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Analytic Smoothing Effect of the Time Variable for the Spatially Homogeneous Landau Equation

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Abstract. In this work, we study the Cauchy problem of the spatially homogeneous Landau equation with hard potentials in a close-to-equilibrium framework. We prove that the solution to the Cauchy problem enjoys the analytic regularizing effect of the time variable with an L^2 initial datum for positive time. So that the smoothing effect of the Cauchy problem for the spatially homogeneous Landau equation with hard potentials is exactly same as heat equation.

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1 Introduction

In this work, we are concerned with the following Cauchy problem of spatially homogenous Landau equation

$$\begin{cases} \partial_t F = Q(F,F), \\ F|_{t=0} = F_0, \end{cases} \tag{1.1}$$

where $F=F(t,v)\geq 0$ is the density distribution function at time $t\geq 0$, with the velocity variable $v\in\mathbb{R}^3$. The Landau bilinear collision operator is defined by

$$Q(G,F)(v)=\sum_{i,j=1}^{3}\partial_i\left(\int_{\mathbb{R}^3}a_{ij}(v-v_*)[G(v_*)\partial_j F(v)-\partial_j G(v_*)F(v)]\mathrm{d}v_*\right),$$

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where

$$a_{ij}(v)=(\delta_{ij}|v|^2-v_iv_j)|v|^\gamma,\quad \gamma\geq -3,$$

is a symmetric non-negative matrix such that $a_{ij}(v)v_iv_j=0$. Here, γ is a parameter which leads to the classification of the hard potential if $\gamma>0$, Maxwellian molecules if $\gamma=0$, soft potential if $\gamma\in]-3,0[$ and Coulombian potential if $\gamma=-3$.

The Landau equation was introduced as a limit of the Boltzmann equation when the collisions become grazing in [1,2]. The global existence, and uniqueness of classical solutions for the spatially homogeneous Landau equation with hard potentials, regularizing effects, and large-time behavior have been addressed by Desvillettes and Villani [3,4]. Moreover, they proved the smoothness of the solution in $C^\infty(]0,\infty[;\mathcal{S}(\mathbb{R}^3))$. Carrapatoso [5] proved an exponential in-time convergence to the equilibrium. In [6], the authors proved the solution is analytic of v variables for any $t>0$ and the Gevrey regularity in [7,8].

Let μ be the Maxwellian distribution

$$\mu(v)=(2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}},$$

we shall linearize the Landau equation (1.1) around μ with the fluctuation of the density distribution function

$$F(t,v)=\mu(v)+\sqrt{\mu}(v)f(t,v),$$

since $Q(\mu,\mu)=0$, the Cauchy problem (1.1) for $f=f(t,v)$ takes the form

$$\begin{cases}\partial_t f+\mathcal{L}(f)=\Gamma(f,f),\\ f|_{t=0}=f_0,\end{cases}\tag{1.2}$$

with $F_0(v)=\mu+\sqrt{\mu}f_0(v)$, where

$$\Gamma(g,h)=\mu^{\frac{-1}{2}}Q(\mu^{\frac{1}{2}}g,\mu^{\frac{1}{2}}h),$$
$$\mathcal{L}(f)=\mathcal{L}_1f+\mathcal{L}_2f,\quad \mathcal{L}_1f=-\Gamma(\mu^{\frac{1}{2}},f),\quad \mathcal{L}_2f=-\Gamma(f,\mu^{\frac{1}{2}}).$$

In the case of the Maxwellian molecules, Villani [4] has proved a linear functional inequality between entropy and entropy dissipation by constructive methods, from which one deduces an exponential convergence of the solution to the Maxwellian equilibrium in relative entropy, which in turn implies an exponential convergence in L^1-distance. In [9], Desvillettes and Villani have proved a functional inequality for entropy dissipation is not linear, from which one obtains a polynomial in time convergence of solutions towards the equilibrium in relative entropy, which implies the same type of convergence in L^1-distance. In [10], the authors studied the spatially homogeneous Landau equation and non-cutoff Boltzmann equation in a close-to-equilibrium framework and proved the Gelfand-Shilov smoothing effect (see also [11,12]). Guo [13] constructed global classical solutions for the spatially inhomogeneous Landau equation near a global Maxwellian in a periodic box, and the smoothness of the solutions have been studied in [14–16]. The analytic smoothing effect of the velocity variable for the nonlinear Landau equation has been studied in [17, 18]. The variant regularity results in a close to equilibrium setting were considered by [19–21].

Let us give the definition of analytic function spaces $\mathcal{A}(\Omega)$ where $\Omega\subset\mathbb{R}^n$ is a open domain. We say that $u\in\mathcal{A}(\Omega)$ if $u\in C^\infty(\Omega)$ and there exists a constant C such that for all multi-indices $\alpha\in\mathbb{N}^n$,

$$\|\partial^\alpha u\|_{L^\infty(\Omega)}\le C^{|\alpha|+1}\alpha!.$$

Remark that, by using the Sobolev embedding, we can replace the L^∞ norm by the L^2 norm, or norm in any Sobolev space in the above definition.

In this work, we consider the Cauchy problem (1.2) with $\gamma\ge 0$, show that the solution of the Cauchy problem (1.2) with initial datum in $L^2(\mathbb{R}^3)$ enjoys the analytic regularizing effect of time variable. Our main result reads as follows.

Theorem 1.1. *Assume $f_0\in L^2(\mathbb{R}^3)$ and $T>0$, let f be the solution of the Cauchy problem* (1.2) *with $\|f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}$ small enough. Then there exists a constant $C>0$ such that for any $k\in\mathbb{N}$, we have*

$$\|\partial_t^k f(t)\|_{L^2(\mathbb{R}^3)}\le\frac{C^{k+1}}{t^k}k!,\qquad\forall t\in]0,T]. \tag{1.3}$$

Remark 1.1. In the paper [17], for $f_0\in L^2(\mathbb{R}^3)$ with $\|f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}\le\epsilon$ small enough, the solution of the Cauchy problem (1.2) satisfies $f(t)\in\mathcal{A}(\mathbb{R}^3)$ for all $0<t\le T$, i. e. there exists a constant $C>0$ such that

$$\|t^{\frac{|\alpha|}{2}}\partial_v^\alpha f(t)\|_{L^2(\mathbb{R}^3)}\le C^{|\alpha|+1}\alpha!,\qquad\forall\alpha\in\mathbb{N}^3,\ \forall t\in]0,T],$$

which implies that $f\in C^\infty(]0,T[;\mathcal{A}(\mathbb{R}^3))$, so that we prove only the estimate (1.3) for the smooth solution of (1.2). Combine with the results of [17], we have proved that, if f is the solution of the nonlinear Cauchy problem (1.2) with $\|f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}$ small enough, then we have

$$f\in\mathcal{A}(]0,T[\times\mathbb{R}^3),$$

which implies that the smoothing effect properties of Cauchy problem for the spatially homogeneous Landau equation with hard potentials is exactly same as the semilinear heat equation.

