_0^\tau\mathcal{F}[\delta(x+u)*f_4(x)]\mathrm{d}u\mathrm{d}\tau\mathrm{d}y. \tag{3.28}$$

Substituting (3.24)-(3.28) into (3.23), we obtain

$$\begin{aligned}\widehat{u}(\xi,t)=&-\frac{1}{2}t\frac{\mathcal{F}[\delta(x+t)*f_2(x)]+\mathcal{F}[\delta(x-t)*f_2(x)]}{2}\\&-\frac{1}{2}t\Big(-\frac{1}{2}\int_0^t\int_0^y\mathcal{F}[\delta(x+u)*f_4(x)]\mathrm{d}u\mathrm{d}y-\frac{1}{2}\int_0^t\int_0^y\mathcal{F}[\delta(x-u)*f_4(x)]\mathrm{d}u\mathrm{d}y\Big)\\&+\frac{\mathcal{F}[\delta(x+t)*f_1(x)]+\mathcal{F}[\delta(x-t)*f_1(x)]}{2}+\frac{1}{4}t\int_{-t}^t\mathcal{F}[\delta(x+\theta)*\triangle f_1(x)]\mathrm{d}\theta\\&+\frac{1}{4}t\int_{-t}^t\mathcal{F}[\delta(x+\theta)*f_3(x)]\mathrm{d}\theta+\frac{3}{4}\int_{-t}^t\mathcal{F}[\delta(x+\theta)*f_2(x)]\mathrm{d}\theta\\&-\frac{1}{4}\int_{-t}^t\int_0^y\int_0^\tau\mathcal{F}[\delta(x+u)*f_4(x)]\mathrm{d}u\mathrm{d}\tau\mathrm{d}y.\end{aligned} \tag{3.29}$$

Taking the inverse Fourier transform on (3.29), we have

$$\begin{aligned}u(x,t)=&-\frac{1}{2}t\frac{(\delta(x+t)*f_2(x))+(\delta(x-t)*f_2(x))}{2}\\&-\frac{1}{2}t\Big(-\frac{1}{2}\int_0^t\int_0^y(\delta(x+u)*f_4(x))\mathrm{d}u\mathrm{d}y-\frac{1}{2}\int_0^t\int_0^y(\delta(x-u)*f_4(x))\mathrm{d}u\mathrm{d}y\Big)\\&+\frac{(\delta(x+t)*f_1(x))+(\delta(x-t)*f_1(x))}{2}+\frac{1}{4}t\int_{-t}^t(\delta(x+\theta)*\triangle f_1(x))\mathrm{d}\theta\\&+\frac{1}{4}t\int_{-t}^t(\delta(x+\theta)*f_3(x))\mathrm{d}\theta+\frac{3}{4}\int_{-t}^t(\delta(x+\theta)*f_2(x))\mathrm{d}\theta\\&-\frac{1}{4}\int_{-t}^t\int_0^y\int_0^\tau(\delta(x+u)*f_4(x))\mathrm{d}u\mathrm{d}\tau\mathrm{d}y\\=&-\frac{1}{4}t[f_2(x+t)+f_2(x-t)]+\frac{1}{4}t\int_{x-t}^{x+t}\int_0^y f_4(u)\mathrm{d}u\mathrm{d}y\\&+\frac{1}{2}[f_1(x+t)+f_1(x-t)]+\frac{1}{4}t\int_{x-t}^{x+t}\triangle f_1(y)\mathrm{d}y+\frac{1}{4}t\int_{x-t}^{x+t}f_3(z)\mathrm{d}z\\&+\frac{3}{4}\int_{x-t}^{x+t}f_2(\theta)\mathrm{d}\theta-\frac{1}{4}\int_{x-t}^{x+t}\int_0^y\int_0^\tau f_4(\omega)\mathrm{d}\omega\mathrm{d}\tau\mathrm{d}y.\end{aligned} \tag{3.30}$$

Case II: $n\geq 3$ **and** n **is odd.**

According to Lemma 2.9, differentiating equation (2.12) with respect to t, we have

$$\cos(|\xi|t)=c_n\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(0,t)}e^{-ix\cdot\xi}\mathrm{d}\sigma(x)\Big). \tag{3.31}$$

For any function $\theta\in S(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{\sin(|\xi|t)}{|\xi|}\theta(\xi)\mathrm{d}\xi=&c_n\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(0,t)}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}\mathrm{d}\xi\mathrm{d}\sigma(x)\Big)\\ =&c_n\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(0,t)}\widehat{\theta}(x)\mathrm{d}\sigma(x)\Big).\end{aligned}\tag{3.32}$$

By (3.32), we conclude that

$$c_n\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\widehat{\int_{\partial B(0,t)}\mathrm{d}\sigma(x)}\Big)(\xi)=\frac{\sin(|\xi|t)}{|\xi|}.$$

By the Fourier inversion and convolution formulas, we can obtain

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_2}(\xi)\frac{\sin(|\xi|t)}{|\xi|}e^{i\xi\cdot x}\mathrm{d}\xi=c_n\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(x,t)}f_2(y)\mathrm{d}\sigma(y)\Big)\\ =&\frac{1}{(n-2)!!}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_2(x)\Big),\end{aligned}\tag{3.33}$$

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_2}(\xi)\cos(|\xi|t)e^{i\xi\cdot x}\mathrm{d}\xi=c_n\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(x,t)}f_2(y)\mathrm{d}\sigma(y)\Big)\\ =&\frac{1}{(n-2)!!}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_2(x)\Big),\end{aligned}\tag{3.34}$$

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_1}(\xi)\cos(|\xi|t)e^{i\xi\cdot x}\mathrm{d}\xi=c_n\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(x,t)}f_1(y)\mathrm{d}\sigma(y)\Big)\\ =&\frac{1}{(n-2)!!}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_1(x)\Big),\end{aligned}\tag{3.35}$$