2 Analysis of Landau collision operator

The operators $\mathcal{L}_1,\mathcal{L}_2$ and Γ are defined in [13] as follow:

$$\mathcal{L}_1 f=-\sum_{i,j=1}^3\left\{\partial_i[(a_{ij}*\mu)\partial_j f]+(a_{ij}*\mu)\frac{v_i}{2}\frac{v_j}{2}f-\partial_i\left[(a_{ij}*\mu)\frac{v_j}{2}\right]f\right\}, \tag{2.1}$$

$$\mathcal{L}_2 f = -\sum_{i,j=1}^{3} \mu^{-\frac{1}{2}} \partial_i \left\{ \mu \left[a_{ij} * \left(\mu^{\frac{1}{2}} \partial_j f + \mu^{\frac{1}{2}} \frac{v_j}{2} f \right) \right] \right\},$$

$$\Gamma(f,g) = \sum_{i,j=1}^{3} \left\{ \partial_i [(a_{ij} * (\mu^{\frac{1}{2}} f)) \partial_j g] - \left[a_{ij} * \left(\frac{v_i}{2} \mu^{\frac{1}{2}} f \right) \right] \partial_j g \right.$$
$$\left. - \partial_i [(a_{ij} * (\mu^{\frac{1}{2}} \partial_j f)) g] + \left[a_{ij} * \left(\frac{v_i}{2} \mu^{\frac{1}{2}} \partial_j f \right) \right] g \right\}.$$

Since the use of a different normalization for the Maxwellian, these representations are different in a few places by a factor of $1/2$ from that in [13]. The linear operator $\mathcal{L}$ is nonnegative.

For later use, we derive some results for the linear operator $\mathcal{L}$. For simplicity, with $s \in \mathbb{R}$, we define

$$\|f\|_{p,s} = \|(1+|\cdot|)^s f\|_{L^p(\mathbb{R}^3)}, \quad 1 \le p \le \infty,$$
$$\|f\|_{L^2_A}^2 = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} \left(\bar{a}_{ij} \partial_i f \partial_j f + \bar{a}_{ij} \frac{1}{4} v_i v_j f^2 \right) \mathrm{d}v,$$

where $\bar{a}_{ij} = a_{ij} * \mu$. From Corollary 1 in [13], there exists $C_1 > 0$ such that

$$\|f\|_{L^2_A}^2 \ge C_1 (\|\mathbf{P}_v \bigtriangledown f\|_{2,\gamma/2}^2 + \|(\mathbf{I} - \mathbf{P}_v) \bigtriangledown f\|_{2,1+\gamma/2}^2 + \|f\|_{2,1+\gamma/2}^2),$$

where for any vector-valued function $g = (g_1, g_2, g_3)$, define the projection to the vector $v \in \mathbb{R}^3$ as

$$(\mathbf{P}_v g)_i = \sum_{j=1}^{3} g_j v_j \frac{v_i}{|v|^2}, \quad 1 \le i \le 3.$$

Noticing that $f = \mathbf{P}_v \bigtriangledown f + (\mathbf{I} - \mathbf{P}_v) \bigtriangledown f$, we have

$$\|f\|_{L^2_A} \ge C_1 (\|\bigtriangledown f\|_{2,\gamma/2} + \|f\|_{2,1+\gamma/2}). \tag{2.2}$$

From representation (2.1), we can get the coercivity of the operator $\mathcal{L}_1$.

Lemma 2.1. *Let $f \in \mathcal{S}(\mathbb{R}^3)$, then there exists a constant $C_2 > 0$ such that*

$$(\mathcal{L}_1 f, f)_{L^2} \ge \|f\|_{L^2_A}^2 - C_2 \|f\|_{2,\gamma/2}^2.$$

Proof. By the representation (2.1) and integrating by parts, we have

$$(\mathcal{L}_1 f, f)_{L^2} = \sum_{i,j=1}^{3} \left[((a_{ij} * \mu) \partial_j f, \partial_i f)_{L^2} + \frac{1}{4} ((a_{ij} * \mu) v_i v_j f, f)_{L^2} \right] - \frac{1}{2} \sum_{i,j=1}^{3} (\partial_i [(a_{ij} * \mu) v_j] f, f)_{L^2}$$
$$= \|f\|_{L^2_A} - \frac{1}{2} \sum_{i,j=1}^{3} (\partial_i [(a_{ij} * \mu) v_j] f, f)_{L^2}.$$

Using

$$\sum_{i=1}^{3} a_{ij}(v)v_i = \sum_{j=1}^{3} a_{ij}(v)v_j = 0,$$

it follows that

$$\begin{aligned}\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_i[(a_{ij}*\mu)v_j]f^2\mathrm{d}v &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_i\Big(\int_{\mathbb{R}^3}a_{ij}(v-v')v'_j\mu(v')\mathrm{d}v'\Big)f^2\mathrm{d}v\\ &= \sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\partial_i[a_{ij}*(v_j\mu)]f^2\mathrm{d}v.\end{aligned}$$

Expanding $\partial_i a_{ij}(v-v')$ to get

$$\partial_i a_{ij}(v-v') = \partial_i a_{ij}(v) + \sum_{l=1}^{3}\Big(\int_0^1 \partial_l\partial_i a_{ij}(v-sv')\mathrm{d}s\Big)v'_l,$$

then by

$$\int_{\mathbb{R}^3} v'_j\mu(v')\mathrm{d}v' = 0,$$

we can deduce that

$$\partial_i a_{ij}*(v_j\mu) = \sum_{l=1}^{3}\int_{\mathbb{R}^3}\int_0^1 \partial_l\partial_i a_{ij}(v-sv')\mathrm{d}s v'_l v'_j\mu(v')\mathrm{d}v',$$

and using

$$|\partial^\beta a_{ij}(v)| \le c(1+|v|)^{\gamma+2-|\beta|}, \quad \forall \beta\in\mathbb{N}^3,$$

we can conclude that

$$\begin{aligned}\frac{1}{2}\Big|\sum_{i,j=1}^{3}(\partial_i[(a_{ij}*\mu)v_j]f,f)_{L^2}\Big| &= \frac{1}{2}\Big|\sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\partial_i a_{ij}(v-v')v'_j\mu(v')\mathrm{d}v' f^2(v)\mathrm{d}v\Big|\\ &\le \frac{1}{2}\sum_{i,j=1}^{3}\sum_{l=1}^{3}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}v'_l v'_j\mu(v')\int_0^1\partial_l\partial_i a_{ij}(v-sv')\mathrm{d}s\mathrm{d}v' f^2(v)\mathrm{d}v\Big|\\ &\le C_2\int_{\mathbb{R}^3}(1+|v|)^\gamma f^2(v)\mathrm{d}v.\end{aligned}$$

We thus complete the proof of the Lemma 2.1. □

We recall the trilinear estimate, which has been addressed in [17].

Lemma 2.2. ([17]) *Let $F,G,H\in\mathcal{S}(\mathbb{R}^3)$, then there exists a constant $C_3>0$ such that*

$$|\langle\Gamma(F,G),H\rangle_{L^2}| \le C_3\|F\|_{L^2}\|G\|_{L^2_A}\|H\|_{L^2_A}.$$

Let $F=\sqrt{\mu}, G=f, H=g$ and $F=f, G=\sqrt{\mu}, H=g$ in Lemma 2.2, then we have the following estiamtes for the operators $\mathcal{L}_1$ and $\mathcal{L}_2$.

Corollary 2.1. *Let $f,g\in\mathcal{S}(\mathbb{R}^3)$, then there exists a constant $C_4>0$ such that*

$$|(\mathcal{L}_1 f,g)_{L^2}|\le C_4\|f\|_{L^2_A}\|g\|_{L^2_A},$$

$$|(\mathcal{L}_2 f,g)_{L^2}|\le C_4\|f\|_{L^2}\|g\|_{L^2_A}.$$

3 Energy estimates

We need the following interpolation inequality for $g\in\mathcal{S}(\mathbb{R}^3)$.

Lemma 3.1. *Let $g\in\mathcal{S}(\mathbb{R}^3)$, then for all $0<\delta<1$ we have*

$$\|g\|^2_{2,\gamma/2}\le\delta\|g\|^2_{L^2_A}+C_\delta\|g\|^2_{L^2}. \tag{3.1}$$

Proof. From Hölder's inequality and inequality (2.2), it follows that

$$\begin{aligned}\|g\|^2_{2,\gamma/2}&=\int_{\mathbb{R}^3}(1+|v|)^\gamma g^{\frac{2\gamma}{\gamma+2}}(v)g^{\frac{4}{\gamma+2}}(v)\mathrm{d}v\\ &\le\|g\|^{\frac{2\gamma}{\gamma+2}}_{2,\gamma/2+1}\|g\|^{\frac{4}{\gamma+2}}_{L^2}\le\left(\frac{1}{C_1}\|g\|_{L^2_A}\right)^{\frac{2\gamma}{\gamma+2}}\|g\|^{\frac{4}{\gamma+2}}_{L^2},\end{aligned}$$

then by using the Young inequality

$$ab\le\frac{1}{p}a^p+\frac{1}{q}b^q,\quad (a,b\ge0,\ \frac{1}{p}+\frac{1}{q}=1)$$

and the fact $\gamma\ge0$, we get

$$\begin{aligned}\left(\frac{1}{C_1}\|g\|_{L^2_A}\right)^{\frac{2\gamma}{\gamma+2}}\|g\|^{\frac{4}{\gamma+2}}_{L^2}&\le\frac{\gamma}{\gamma+2}\delta\|g\|^2_{L^2_A}+\frac{2}{\gamma+2}C_1^{-\gamma}\delta^{-\gamma/2}\|g\|^2_{L^2}\\ &\le\delta\|g\|^2_{L^2_A}+C_1^{-\gamma}\delta^{-\gamma/2}\|g\|^2_{L^2}.\end{aligned}$$

Let $C_\delta=C_1^{-\gamma}\delta^{-\gamma/2}$, then it follows that (3.1) holds. □

We study now the energy estimates of the solution of the Cauchy problem (1.2), we have

Lemma 3.2. *Assume $f_0\in L^2(\mathbb{R}^3)$ and $T>0$, let f be the solution of the Cauchy problem* (1.2) *with $\|f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}$ small enough. Then there exists a constant $B_0>0$ such that*

$$\|f\|^2_{L^\infty([0,T];L^2(\mathbb{R}^3))}+\|f\|^2_{L^2([0,T];L^2_A(\mathbb{R}^3))}\le B_0^2\|f_0\|^2_{L^2(\mathbb{R}^3)}\le\epsilon^2B_0^2. \tag{3.2}$$

We will take ϵ small such that $0<\epsilon B_0\leq 1$.