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_4}(\xi)\frac{\cos(|\xi|t)}{|\xi|^2}e^{i\xi\cdot x}\mathrm{d}\xi=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_4}(\xi)\Big[-\int_0^t\frac{\sin(s|\xi|)}{|\xi|}\mathrm{d}s+\frac{1}{|\xi|^2}\Big]e^{i\xi\cdot x}\mathrm{d}\xi\\ =&-\int_0^t c_n\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n s}\int_{\partial B(x,s)}f_4(y)\mathrm{d}\sigma(y)\Big)\mathrm{d}s+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_4}(\xi)\frac{1}{|\xi|^2}e^{i\xi\cdot x}\mathrm{d}\xi\\ =&-\int_0^t\frac{1}{(n-2)!!}\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_tf_4(x)\Big)\mathrm{d}s+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_4}(\xi)\frac{1}{|\xi|^2}e^{i\xi\cdot x}\mathrm{d}\xi,\end{aligned}\tag{3.36}$$

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_3}(\xi)\frac{\sin(|\xi|t)}{|\xi|}e^{i\xi\cdot x}\mathrm{d}\xi=c_n\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(x,t)}f_3(y)\mathrm{d}\sigma(y)\Big)\\ =&\frac{1}{(n-2)!!}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_3(x)\Big),\end{aligned}\tag{3.37}$$

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_4}(\xi)\frac{\sin(|\xi|t)}{|\xi|^3}e^{i\xi\cdot x}\mathrm{d}\xi=-\int_0^t\int_0^v c_n\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n s}\int_{\partial B(x,s)}f_4(y)\mathrm{d}\sigma(y)\Big)\mathrm{d}s\mathrm{d}v\\ &\quad+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f_4}(\xi)\frac{t}{|\xi|^2}e^{i\xi\cdot x}\mathrm{d}\xi\end{aligned}$$

$$=-\int_0^t\int_0^v \frac{1}{(n-2)!!}\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_sf_4(x)\Big)\mathrm{d}s\mathrm{d}v+\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f}_4(\xi)\frac{t}{|\xi|^2}e^{i\xi\cdot x}\mathrm{d}\xi, \tag{3.38}$$

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f}_1(\xi)|\xi|\sin(|\xi|t)e^{i\xi\cdot x}\mathrm{d}\xi=-\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f}_1(\xi)\Big(\frac{\partial^2}{\partial t^2}\Big)\frac{\sin(|\xi|t)}{|\xi|}e^{i\xi\cdot x}\mathrm{d}\xi$$

$$=-c_n\Big(\frac{\partial^2}{\partial t^2}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(\frac{1}{\rho_n t}\int_{\partial B(x,t)}f_1(y)\mathrm{d}\sigma(y)\Big)$$

$$=-\frac{1}{(n-2)!!}\Big(\frac{\partial^2}{\partial t^2}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_1(x)\Big). \tag{3.39}$$

Combining (3.33)-(3.38) and (3.39), we can find the solution formula for the n-dimensional biwave equation

$$\begin{aligned}u(x,t)=\frac{1}{(n-2)!!}\Big[&\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_1(x)\Big)-\frac{t}{2}\Big(\frac{\partial^2}{\partial t^2}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_1(x)\Big)\\&+\frac{3}{2}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_2(x)\Big)-\frac{t}{2}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_2(x)\Big)\\&+\frac{t}{2}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-3}{2}}\Big(t^{n-2}A_tf_3(x)\Big)+\frac{t}{2}\int_0^t\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_sf_4(x)\Big)\mathrm{d}s\\&-\frac{t}{2}\int_0^t\int_0^v\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_sf_4(x)\Big)\mathrm{d}s\mathrm{d}v\Big].\end{aligned}$$

Case III: $n\geq 2$ **and** n **is even.**

Lemma 3.1. *Given a function* $\varphi:\mathbb{R}^{n+1}\to\mathbb{R}$*, which does not depend on the last variable, i.e.,* $\varphi(x_1,x_2,\cdots,x_{n+1})=\psi(x_1,x_2,\cdots,x_n)$*, then*

$$A_t\varphi(x,0)=\frac{2}{\rho_{n+1}}\int_{B_n(0,1)}\frac{\psi(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y.$$

Moreover,

$$\widetilde{A}_t\varphi(x)=\frac{2}{\rho_{n+1}}\int_{B_n(0,1)}\frac{\varphi(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y,$$

which is called as modified spherical mean of φ*.*

By Remark 2.1, we know $\rho_{n+1}=\omega_{n-1}$, then

$$\widetilde{A}_t\varphi(x)=\frac{2}{\rho_{n+1}}\int_{B_n(0,1)}\frac{\varphi(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y=\frac{2}{\omega_n}\int_{|y|<1}\frac{\varphi(x+ty)}{\sqrt{1-|y|^2}}\mathrm{d}y.$$

The Hadamard's method of descent gives the solution formula of the biwave equation in the even dimensional space $\mathbb{R}^n$, we can get

$$u(x,t)=\frac{1}{(n-1)!!}\Big[\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^{n-1}\widetilde{A}_tf_1(x)\Big)-\frac{t}{2}\Big(\frac{\partial^2}{\partial t^2}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^{n-1}\widetilde{A}_tf_1(x)\Big)$$

$$+\frac{3}{2}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^{n-1}\widetilde{A}_t f_2(x)\Big)-\frac{t}{2}\Big(\frac{\partial}{\partial t}\Big)\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^{n-1}\widetilde{A}_t f_2(x)\Big)$$
$$+\frac{t}{2}\Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{\frac{n-2}{2}}\Big(t^{n-1}\widetilde{A}_t f_3(x)\Big)+\frac{t}{2}\int_0^t\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}\Big(s^{n-1}\widetilde{A}_s f_4(x)\Big)\mathrm{d}s$$
$$-\frac{t}{2}\int_0^t\int_0^v\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}\Big(s^{n-1}\widetilde{A}_s f_4(x)\Big)\mathrm{d}s\mathrm{d}v\Big].$$

4 $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates of the biwave equations

4.1 $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ estimates

In this subsection, we mainly discuss $L^\infty(\mathbb{R}^n)-W^{N,1}(\mathbb{R}^n)$ estimates of the biwave equations, with the help of Lemma 2.7 and the solution formulas (3.8), (3.12), (3.16), we are interested in the case $t>1$.