Proof. By (1.2), we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{L^2}^2+(\mathcal{L}_1 f,f)_{L^2}=(\Gamma(f,f),f)_{L^2}-(\mathcal{L}_2 f,f)_{L^2}.$$

Using Lemma 2.1 and taking $\delta=\frac{1}{8C_2}$ in (3.1), for all $0\leq t\leq T$, we can conclude

$$(\mathcal{L}_1 f,f)_{L^2}\geq\|f\|_{L^2_A}^2-C_2\|f\|_{2,\gamma/2}^2\geq\frac{7}{8}\|f\|_{L^2_A}^2-\tilde{C}_2\|f\|_{L^2}^2,$$

and $\tilde{C}_2$ depends on C_1. Since $\|f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}\leq\epsilon$, using Lemma 2.2 and taking ϵ such that $C_3\epsilon\leq\frac{1}{8}$, for all $0\leq t\leq T$, we have

$$(\Gamma(f,f),f)_{L^2}\leq C_3\|f\|_{L^2}\|f\|_{L^2_A}^2\leq\frac{1}{8}\|f\|_{L^2_A}^2,$$

Corollary 2.1 and Hölder's inequality implies

$$|(\mathcal{L}_2 f,f)_{L^2}|\leq C_4\|f\|_{L^2}\|f\|_{L^2_A}\leq\frac{1}{8}\|f\|_{L^2_A}^2+2C_4^2\|f\|_{L^2}^2.$$

Combining the above estimates, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{L^2}^2+\|f\|_{L^2_A}^2\leq\left(2\tilde{C}_2+4C_4^2\right)\|f\|_{L^2}^2,$$

integrating from 0 to t to get

$$\|f(t)\|_{L^2}^2+\int_0^t\|f(\tau)\|_{L^2_A}^2\mathrm{d}\tau\leq\left(2\tilde{C}_2+4C_4^2\right)\int_0^t\|f(\tau)\|_{L^2}^2\mathrm{d}\tau+\|f_0\|_{L^2}^2, \tag{3.3}$$

then by Gronwall inequality, we get for $0\leq t\leq T$

$$\|f(t)\|_{L^2}^2\leq[1+\left(2\tilde{C}_2+4C_4^2\right)Te^{\left(2\tilde{C}_2+4C_4^2\right)T}]\|f_0\|_{L^2}^2. \tag{3.4}$$

Substituting (3.4) into (3.3) and taking $B_0\geq 1+\left(2\tilde{C}_2+4C_4^2\right)Te^{2\left(\tilde{C}_2+4C_4^2\right)T}$, one can obtain

$$\|f(t)\|_{L^2}^2+\int_0^t\|f(\tau)\|_{L^2_A}^2\mathrm{d}\tau\leq\left(2\tilde{C}_2+4C_4^2\right)Te^{\left(2\tilde{C}_2+4C_4^2\right)T}\|f_0\|_{L^2}^2\leq B_0^2\epsilon^2.$$

□

Lemma 3.3. *Assume $f_0\in L^2(\mathbb{R}^3)$ and $T>0$, let f be the solution of the Cauchy problem* (1.2) *with $\|f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}$ small enough. Then there exists a constant $B_1>0$ such that*

$$\|\tau\partial_\tau f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}^2+\|\tau\partial_\tau f\|_{L^2([0,T];L^2_A(\mathbb{R}^3))}^2\leq\epsilon^2 B_1^2. \tag{3.5}$$

We also take ϵ small such that $0<\epsilon B_1\leq 1$.

Proof. Since the solution of (1.2) belongs to $C^\infty(]0,T[;\mathcal{S}(\mathbb{R}^3))$, we have that

$$\partial_t(t\partial_t f)+\mathcal{L}_1(t\partial_t f)=\partial_t f-\mathcal{L}_2(t\partial_t f)+t\partial_t\Gamma(f,f),$$

and for $0\leq t\leq T$

$$\begin{aligned}&\frac{1}{2}\|t\partial_t f\|_{L^2}^2+\int_0^t(\mathcal{L}_1(\tau\partial_\tau f),\tau\partial_\tau f)_{L^2}\mathrm{d}\tau\\ =&\int_0^t\tau\|\partial_\tau f\|_{L^2}^2\mathrm{d}\tau-\int_0^t(\mathcal{L}_2(\tau\partial_\tau f),\tau\partial_\tau f)_{L^2}\mathrm{d}\tau+\int_0^t(\tau\partial_\tau\Gamma(f,f),\tau\partial_\tau f)_{L^2}\mathrm{d}\tau\\ =&R_1+R_2+R_3.\end{aligned}$$

Firstly, using Lemma 2.1 and (3.1) with $\delta=\frac{1}{8C_2}$, for all $0\leq t\leq T$, we can conclude

$$\begin{aligned}\int_0^t\tau^2(\mathcal{L}_1(\partial_\tau f),\partial_\tau f)_{L^2}\mathrm{d}\tau&\geq\|\tau\partial_\tau f\|_{L^2([0,t];L^2_A)}^2-C_2\int_0^t\tau^2\|\partial_\tau f\|_{2,\gamma/2}^2\mathrm{d}\tau\\ &\geq\frac{7}{8}\|\tau\partial_\tau f\|_{L^2([0,t];L^2_A)}^2-\tilde{C}_2T\int_0^t\tau\|\partial_\tau f\|_{L^2}^2\mathrm{d}\tau.\end{aligned}$$

For the term R_1, since f is solution of (1.2), i.e.

$$\partial_t f=\Gamma(f,f)-\mathcal{L}(f),$$

using Lemma 2.2 and Corollary 2.1, for all $0\leq t\leq T$, we have

$$\begin{aligned}\int_0^t\tau\|\partial_\tau f\|_{L^2}^2\mathrm{d}\tau=&\int_0^t\tau(\Gamma(f,f),\partial_\tau f)_{L^2}\mathrm{d}\tau-\int_0^t\tau(\mathcal{L}(f),\partial_\tau f)_{L^2}\mathrm{d}\tau\\ \leq&C_3\int_0^t\|f\|_{L^2}\|f\|_{L^2_A}\|\tau\partial_\tau f\|_{L^2_A}\mathrm{d}\tau+C_4\int_0^t\left(\|f\|_{L^2}+\|f\|_{L^2_A}\right)\|\tau\partial_\tau f\|_{L^2_A}\mathrm{d}\tau.\end{aligned}$$

Using Cauchy-Schwarz inequality, for $0<\delta<1$,

$$\begin{aligned}\int_0^t\tau\|\partial_\tau f\|_{L^2}^2\mathrm{d}\tau\leq&\delta\|\tau\partial_\tau f\|_{L^2([0,t];L^2_A)}^2+\frac{C_3^2}{2\delta}\|f\|_{L^\infty([0,t];L^2)}^2\int_0^t\|f\|_{L^2_A}^2\mathrm{d}\tau\\ &+\frac{C_4^2}{\delta}\left(T\|f\|_{L^\infty([0,t];L^2)}^2+\int_0^t\|f\|_{L^2_A}^2\mathrm{d}\tau\right).\end{aligned}$$