By Lemma 2.7 and the solution formulas (3.5), (3.9), (3.13), we get

$$|v(x,t)|\leq C\big(||g_1|||_{W^{[n/2]+1,1}(\mathbb{R}^n)}+||g_2||_{W^{[n/2],1}(\mathbb{R}^n)}\big)t^{-\frac{(n-1)}{2}}. \tag{4.1}$$

Then, we have

$$|u(x,t)|\leq C\big(||f_1||_{W^{[n/2]+1,1}(\mathbb{R}^n)}+||f_2||_{W^{[n/2],1}(\mathbb{R}^n)}\big)t^{-\frac{(n-1)}{2}}$$
$$+C\int_0^t||v(x,s)||_{W^{[n/2],1}(\Omega)}(t-s)^{-\frac{(n-1)}{2}}\mathrm{d}s, \tag{4.2}$$

where

$$||v(\cdot,s)||_{W^{[n/2],1}(\Omega)}=\int_\Omega\sum_{|\alpha|\leq n/2}|D^\alpha v(x,s)|\mathrm{d}x,\quad \alpha=(\alpha_1,\cdots,\alpha_n),$$
$$D^\alpha=\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},\quad |\alpha|=\alpha_1+\cdots+\alpha_n,\quad \Omega\subset\mathbb{R}^n$$

is an open set.

Case I: $n=1$.

Combining (3.5) and (3.8), we have

$$u(x,t)=\frac{1}{2}[f_1(x+t)+f_1(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}f_2(y)\mathrm{d}y+\frac{1}{4}\int_0^t\int_{x-t'}^{x+t'}[f_3(y+s)+f_3(y-s)]\mathrm{d}y\mathrm{d}s$$
$$-\frac{1}{4}\int_0^t\int_{x-t'}^{x+t'}[\triangle f_1(y+s)+\triangle f_1(y-s)]\mathrm{d}y\mathrm{d}s+\frac{1}{4}\int_0^t\int_{x-t'}^{x+t'}\int_{y-s}^{y+s}f_4(\omega)\mathrm{d}\omega\mathrm{d}y\mathrm{d}s$$
$$-\frac{1}{4}\int_0^t\int_{x-t'}^{x+t'}\int_{y-s}^{y+s}\triangle f_2(\omega)\mathrm{d}\omega\mathrm{d}y\mathrm{d}s.$$

Then, we get directly

$$
\begin{aligned}
|u(x,t)| \le & C\big(||f_1||_{L^\infty(\mathbb{R})}+||f_2||_{L^1(\mathbb{R})}\big)+C||f_4||_{L^1(\mathbb{R})}t^2 \\
& +C\big(||f_2||_{L^\infty(\mathbb{R})}+||f_1||_{W^{2,1}(\mathbb{R})}+||f_3||_{L^1(\mathbb{R})}\big)t.
\end{aligned} \tag{4.3}
$$

Case II: $n\ge 3$ **and** n **is odd.**

By (3.9), we know

$$
v(x,s)=\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_s g_1(x)\Big)+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}A_s g_2(x)\Big)\Big].
$$

Then

$$
\begin{aligned}
D^\alpha v(x,s)=&\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}D^\alpha A_s g_1(x)\Big) \\
&+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}D^\alpha A_s g_2(x)\Big)\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
D^\alpha A_s g_1(x)=&\frac{1}{\omega_{n-1}}\int_{|\xi|=1}D^\alpha g_1(x+s\xi)\mathrm{d}\xi, \\
D^\alpha A_s g_2(x)=&\frac{1}{\omega_{n-1}}\int_{|\xi|=1}D^\alpha g_2(x+s\xi)\mathrm{d}\xi.
\end{aligned}
$$

Similar to [12], we assume $g_1=0$, by Lemma 2.1 (ii), we know

$$
D^\alpha v(x,s)=\frac{1}{1\cdot 3\cdot 5\cdots(n-2)}\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-3}{2}}\Big(s^{n-2}D^\alpha A_s g_2(x)\Big)=\sum_{k=0}^{\frac{n-3}{2}}a_k s^{k+1}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha(A_s g_2(x)),
$$

and

$$
\begin{aligned}
\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha\Big(A_s g_2(x)\Big)=&-\int_s^\infty\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\Big)^{k+1}D^\alpha\Big(A_s g_2(x)\Big)\mathrm{d}(\tau-s) \\
=&\frac{(-1)^{k-j}}{(j-k-1)!}\int_s^\infty(\tau-s)^{j-k-1}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\Big)^j D^\alpha(A_\tau g_2(x))\mathrm{d}\tau.
\end{aligned}
$$

Then, we have

$$
\Big|s^{k+1}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha(A_s g_2(x))\Big|\le C\int_s^\infty s^{k+1}(\tau-s)^{j-k-1}\Big|\int_{|\xi|=1}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\Big)^j D^\alpha g_2(x+\tau\xi)\mathrm{d}\xi\Big|\mathrm{d}\tau,
$$

due to $\mathrm{d}x=\tau^{n-1}\mathrm{d}\xi$, when $\tau\ge s$,

$$
s^{k+1}(\tau-s)^{j-k-1}\le s^{k+1}\tau^{j-k-1}=s^{k+1}\tau^{j-k-1-(n-1)}\tau^{n-1}\le C\tau^{n-1}s^m,
$$

where $m+n=j+1$, $j\leq n+k$. On the other hand, due to $j\geq k+1$, if $k+1\leq j\leq n+k$, we can obtain

$$\begin{aligned}s^{k+1}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k\Big|D^\alpha(A_s g_2(x))\Big|\leq& C\int_s^\infty s^{k+1}(\tau-s)^{j-k-1}\Big|\int_{|\xi|=1}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\Big)^j D^\alpha g_2(x+\tau\xi)\mathrm{d}\xi\Big|\mathrm{d}\tau\\ \leq& Cs^{1-n+j}\int_s^\infty\Big|\int_{|\xi|=1}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau}\Big)^j D^\alpha g_2(x+\tau\xi)\mathrm{d}\xi\Big|\tau^{n-1}\mathrm{d}\tau\\ =&Cs^{1-n+j}\int_{|y|>s}\sum_{|\beta|=j}\Big|D^\beta D^\alpha g_2\Big|(x+y)\mathrm{d}y,\end{aligned}\tag{4.4}$$

where $\tau\xi=y,\xi=(\xi_1,\cdots,\xi_n)$, by the divergence Theorem, $\forall q=1,\cdots,n$, we have

$$\int_{|\xi|=1}f(s\xi)\xi_q\mathrm{d}s(\xi)=s^{1-n}\int_{|y|\leq s}\partial_q f(y)\mathrm{d}y,$$

then

$$\begin{aligned}\omega_{n-1}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha(A_s g_2(x))=&\int_{|\xi|=1}\sum_{|\beta|=k}s^{-|\beta|-1}\partial^\beta(D^\alpha g_2(x+s\xi))(s\xi)^\beta\xi_q(s\xi^q)\mathrm{d}s(\xi)\\ =&s^{-n-k}\int_{|y|\leq s}\sum_{|\beta|=k}\partial_q(\partial^\beta(D^\alpha g_2(x+y))\cdot y^\beta y_q)\mathrm{d}y.\end{aligned}$$

Since $|y|\leq s$, we have

$$\omega_{n-1}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha(A_s g_2(x))\leq s^{-n-k}\int_{|y|\leq s}\sum_{|\beta|=k,k+1}s^{|\beta|}\Big|\partial^\beta D^\alpha g_2(x+y)\Big|\mathrm{d}y.$$

Notice that $0\leq k\leq\frac{n-3}{2}$, we need to take $\frac{n-1}{2}\leq j\leq n$, when $j=\frac{n-1}{2}$, and by (4.4), we obtain

$$\begin{aligned}\Big|D^\alpha v(x,s)\Big|=&\sum_{k=0}^{(n-3)/2}a_k s^{k+1}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha(A_s g_2(x))\\ \leq& Cs^{1-n+j}\int_{|y|>s}\sum_{|\beta|=j}\Big|D^\beta D^\alpha g_2\Big|(x+y)\mathrm{d}y\\ \leq& Cs^{\frac{1-n}{2}}\int_{|y|>1}\sum_{|\beta|\leq(n-1)/2}\Big|D^\beta D^\alpha g_2\Big|(x+y)\mathrm{d}y.\end{aligned}\tag{4.5}$$

If $j=n-1$, we have

$$\Big|D^\alpha v(x,s)\Big|\leq C\int_{|y|>1}\sum_{|\beta|\leq(n-1)}\Big|D^\beta D^\alpha g_2\Big|(x+y)\mathrm{d}y.\tag{4.6}$$

Now, we estimate $s^k\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha g_1(x)$ $(0\leq k\leq\frac{n-1}{2})$. If $j=\frac{n+1}{2}$, we have

$$s^k\Big|\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k D^\alpha g_1(x)\Big|\leq Cs^{1-n+(n+1)/2}\cdot s^{-1}\int_{|y|>1}\sum_{|\beta|\leq(n+1)/2}\Big|D^\beta D^\alpha g_1\Big|(x+y)\mathrm{d}y$$

$$=Cs^{\frac{1-n}{2}}\int_{|y|>1}\sum_{|\beta|\le(n+1)/2}\left|D^\beta D^\alpha g_1\right|(x+y)\mathrm{d}y. \tag{4.7}$$

If $j=n$, we get

$$s^k\left|\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^k D^\alpha g_1(x)\right|\le C\int_{|y|>1}\sum_{|\beta|\le n}\left|D^\beta D^\alpha g_1\right|(x+y)\mathrm{d}y. \tag{4.8}$$

Case III: $n\geq 2$ **and** n **is even.**

By (3.13), we know

$$\begin{aligned}v(x,s)=&\frac{2}{1\cdot3\cdot5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{g_1(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\\&+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big].\end{aligned}$$

Then

$$\begin{aligned}D^\alpha v(x,s)=&\frac{2}{1\cdot3\cdot5\cdots(n-1)\omega_n}\Big[\Big(\frac{\partial}{\partial s}\Big)\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{D^\alpha g_1(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\\&+\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{D^\alpha g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big].\end{aligned}$$

Let $g_1=0$, we have

$$\begin{aligned}D^\alpha v(x,s)=&\frac{2}{1\cdot3\cdot5\cdots(n-1)\omega_n}\Big(\frac{1}{s}\frac{\partial}{\partial s}\Big)^{\frac{n-2}{2}}s^{n-1}\int_{|y|<1}\frac{D^\alpha g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\\=&\sum_{k=0}^{(n-2)/2}a_ks^{k+1}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k\Big(\int_{|y|<1}\frac{D^\alpha g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\Big),\end{aligned}$$

and

$$\begin{aligned}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k\int_{|y|<1}\frac{D^\alpha g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y=&\int_{|y|<1}\sum_{|\beta|=k}\frac{\partial^\beta D^\alpha g_2(x+sy)}{\sqrt{1-|y|^2}}y^\beta\mathrm{d}y\\=&\int_{|z|\le s}\sum_{|\beta|=k}\frac{\partial^\beta D^\alpha g_2(x+z)}{\sqrt{s^2-|z|^2}}\Big(\frac{z}{s}\Big)^\beta s^{1-n}\mathrm{d}z\\=&\int_0^s\mathrm{d}r\int_{Sr(\xi)}\sum_{|\beta|=k}\frac{\partial^\beta D^\alpha g_2(x+r\xi)}{\sqrt{s^2-r^2}}\Big(\frac{r\xi}{s}\Big)^\beta r^{n-1}s^{1-n}\mathrm{d}s(\xi)\\=&\int_0^s\int_{|\xi|=1}s^{1-n}\frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta,\end{aligned}$$

where $sy=z$, $z=r\xi$. When $s>1$, we divide the integral into two parts $\int_0^{s-\frac{1}{2}}$ and $\int_{s-\frac{1}{2}}^{t}$, we have

$$\begin{aligned}
&\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)^k \int_{|y|<1} \frac{D^\alpha g_2(x+sy)}{\sqrt{1-|y|^2}}\mathrm{d}y\\
=&\int_0^{s-\frac{1}{2}}\int_{|\xi|=1} s^{1-n}\frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta\\
&\quad+\int_{s-\frac{1}{2}}^{s}\int_{|\xi|=1} s^{1-n}\frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta.
\end{aligned}$$