Then, (3.2) implies, there exists $C_\delta>0$ such that

$$R_1=\int_0^t\tau\|\partial_\tau f\|_{L^2}^2\mathrm{d}\tau\leq C_\delta B_0^2\epsilon^2+\delta\|\tau\partial_\tau f\|_{L^2([0,t];L^2_A)}^2. \tag{3.6}$$

For the term R_2, using Corollary 2.1, for all $0\le t\le T$, we have

$$\begin{aligned}|R_2|=&\left|\int_0^t \tau^2(\mathcal{L}_2(\partial_\tau f),\partial_\tau f)_{L^2}\mathrm{d}\tau\right|\le C_4\int_0^t \tau^2\|\partial_\tau f\|_{L^2}\|\partial_\tau f\|_{L^2_A}\mathrm{d}\tau\\ \le&\frac{1}{8}\|\tau\partial_\tau f\|^2_{L^2([0,t];L^2_A)}+2C_4^2T\int_0^t\tau\|\partial_\tau f\|^2_{L^2}\mathrm{d}\tau,\end{aligned}$$

then, using (3.6) to get

$$|R_2|\le\frac{1}{8}\|\tau\partial_\tau f\|^2_{L^2([0,t];L^2_A)}+2C_4^2T\left(C_\delta B_0^2\epsilon^2+\delta\|\tau\partial_\tau f\|^2_{L^2([0,t];L^2_A)}\right),$$

so taking $2C_4^2T\delta=\frac{1}{8}$, one has

$$|R_2|\le\frac{1}{4}\|\tau\partial_\tau f\|^2_{L^2([0,t];L^2_A)}+\tilde{C}_4B_0^2\epsilon^2.$$

Finally, for the term R_3, Lemma 2.2 implies

$$\begin{aligned}|R_3|=&\left|\int_0^t\tau^2\partial_\tau(\Gamma(f,f),\partial_\tau f)_{L^2}\mathrm{d}\tau\right|\\ \le&\int_0^t\tau^2|(\Gamma(\partial_\tau f,f),\partial_\tau f)|\mathrm{d}\tau+\int_0^t\tau^2|(\Gamma(f,\partial_\tau f),\partial_\tau f)|\mathrm{d}\tau\\ \le&C_3\int_0^t\tau^2\|\partial_\tau f\|_{L^2}\|f\|_{L^2_A}\|\partial_\tau f\|_{L^2_A}\mathrm{d}\tau+C_3\int_0^t\|f\|_{L^2}\|\tau\partial_\tau f\|^2_{L^2_A}\mathrm{d}\tau\\ \le&\frac{1}{8}\|\tau\partial_\tau f\|^2_{L^2([0,t];L^2_A)}+2C_3^2\int_0^t\|f\|^2_{L^2_A}\|\tau\partial_\tau f\|^2_{L^2}\mathrm{d}\tau+C_3\int_0^t\|f\|_{L^2}\|\tau\partial_\tau f\|^2_{L^2_A}\mathrm{d}\tau\\ \le&\frac{1}{8}\|\tau\partial_\tau f\|^2_{L^2([0,t];L^2_A)}+2C_3^2\|\tau\partial_\tau f\|^2_{L^\infty([0,t];L^2)}\int_0^t\|f\|^2_{L^2_A}\mathrm{d}\tau\\ &+C_3\|f\|^2_{L^\infty([0,t];L^2)}\int_0^t\|\tau\partial_\tau f\|^2_{L^2_A}\mathrm{d}\tau.\end{aligned}$$

Using (3.2) and taking $\epsilon>0$ small such that

$$2C_3^2B_0^2\epsilon^2\le\frac{1}{4},\quad C_3B_0^2\epsilon^2\le\frac{1}{8}.$$

We get then, for all $0\le t\le T$,

$$\left|\int_0^t\tau^2\partial_\tau(\Gamma(f,f),\partial_\tau f)_{L^2}\mathrm{d}\tau\right|\le\frac{1}{4}\|\tau\partial_\tau f\|^2_{L^2([0,t];L^2_A)}+\frac{1}{4}\|\tau\partial_\tau f\|^2_{L^\infty([0,t];L^2)}.$$

Combining the above estimates, taking $\delta=\frac{1}{8}$ in (3.6), one has

$$\frac{1}{4}\|\tau\partial_\tau f\|^2_{L^\infty([0,T];L^2)}+\frac{3}{8}\|\tau\partial_\tau f\|^2_{L^2([0,T];L^2_A)}\le C_5\epsilon^2+\tilde{C}_2T\int_0^T\tau\|\partial_\tau f\|^2_{L^2}\mathrm{d}\tau,$$

using (3.6) with $\tilde{C}_2T\delta\le\frac{1}{8}$ and taking $B_1\ge2\sqrt{C_5}$, then it follows that

$$\|\tau\partial_\tau f\|^2_{L^\infty([0,T];L^2)}+\|\tau\partial_\tau f\|^2_{L^2([0,T];L^2_A)}\le4C_5\epsilon^2\le B_1^2\epsilon^2,$$

with B_1 depends only on C_1,C_2,C_3,C_4 and T, which end the proof of Lemma 3.3. □

4 Proof of main theorem

In this section, we shall show the analytic regularity of time variable for $t>0$. We construct the following estimate, from which we can deduce the inequality (1.3) directly.

Proposition 4.1. *Assume $f_0 \in L^2(\mathbb{R}^3)$ and $T>0$, let f be the solution of the Cauchy problem* (1.2) *with $\|f\|_{L^\infty([0,T];L^2(\mathbb{R}^3))}$ small enough. Then there exists a constant $B>0$ such that, for any $k\in\mathbb{N}_+$*

$$\|\tau^k\partial_\tau^k f\|^2_{L^\infty([0,T];L^2(\mathbb{R}^3))}+\|\tau^k\partial_\tau^k f\|^2_{L^2([0,T];L^2_A(\mathbb{R}^3))}\le B^{2(k-1)}((k-2)!)^2. \tag{4.1}$$