For the first part, we have

$$\begin{aligned}
&\int_0^{s-\frac{1}{2}}\int_{|\xi|=1} s^{1-n}\frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta\\
=&\int_0^{s-\frac{1}{2}}\frac{1}{\sqrt{s^2-r^2}}\mathrm{d}r\int_{|\xi|=1} s^{1-n}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta r^{n-1}\mathrm{d}s(\xi)\\
=&\int_0^{s-\frac{1}{2}}\frac{1}{\sqrt{s^2-r^2}}\mathrm{d}r\int_{S_r(\xi)} s^{1-n}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta \mathrm{d}s_r(\xi)\\
=&\int_0^{s-\frac{1}{2}}\frac{1}{\sqrt{s^2-r^2}}\mathrm{d}r s^{1-n}\int_{|y|\le s-\frac{1}{2}}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+y)\mathrm{d}y\\
=&O[(s^2-(s-\frac{1}{2})^2)^{-\frac{1}{2}}\cdot s^{1-n}]=O\Big(\frac{1}{\sqrt{s}}\cdot s^{1-n}\Big)=O(s^{\frac{1}{2}-n}).
\end{aligned}\tag{4.9}$$

For the second part, we have

$$\begin{aligned}
&\int_{s-\frac{1}{2}}^{s}\int_{|\xi|=1} s^{1-n}\frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta\\
=&s^{1-n}\int_{s-\frac{1}{2}}^{s}\frac{r\mathrm{d}r}{\sqrt{s^2-r^2}}\int_{|\xi|=1} r^{n-2}\sum_{|\beta|=k}\partial^\beta D^\alpha g_2(x+r\xi)\Big(\frac{r\xi}{s}\Big)^\beta \mathrm{d}s(\xi)\\
=&s^{1-n}\int_{s-\frac{1}{2}}^{s}\frac{r\mathrm{d}r}{\sqrt{s^2-r^2}}\Big(r^{n-2}\Big(\frac{\mathrm{d}}{\mathrm{d}r}\Big)^k A_r(D^\alpha g_2(x))\Big).
\end{aligned}\tag{4.10}$$

Applying (4.4) with $j=k+1$,

$$\begin{aligned}
&r^{n-2}\Big(\frac{\mathrm{d}}{\mathrm{d}r}\Big)^k A_r(D^\alpha g_2(x))=r^{n-2-k-1}r^{k+1}\Big(\frac{\mathrm{d}}{\mathrm{d}r}\Big)^k A_r(D^\alpha g_2(x))\\
=&r^{n-k-3}r^{k+1}\Big(\frac{\mathrm{d}}{\mathrm{d}r}\Big)^k A_r(D^\alpha g_2(x))
\end{aligned}$$

$$
\begin{aligned}
&\le r^{n-k-3} r^{1-n+k+1} \int_{|y|>r} \sum_{|\beta|=k+1} \Big| D^\beta (D^\alpha g_2) \Big| (x+y) \mathrm{d}y \\
&= r^{-1} \int_{|y|>r} \sum_{|\beta|=k+1} \Big| D^\beta D^\alpha g_2 \Big| (x+y) \mathrm{d}y,
\end{aligned}
$$

where

$$
\int_{s-\frac{1}{2}}^{s} \frac{r\mathrm{d}r}{\sqrt{s^2-r^2}} = \frac{1}{2} \int_{s-\frac{1}{2}}^{s} \frac{\mathrm{d}r^2}{\sqrt{s^2-r^2}} = O(s^{\frac{1}{2}}) \quad \text{and} \quad r \sim s,
$$

we can obtain

$$
\begin{aligned}
&\int_{s-\frac{1}{2}}^{s} \int_{|\xi|=1} s^{1-n} \frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}} \sum_{|\beta|=k} \partial^\beta D^\alpha g_2(x+r\xi) \Big(\frac{r\xi}{s} \Big)^\beta \\
&= O(s^{-1} \cdot s^{1-n} \cdot s^{\frac{1}{2}}) = O(s^{\frac{1}{2}-n}).
\end{aligned}
$$

When $s>1$, we get

$$
\begin{aligned}
&\Big| \int_0^s \int_{|\xi|=1} s^{1-n} \frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}} \sum_{|\beta|=k} \partial^\beta D^\alpha g_2(x+r\xi) \Big(\frac{r\xi}{s} \Big)^\beta \Big| \\
&\le s^{\frac{1}{2}-n} \int_{|y|>s} \sum_{|\beta|=k+1} \Big| D^\beta D^\alpha g_2(x+y) \Big| \mathrm{d}y,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| s^{k+1} \int_0^s \int_{|\xi|=1} s^{1-n} \frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}} \sum_{|\beta|=k} \partial^\beta D^\alpha g_2(x+r\xi) \Big(\frac{r\xi}{s} \Big)^\beta \right| \\
&\le s^{k+1} \left| \int_0^s \int_{|\xi|=1} s^{1-n} \frac{r^{n-1}\mathrm{d}r\mathrm{d}s(\xi)}{\sqrt{s^2-r^2}} \sum_{|\beta|=k} \partial^\beta D^\alpha g_2(x+r\xi) \Big(\frac{r\xi}{s} \Big)^\beta \right| \\
&\le s^{k+1+\frac{1}{2}-n} \int_{|y|>s} \sum_{|\beta|=k+1} \Big| D^\beta D^\alpha g_2(x+y) \Big| \mathrm{d}y \\
&\le C s^{\frac{1}{2}-n+[(n-2)/2+1]} \int_{|y|>s} \sum_{|\beta|=k+1} \Big| D^\beta D^\alpha g_2(x+y) \Big| \mathrm{d}y \\
&= C s^{-\frac{(n-1)}{2}} \| D^\alpha g_2(x) \|_{W^{[n/2],1}}.
\end{aligned} \tag{4.11}
$$