We have that (4.1) implies immediately (1.3), so it is enough to prove this Proposition 4.1 for Theorem 1.1. We prove this proposition by induction for the index k. For $k=1$, it is enough to take, in (3.5),

$$0<\epsilon B_1\le 1,$$

and by convention $(-1)!=1$, $0!=1$. Now for $k\ge 2$, since μ is a function with respect to the variable v, we have

$$t^k\partial_t^k\mathcal{L}f=\mathcal{L}(t^k\partial_t^k f).$$

Then by (1.2), one can obtain,

$$\begin{aligned}&\partial_t(t^k\partial_t^k f)+\mathcal{L}_1(t^k\partial_t^k f)\\ =&kt^{k-1}\partial_t^k f-\mathcal{L}_2(t^k\partial_t^k f)+\Gamma(f,t^k\partial_t^k f)+\Gamma(t^k\partial_t^k f,f)+\sum_{1\le j\le k-1}C_k^j\,\Gamma(t^j\partial_t^j f,t^{k-j}\partial_t^{k-j}f),\end{aligned}$$

where $C_k^j=\frac{k!}{j!(k-j)!}$. Then taking the $L^2(\mathbb{R}^3)$ inner product of both sides with respect to $t^k\partial_t^k f$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|t^k\partial_t^k f\|^2_{L^2}+(\mathcal{L}_1(t^k\partial_t^k f),t^k\partial_t^k f)_{L^2}\\ =&kt^{2k-1}\|\partial_t^k f\|^2_{L^2}-(\mathcal{L}_2(t^k\partial_t^k f),t^k\partial_t^k f)_{L^2}\\ &\quad+(\Gamma(f,t^k\partial_t^k f),t^k\partial_t^k f)_{L^2}+(\Gamma(t^k\partial_t^k f,f),t^k\partial_t^k f)_{L^2}\\ &\quad+\sum_{1\le j\le k-1}C_k^j\,\Gamma(t^j\partial_t^j f,t^{k-j}\partial_t^{k-j}f),t^k\partial_t^k f)_{L^2}.\end{aligned}$$

For all $0<t\le T$, integrating from 0 to t, using Lemma 2.1 and (3.1) with $\delta=\frac{1}{8C_2}$, we can conclude

$$\begin{aligned}\int_0^t\tau^{2k}(\mathcal{L}_1(\partial_\tau^k f),\partial_\tau^k f)_{L^2}\mathrm{d}\tau&\ge\|\tau^k\partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}-C_2\int_0^t\tau^{2k}\|\partial_\tau^k f\|^2_{2,\gamma/2}\mathrm{d}\tau\\ &\ge\frac{7}{8}\|\tau^k\partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}-\tilde{C}_2\int_0^t\tau^{2k}\|\partial_\tau^k f\|^2_{L^2}\mathrm{d}\tau.\end{aligned}$$

Using Corollary 2.1, for all $0\le t\le T$, we have

$$\begin{aligned}\left|\int_0^t \tau^{2k}(\mathcal{L}_2(\partial_\tau^k f),\partial_\tau^k f)_{L^2}\mathrm{d}\tau\right| &\le C_4\int_0^t \tau^{2k}\|\partial_\tau^k f\|_{L^2}\|\partial_\tau^k f\|_{L^2_A}\mathrm{d}\tau\\ &\le \frac{1}{8}\|\tau^k\partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}+2C_4^2\int_0^t \tau^{2k}\|\partial_\tau^k f\|^2_{L^2}\mathrm{d}\tau.\end{aligned}$$

Finally, using Lemma 2.2, we have

$$\begin{aligned}&\frac{1}{2}\|t^k\partial_t^k f\|^2_{L^2}+\frac{3}{4}\int_0^t\|\tau^k\partial_\tau^k f\|^2_{L^2_A}\mathrm{d}\tau\\ &\le k\int_0^t \tau^{2k-1}\|\partial_\tau^k f\|^2_{L^2}\mathrm{d}\tau+\tilde{C}_3\int_0^t\|\tau^k\partial_\tau^k f\|^2_{L^2}\mathrm{d}\tau\\ &\quad +C_3\sum_{0\le j\le k}C_k^j\int_0^t\|\tau^j\partial_\tau^j f\|_{L^2}\|\tau^{k-j}\partial_\tau^{k-j}f\|_{L^2_A}\|\tau^k\partial_\tau^k f\|_{L^2_A}\mathrm{d}\tau,\end{aligned}\tag{4.2}$$

with $\tilde{C}_3=\tilde{C}_2+2C_4^2$ depends only on C_1,C_2,C_3,C_4 and T.

We prove now (4.1) by induction on k. Assume that for $k\ge 2$, (4.1) holds true for $1\le m\le k-1$,

$$\|\tau^m\partial_\tau^m f\|^2_{L^\infty([0,T];L^2(\mathbb{R}^3))}+\|\tau^m\partial_\tau^m f\|^2_{L^2([0,T];L^2_A(\mathbb{R}^3))}\le B^{2(m-1)}((m-2)!)^2.\tag{4.3}$$

And we shall prove that (4.1) holds true for $m=k$. We estimate the terms of the RHS of (4.2) by the following lemmas.

Lemma 4.1. *Assume that* (4.3) *holds true for any* $1\le m\le k-1$, *and* f *satisfies* (3.2), *then*

$$k\int_0^t \tau^{2k-1}\|\partial_\tau^k f\|^2_{L^2}\mathrm{d}\tau\le\frac{1}{8}\|\tau^k\partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}+A_1B^{2(k-2)}((k-2)!)^2,\tag{4.4}$$

with A_1 *depends only on* C_1,C_2,C_3,C_4 *and* T.

We have also

Lemma 4.2. *Assume that* (4.3) *holds true for any* $1\le m\le k-1$, *then*

$$\begin{aligned}&C_3\sum_{1\le j\le k-1}C_k^j\int_0^t\|\tau^j\partial_\tau^j f\|_{L^2}\|\tau^{k-j}\partial_\tau^{k-j}f\|_{L^2_A}\|\tau^k\partial_\tau^k f\|_{L^2_A}\mathrm{d}\tau\\ &\le\frac{1}{8}\|\tau^k\partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}+A_2B^{2(k-2)}((k-2)!)^2,\end{aligned}\tag{4.5}$$

with A_2 *depends only on* C_1,C_2,C_3,C_4 *and* T.