Then, if $s>1$,

$$
\sum_{|\alpha| \le \frac{n}{2}} \Big| D^\alpha v(x,s) \Big| \le C \Big(\sum_{|\alpha| \le \frac{n}{2}} \| D^\alpha g_1 \|_{W^{[n/2]+1,1}(\Omega)} + \sum_{|\alpha| \le \frac{n}{2}} \| D^\alpha g_2 \|_{W^{[n/2],1}(\Omega)} \Big) s^{-\frac{(n-1)}{2}}. \tag{4.12}
$$

Consequently, when $t>1$, $s>1$, we get

$$
|u(x,t)| \le C \big(\|f_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} + \|f_2\|_{W^{[n/2],1}(\mathbb{R}^n)} \big) t^{-\frac{(n-1)}{2}}
$$

$$+C\int_1^t (t-s)^{-\frac{(n-1)}{2}} s^{-\frac{(n-1)}{2}} \mathrm{d}s \int_\Omega \sum_{|\alpha|\le \frac{n}{2}} \big(\|D^\alpha g_1(x)\|_{W^{[n/2],1}} + \|D^\alpha g_2(x)\|_{W^{[n/2],1}}\big)\mathrm{d}x, \tag{4.13}$$

where Ω is an n-dimensional sphere with x as the center and $t-s$ as the radius.

Finally, we get

$$\begin{aligned}|u(x,t)| \le & C\big(||f_1||_{W^{[n/2]+1,1}(\mathbb{R}^n)} + \|f_2\|_{W^{[n/2],1}(\mathbb{R}^n)}\big)t^{-\frac{(n-1)}{2}} \\ & + C\int_1^t (t-s)^{-\frac{(n-1)}{2}} s^{-\frac{(n-1)}{2}} \sum_{|\alpha|\le \frac{n}{2}} \big(\|D^\alpha g_1(x)\|_{W^{[n/2],1}(\Omega)} \\ & + \|D^\alpha g_2(x)\|_{W^{[n/2],1}(\Omega)}\big)(t-s)^n \mathrm{d}s. \end{aligned}\tag{4.14}$$

In the following, it suffices to estimate

$$\int_1^t (t-s)^{-\frac{(n-1)}{2}} s^{-\frac{(n-1)}{2}} (t-s)^n \mathrm{d}s = \int_1^t \Big(\frac{t-s}{s}\Big)^{\frac{n-1}{2}} (t-s)\mathrm{d}s. \tag{4.15}$$

Let $(\frac{t-s}{s})^{\frac{1}{2}} = \tau$, then $s = \frac{t}{1+\tau^2}$, $\mathrm{d}s = -\frac{2\tau t}{(1+\tau^2)^2}\mathrm{d}\tau$. Thus

$$\int_1^t (t-s)^{-\frac{(n-1)}{2}} s^{-\frac{(n-1)}{2}} (t-s)^n \mathrm{d}s = \int_0^{\sqrt{t-1}} \frac{2\tau^{n+2}t^2}{(1+\tau^2)^3}\mathrm{d}\tau.$$

If $0<\tau<1$, we have

$$\int_0^1 \frac{2\tau^{n+2}t^2}{(1+\tau^2)^3}\mathrm{d}\tau \le \int_0^1 \frac{2t^2}{(1+\tau^2)^3}\mathrm{d}\tau \le 2t^2. \tag{4.16}$$

If $\tau>1$, we have

$$\int_1^{\sqrt{t-1}} \frac{2\tau^n t^2}{(1+\tau^2)^3}\mathrm{d}\tau \le 2t^2 \int_1^{\sqrt{t-1}} \tau^{n-4}\mathrm{d}\tau. \tag{4.17}$$

For (4.17), we have,

When $n=2$,

$$2t^2\int_1^{\sqrt{t-1}} \tau^{n-4}\mathrm{d}\tau = 2t^2\int_1^{\sqrt{t-1}} \frac{1}{\tau^2}\mathrm{d}\tau = 2t^2\Big(1-\frac{1}{\sqrt{t-1}}\Big) \le 2t^2.$$

When $n=3$,

$$2t^2\int_1^{\sqrt{t-1}} \tau^{n-4}\mathrm{d}\tau = 2t^2\int_1^{\sqrt{t-1}} \frac{1}{\tau}\mathrm{d}\tau = t^2\ln(t-1) \lesssim t^2\ln t.$$

When $n>3$,

$$2t^2\int_1^{\sqrt{t-1}} \tau^{n-4}\mathrm{d}\tau = \frac{2t^2}{n-3}\Big[(t-1)^{\frac{n-3}{2}}-1\Big] \le \frac{2t^2}{n-3}(t-1)^{\frac{n-3}{2}} \le \frac{2t^{\frac{n+1}{2}}}{n-3}.$$

Combining above analysis and computations, we get

When $n=2$,

$$|u(x,t)|\leq C\big(||f_1||_{W^{[n/2]+1,1}(\mathbb{R}^n)}+||f_2||_{W^{[n/2],1}(\mathbb{R}^n)}\big)t^{-\frac{(n-1)}{2}} \\ +C\sum_{|\alpha|\leq\frac{n}{2}}\big(||D^\alpha g_1(x)||_{W^{[n/2],1}(\Omega)}+||D^\alpha g_2(x)||_{W^{[n/2],1}(\Omega)}\big)t^2.$$