Lemma 4.3. *Assume that f satisfies* (3.2)*, then, for* $0<t\leq T$*,*

$$\begin{aligned}&C_3\int_0^t\|\tau^k\partial_\tau^k f\|_{L^2}\|f\|_{L^2_A}\|\tau^k\partial_\tau^k f\|_{L^2_A}\mathrm{d}\tau\\&\leq\frac{1}{8}\|\tau^k\partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}+2C_3^2B_0^2\epsilon^2\ \|\tau^k\partial_\tau^k f\|^2_{L^\infty([0,t];L^2)},\end{aligned}\tag{4.6}$$

and

$$\int_0^t\|f\|_{L^2}\|\tau^k\partial_\tau^k f\|^2_{L^2_A}\mathrm{d}\tau\leq B_0\epsilon\|\tau^k\partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}.\tag{4.7}$$

We will give the proofs of these three lemmas in the next section.

End of proof of Proposition 4.1

Choose $0<\epsilon<1$ small such that

$$C_3B_0\epsilon\leq\frac{1}{8},\quad 2C_3^2B_0^2\epsilon^2\leq\frac{1}{4}.$$

Since (4.3) holds true for any $1\leq m\leq k-1$, and f satisfies (3.2), then combine (4.2), (4.4)-(4.7), we get, for $0<t\leq T$,

$$\|t^k\partial_t^k f\|^2_{L^2}+\int_0^t\|\tau^k\partial_\tau^k f\|^2_{L^2_A}\mathrm{d}\tau\leq 4(A_1+A_2)(B^{k-2}(k-2)!)^2+4\tilde{C}_3\int_0^t\|\tau^k\partial_\tau^k f\|^2_{L^2}\mathrm{d}\tau,$$

with $\tilde{C}_3$ depends only on C_1,C_2,C_3,C_4 and T. By using Gronwall inequality, we get for $0<t\leq T$,

$$\|t^k\partial_t^k f\|^2_{L^2}\leq 4e^{4\tilde{C}_3T}(A_1+A_2)B^{2(k-2)}((k-2)!)^2,$$

which deduce

$$\begin{aligned}&\|\tau^k\partial_\tau^k f\|^2_{L^\infty([0,T],L^2)}+\|\tau^k\partial_\tau^k f\|^2_{L^2([0,T],L^2_A)}\\&\leq 4(e^{4\tilde{C}_3T}\ 4\tilde{C}_3T+1)(A_1+A_2)B^{2(k-2)}((k-2)!)^2.\end{aligned}$$

We prove then

$$\|\tau^k\partial_\tau^k f\|^2_{L^\infty([0,T],L^2)}+\|\tau^k\partial_\tau^k f\|^2_{L^2([0,T],L^2_A)}\leq B^{2(k-1)}((k-2)!)^2,$$

if we choose the constant B such that

$$4(e^{4\tilde{C}_3T}\ 4\tilde{C}_3T+1)(A_1+A_2)\leq B^2,$$

so that the constant B depends only on C_1,C_2,C_3,C_4,T and small ϵ. We finish the proof of Proposition 4.1. □

5 Proofs of technical lemmas

Before give the proof of Lemmas 4.1-4.3, we need the following lemma.

Lemma 5.1. *For all* $k\in\mathbb{N}, k\geq 5$, *we have*

$$\sum_{2\leq j\leq k-3}\frac{k(k-1)}{j(j-1)(k-j-1)(k-j-2)}\leq 12. \tag{5.1}$$

Proof. Without loss of generality, we may assume $k-1$ is even, then the summation can be rewritten as

$$\sum_{2\leq j\leq \frac{k-3}{2}}\frac{k(k-1)}{j(j-1)(k-1-j)(k-2-j)}+\sum_{\frac{k-1}{2}\leq j\leq k-3}\frac{k(k-1)}{j(j-1)(k-1-j)(k-2-j)}.$$

For the first term in above, since $j\leq\frac{k-3}{2}$, we have $k-j\geq\frac{k+3}{2}$. Then it follows that

$$\sum_{2\leq j\leq \frac{k-3}{2}}\frac{k(k-1)}{j(j-1)(k-1-j)(k-2-j)}\leq\sum_{2\leq j\leq \frac{k-3}{2}}\frac{4}{j(j-1)}\leq 4.$$

For the second term, by $j\geq\frac{k-1}{2}$, we have

$$\sum_{\frac{k-1}{2}\leq j\leq k-3}\frac{k(k-1)}{j(j-1)(k-1-j)(k-2-j)}\leq\sum_{\frac{k-1}{2}\leq j\leq k-3}\frac{8}{(k-1-j)(k-2-j)}\leq 8.$$

Thus (5.1) holds true. □

Proof of Lemma 4.1. For $k\geq 2$, by (1.2), one has

$$\begin{aligned}\partial_t^k f=&\partial_t(\partial_t^{k-1}f)=-\mathcal{L}(\partial_t^{k-1}f)+\partial_t^{k-1}\Gamma(f,f)\\ =&-\mathcal{L}(\partial_t^{k-1}f)+\sum_{0\leq j\leq k-1}C_{k-1}^j\Gamma(\partial_t^j f,\partial_t^{k-1-j}f).\end{aligned}$$

Then we have

$$\begin{aligned}k\int_0^t\tau^{2k-1}\|\partial_\tau^k f\|_{L^2}^2\mathrm{d}\tau=&k\sum_{0\leq j\leq k-1}C_{k-1}^j\int_0^t\tau^{2k-1}(\Gamma(\partial_\tau^j f,\partial_\tau^{k-1-j}f),\partial_\tau^k f)_{L^2}\mathrm{d}\tau\\ &-k\int_0^t\tau^{2k-1}(\mathcal{L}(\partial_\tau^{k-1}f),\partial_\tau^k f)_{L^2}\mathrm{d}\tau\\ =:&S_1+S_2.\end{aligned}$$

Using Lemma 2.2, we can conclude

$$\begin{aligned}|S_1| \leq & C_3 k \sum_{0\leq j\leq k-1} C_{k-1}^j \int_0^t \|\tau^j \partial_\tau^j f\|_{L^2} \|\tau^{k-1-j}\partial_\tau^{k-1-j} f\|_{L^2_A} \|\tau^k \partial_\tau^k f\|_{L^2_A} \mathrm{d}\tau \\ \leq & 4C_3^2 k^2 \left(\sum_{0\leq j\leq k-1} C_{k-1}^j \|\tau^j \partial_\tau^j f\|_{L^\infty([0,t];L^2)} \|\tau^{k-1-j}\partial_\tau^{k-1-j} f\|_{L^2([0,t];L^2_A)} \right)^2 \\ & + \frac{1}{16} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}.\end{aligned}$$