When $n=3$,

$$|u(x,t)|\leq C\big(||f_1||_{W^{[n/2]+1,1}(\mathbb{R}^n)}+||f_2||_{W^{[n/2],1}(\mathbb{R}^n)}\big)t^{-\frac{(n-1)}{2}} \\ +C\sum_{|\alpha|\leq\frac{n}{2}}\big(||D^\alpha g_1(x)||_{W^{[n/2],1}(\Omega)}+||D^\alpha g_2(x)||_{W^{[n/2],1}(\Omega)}\big)t^2\ln t.$$

When $n>3$,

$$|u(x,t)|\leq C\big(||f_1||_{W^{[n/2]+1,1}(\mathbb{R}^n)}+||f_2||_{W^{[n/2],1}(\mathbb{R}^n)}\big)t^{-\frac{(n-1)}{2}} \\ +C\sum_{|\alpha|\leq\frac{n}{2}}\big(||D^\alpha g_1(x)||_{W^{[n/2],1}(\Omega)}+||D^\alpha g_2(x)||_{W^{[n/2],1}(\Omega)}\big)t^{\frac{n+1}{2}}.$$

Since $g_1(x)=f_3(x)-\triangle f_1(x), g_2(x)=f_4(x)-\triangle f_2(x)$, we finally get

When $n=2$,

$$|u(x,t)|\leq C\big(||f_1||_{W^{6,1}(\mathbb{R}^2)}+||f_2||_{W^{5,1}(\mathbb{R}^2)}\big)t^{3/2} \\ +C\big(||f_3||_{W^{2,1}(\mathbb{R}^2)}+||f_4||_{W^{2,1}(\mathbb{R}^2)}\big)t^2. \tag{4.18}$$

When $n=3$,

$$|u(x,t)|\leq C\big(||f_1||_{W^{7,1}(\mathbb{R}^3)}+||f_2||_{W^{6,1}(\mathbb{R}^3)}\big)t\ln t \\ +C\big(||f_3||_{W^{3,1}(\mathbb{R}^3)}+||f_4||_{W^{3,1}(\mathbb{R}^3)}\big)t^2\ln t. \tag{4.19}$$

When $n>3$,

$$|u(x,t)|\leq C\big(||f_1||_{W^{[3n/2]+3,1}(\mathbb{R}^n)}+||f_2||_{W^{[3n/2]+2,1}(\mathbb{R}^n)}\big)t \\ +C\big(||f_3||_{W^{[n],1}(\mathbb{R}^n)}+||f_4||_{W^{[n],1}(\mathbb{R}^n)}\big)t^{\frac{n+1}{2}}. \tag{4.20}$$

Combining (4.3) and (4.18)-(4.20) gives Theorem 1.4.

4.2 $L^\infty(\mathbb{R}^n)-W^{N,2}(\mathbb{R}^n)$ estimates

In this section, we prove Theorem 1.5 by the energy method. Applying Γ^a on both side of (3.1) and (3.2), we get

$$\begin{cases}\Box\Gamma^a u=\sum\limits_{|b|\leq|a|}\Gamma^b v,\\ \Box\Gamma^a v=0.\end{cases} \tag{4.21}$$

Define

$$E_M(t)=\Big(\sum_{|a|\le M}\int_{\mathbb{R}^n}[(\partial_t\Gamma^a u)^2+|\nabla\Gamma^a u|^2]\mathrm{d}x\Big)^{\frac{1}{2}},$$

$$H_M(t)=\Big(\sum_{|a|\le M}\int_{\mathbb{R}^n}[(\partial_t\Gamma^a v)^2+|\nabla\Gamma^a v|^2]\mathrm{d}x\Big)^{\frac{1}{2}},$$

where $M\le[\frac{n}{2}]+1$, $|\nabla\Gamma^a u|^2=\sum_{i=1}^n(\partial_i\Gamma^a u)^2$. By (3.2), we know

$$E_M(0)=\left(\sum_{|a|\le M}\int_{\mathbb{R}^n}[(\Lambda^a f_2)^2+|\nabla\Lambda^a f_1|^2]\mathrm{d}x\right)^{\frac{1}{2}},$$

$$H_M(0)=\left(\sum_{|a|\le M}\int_{\mathbb{R}^n}[(\Lambda^a(f_4-\triangle f_2))^2+|\Lambda^a(\nabla f_3-\nabla\triangle f_1)|^2]\mathrm{d}x\right)^{\frac{1}{2}}.$$

Multiplying the second equation of (4.21) by $2\partial_t\Gamma^a v$ and integrating by parts, we get

$$2\Box\Gamma^a v\partial_t\Gamma^a v=\partial_t\big((\partial_t\Gamma^a v)^2+|\nabla\Gamma^a v|^2\big)-2\sum_{i=1}^n\partial_i(\partial_i\Gamma^a v\partial_t\Gamma^a v)=0. \tag{4.22}$$

Then, integrating (4.22) over $\mathbb{R}^n$, we have

$$\partial_t H_M^2(t)=0,$$

which means

$$H_M(t)=H_M(0). \tag{4.23}$$

Multiplying the first equation of (4.21) by $2\partial_t\Gamma^a u$ with $|a|\le M$, and integrating by parts, we arrive at

$$2\Box\Gamma^a u(\partial_t\Gamma^a u)=\partial_t\big((\partial_t\Gamma^a u)^2+|\nabla\Gamma^a u|^2\big)-2\sum_{i=1}^n\partial_i(\partial_t\Gamma^a u\partial_i\Gamma^a u). \tag{4.24}$$

Integrating (4.24) over $\mathbb{R}^n$ and summing all a, we get

$$\begin{aligned}\partial_t E_M^2(t)=&2\sum_{|a|\le M}\int_{\mathbb{R}^n}\Gamma^a v\partial_t\Gamma^a u\mathrm{d}x\lesssim\sum_{|a|\le M}||\Gamma^a v||_{L^2(\mathbb{R}^n)}||\partial_t\Gamma^a u||_{L^2(\mathbb{R}^n)}\\ \lesssim&(1+t)H_M(t)E_M(t).\end{aligned} \tag{4.25}$$