From (4.3), one can obtain

$$\begin{aligned}& \sum_{0\leq j\leq k-1} C_{k-1}^j \|\tau^j \partial_\tau^j f\|_{L^\infty([0,t];L^2)} \|\tau^{k-1-j}\partial_\tau^{k-1-j} f\|_{L^2([0,t];L^2_A)} \\ \leq & \sum_{0\leq j\leq k-1} C_{k-1}^j B^{j-1}(j-2)! B^{k-2-j}(k-3-j)! \\ \leq & B^{k-3}(k-3!) \left(\sum_{2\leq j\leq k-3} \frac{k(k-1)}{j(j-1)(k-1-j)(k-2-j)} + 6 \right). \end{aligned} \tag{5.2}$$

Substituting (5.1) into (5.2) we get

$$\sum_{0\leq j\leq k-1} C_{k-1}^j \|\tau^j \partial_\tau^j f\|_{L^\infty([0,t];L^2)} \|\tau^{k-1-j}\partial_\tau^{k-1-j} f\|_{L^2([0,t];L^2_A)} \leq 18 B^{k-3}(k-3)!,$$

from which we can conclude

$$|S_1| \leq \frac{1}{16} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} + C_6 \left(B^{k-3}(k-2)! \right)^2,$$

with C_6 depends only on C_1, C_2, C_3, C_4 and T, where we use $\frac{k}{k-2} \leq 3$.

For the term S_2, using Corollary 2.1 and (4.3), we have

$$\begin{aligned}|S_2| \leq & C_4 k \int_0^t \left(\|\tau^{k-1}\partial_\tau^{k-1} f\|_{L^2} + \|\tau^{k-1}\partial_\tau^{k-1} f\|_{L^2_A} \right) \|\tau^k \partial_\tau^k f\|_{L^2_A} \mathrm{d}\tau \\ \leq & 4C_4^2 k^2 \left(T \|\tau^{k-1}\partial_\tau^{k-1} f\|^2_{L^\infty([0,t];L^2)} + \|\tau^{k-1}\partial_\tau^{k-1} f\|^2_{L^2([0,t];L^2_A)} \right) + \frac{1}{16} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} \\ \leq & 4C_4^2 k^2 (T+1) \left(B^{k-2}(k-3)! \right)^2 + \frac{1}{16} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} \\ \leq & C_7 \left(B^{k-2}(k-2)! \right)^2 + \frac{1}{16} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)},\end{aligned}$$

with C_7 depends only on C_1, C_2, C_3, C_4 and T.

Taking $A_1 = C_6 + C_7$, so that A_1 depends only on C_1, C_2, C_3, C_4 and T, then combining S_1 and S_2, we get

$$k \int_0^t \tau^{2k-1} \|\partial_\tau^k f\|^2_{L^2} \mathrm{d}\tau \leq \frac{1}{8} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} + A_1 \left(B^{k-2}(k-2)! \right)^2. \qquad \square$$

Proof of Lemma 4.2. Using Hölder's inequality and (4.3), we have

$$
\begin{aligned}
&C_3 \sum_{1\le j\le k-1} C_k^j \int_0^t \|\tau^j \partial_\tau^j f\|_{L^2} \|\tau^{k-j}\partial_\tau^{k-j} f\|_{L^2_A} \|\tau^k \partial_\tau^k f\|_{L^2_A} \mathrm{d}\tau \\
\le& 2C_3^2 \left(\sum_{1\le j\le k-1} C_k^j \|\tau^j \partial_\tau^j f\|_{L^\infty([0,t];L^2)} \|\tau^{k-j}\partial_\tau^{k-j} f\|_{L^2([0,t];L^2_A)} \right)^2 + \frac{1}{8} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} \\
\le& 2C_3^2 \left(B^{k-2}(k-2)! \right)^2 \left(\sum_{2\le j\le k-3} \frac{k(k-1)}{j(j-1)(k-j)(k-j-1)} + 6 \right)^2 + \frac{1}{8} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} \\
\le& 2C_3^2 \left(B^{k-2}(k-2)! \right)^2 \left(\sum_{2\le j\le k-3} \frac{k(k-1)}{j(j-1)(k-j-1)(k-j-2)} + 6 \right)^2 + \frac{1}{8} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}.
\end{aligned}
$$

Then from (5.1), we can get

$$
\begin{aligned}
&C_3 \sum_{1\le j\le k-1} C_k^j \int_0^t \|\tau^j \partial_\tau^j f\|_{L^2} \|\tau^{k-j}\partial_\tau^{k-j} f\|_{L^2_A} \|\tau^k \partial_\tau^k f\|_{L^2_A} \mathrm{d}\tau \\
\le& \frac{1}{8} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} + A_2 \left(B^{k-2}(k-2)! \right)^2,
\end{aligned}
$$

with A_2 depends only on C_1, C_2, C_3, C_4 and T. □

Proof of Lemma 4.3. For the inequality (4.6), using Hölder's inequality and (3.2), one has

$$
\begin{aligned}
&C_3 \int_0^t \|\tau^k \partial_\tau^k f\|_{L^2} \|f\|_{L^2_A} \|\tau^k \partial_\tau^k f\|_{L^2_A} \mathrm{d}\tau \\
\le& \frac{1}{8} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} + 2C_3^2 \|\tau^k \partial_\tau^k f\|^2_{L^\infty([0,t];L^2)} \int_0^t \|f\|^2_{L^2_A} \mathrm{d}\tau \\
\le& \frac{1}{8} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} + 2C_3^2 B_0^2 \epsilon^2.
\end{aligned}
$$

For the inequality (4.7), the inequality (3.2) implies

$$
\int_0^t \|f\|_{L^2} \|\tau^k \partial_\tau^k f\|^2_{L^2_A} \mathrm{d}\tau \le \|f\|_{L^\infty([0,T];L^2)} \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)} \le B_0 \epsilon \|\tau^k \partial_\tau^k f\|^2_{L^2([0,t];L^2_A)}.
$$

□

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