Combining (4.23) and (4.25), we have

$$\partial_t E_M(t)\lesssim(1+t)H_M(0).$$

Integrating above inequality from 0 to t, we get

$$E_M(t) \lesssim E_M(0) + \int_0^t (1+s) H_M(0) \mathrm{d}s \lesssim E_M(0) + (1+t)^2 H_M(0). \tag{4.26}$$

By Lemma 2.10 and Lemma 2.11, we have

$$\begin{aligned}
||u||_{L^\infty(\mathbb{R}^n)} \lesssim & (1+t+|x|)^{-\frac{n-1}{2}} (1+|t-|x||)^{-\frac{1}{2}} \sum_{|a|\leq [\frac{n}{2}]+1} ||\Gamma^a u||_{L^2(\mathbb{R}^n)} \\
\lesssim & (1+t+|x|)^{-\frac{n-1}{2}} (1+|t-|x||)^{-\frac{1}{2}} \sum_{|a|\leq [\frac{n}{2}]+1} ||(\Gamma^a u)'||_{L^2(\mathbb{R}^n)} (1+t) \\
\lesssim & (1+t+|x|)^{-\frac{n-1}{2}} (1+|t-|x||)^{-\frac{1}{2}} (1+t) [E_M(0) + (1+t)^2 H_M(0)] \\
\lesssim & (1+t+|x|)^{-\frac{n-1}{2}} (1+|t-|x||)^{-\frac{1}{2}} (1+t) \Big(||f_1||_{W_\Lambda^{[n/2]+2,2}(\mathbb{R}^n)} + ||f_2||_{W_\Lambda^{[n/2]+1,2}(\mathbb{R}^n)} \Big) \\
& + (1+t+|x|)^{-\frac{n-1}{2}} (1+|t-|x||)^{-\frac{1}{2}} (1+t)^3 \Big(||f_1||_{W_\Lambda^{[n/2]+4,2}(\mathbb{R}^n)} \\
& + ||f_2||_{W_\Lambda^{[n/2]+3,2}(\mathbb{R}^n)} + ||f_3||_{W_\Lambda^{[n/2]+2,2}(\mathbb{R}^n)} + ||f_4||_{W_\Lambda^{[n/2]+1,1}(\mathbb{R}^n)} \Big) \\
\leq & C \Big(||f_1||_{W_\Lambda^{[n/2]+2,2}(\mathbb{R}^n)} + ||f_2||_{W_\Lambda^{[n/2]+1,2}(\mathbb{R}^n)} \Big) (1+t+|x|)^{\frac{3-n}{2}} (1+|t-|x||)^{-\frac{1}{2}} \\
& + C \Big(||f_1||_{W_\Lambda^{[n/2]+4,2}(\mathbb{R}^n)} + ||f_2||_{W_\Lambda^{[n/2]+3,2}(\mathbb{R}^n)} + ||f_3||_{W_\Lambda^{[n/2]+2,2}(\mathbb{R}^n)} \\
& + ||f_4||_{W_\Lambda^{[n/2]+1,1}(\mathbb{R}^n)} \Big) (1+t+|x|)^{\frac{7-n}{2}} (1+|t-|x||)^{-\frac{1}{2}},
\end{aligned}$$

where

$$||f||_{W_\Lambda^{N,2}(\mathbb{R}^n)} = \sum_{|a|\leq N} ||\Lambda^a f||_{L^2(\mathbb{R}^n)} = \sum_{|a|\leq N} \Big(\int_{\mathbb{R}^n} |\Lambda^a f|^2 \mathrm{d}x \Big)^{\frac{1}{2}}, \quad \Lambda = (\partial, r\partial_r, x_i\partial_j - x_j\partial_i, x_i\partial_t)$$

denotes the Klainerman's vector field on $\{t=0\}$. Thus, we completed the proof of Theorem 1.5. □

Acknowledgement

The author was partially supported by the Zhejiang Provincial Outstanding Youth Science Foundation (Grant No. LR22A010004), the Natural Science Foundation of Zhejiang Province (Grant No. LY20A010026), the National Natural Science Foundation of China (Grant Nos. 12071435 and 11871212).

References

[1] Chiang Y. J., Wei C. H., Global existence and lifespan of smooth solutions of biwave maps. *SIAM J. Math. Phys.*, **63** (2) (2022), 1-14.

[2] Chiang Y. J., Biwave maps into manifolds. *SIAM Int. J. Math. Math. Sci.*, **2009** (2009), 1-14.

[3] Chiang Y. J., Some properties of biwave maps. *SIAM J. Geom. Phys.*, **62** (4) (2012), 839-850.

[4] Chiang Y. J., Wolak R. A., Transverval biwave maps. *SIAM Arch. Math.*, **46** (3) (2010), 211-226.

[5] Korzyuka V., Van Vinh N. and Minh N. T., Classical solution of the Cauchy problem for biwave equation: application of Fourier transform. *SIAM Math. Model. Anal.*, **17** (5) (2012), 630-641.

[6] Feng X., Neilan M., Finite element methods for a bi-wave equation modeling d-wave superconductors. *SIAM J. Comput. Math.*, **28** (3) (2010), 331-353.

[7] Feng X., Neilan M., Discontinuous finite element methods for a bi-wave equation modeling d-wave superconductors. *SIAM Math. Comp.*, **80** (275) (2011), 1303-1333.

[8] Fushchych W. I., Roman O. V. and Zhdanov R. Z., Symmetry reduction and some exact solutions of nonlinear biwave equations. *SIAM Rep. Math. Phys.*, **37** (2) (1996), 267-281.

[9] Korzyuk V. I., Cheb E. S., The Cauchy problem for a fourth-order equation with the biwave operator. *SIAM Differ. Equ.*, **43** (5) (2007), 688-695.

[10] Korzyuk V. I., Konopel'ko O. G. and Cheb E. S., Boundary-value problems for fourth-order equations of hyperbolic and composite types. *SIAM J. Math. Sci.*, **171** (1) (2010), 89-115.

[11] Evans L. C., Partial Differential Equations. American Mathematical Society, 1998.

[12] Fang D. Y., Non-linear wave equations. *Hang Zhou: Zhejiang University Press*, (2008), 95-97.