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SINGULARITIES PRODUCED BY THE REFLECTION AND INTERACTION OF TWO PROGRESSING WAVES①

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Abstract We give an example to show that there will be anomalous singularities on the forward half light cone issuing from the reflection point after the reflection at the boundary of two progressing waves carrying singularities. It perfects the results of [1].

Key Words Wave equations; two progressing waves; the reflection and interaction of singularities; mixed problems.

Classifications 35B65; 35L05; 35L20.

1. Introduction to Questions and the Main Results

There have been many works on the propagation of singularities of the solutions to semilinear wave equations so far. In [2] and [3], J. M. Bony considered the case of two progressing waves after intersection, and the elementary fact of his conclusions is that there could be anomalous singularities on the other characteristic hypersurfaces issuing from $H_1 \cap H_2$ after the interaction of two progressing waves propagating on characteristic hypersurfaces H_1 and H_2 as shown in Figure 1. In particular, for the following 2-dimensional wave equation:

$$\Box u = f(u) \tag{1.1}$$

where $u = u(t, x_1, x_2)$, $(t, x_1, x_2) \in R_t \times R_x^2$, we know that there does not exist any anomalous singularities after the interaction of two progressing waves by J. M. Bony's conclusions. But, J. Rauch and M. Reed presented an example to show there are exactly anomalous singularities after the interaction of three progressing waves in [4].

In this paper we consider the case that two progressing waves carrying singularities intersect at the boundary. For this case, Chen Shuxing ([1]) has proved for conormal distributions that there could be anomalous singularities on the forward half light cone issuing from the reflection point after the reflection on the boundary of these two progressing waves. This paper will give an example to show the existence of such singularities.

Denote by (t, x_1, x_2) any point of $R_t \times R_x^2$. We consider the following problem in $(R_t \times R_x^2) \cap \{x_2 > 0\}$

$$\begin{cases} \Box u_1 = 0 & (1.2) \\ \Box u_2 = 0\ , \qquad x_2 > 0 & (1.3) \\ \Box u_3 = u_1 u_2 & (1.4) \\ u_i|_{x_2=0} = 0\ , \qquad i = 1,2,3 & (1.5) \end{cases}$$

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Suppose that u_1, u_2 are as follows

$$u_i(t,x_1,x_2)=\begin{cases}h(t-w_i\cdot x), & t\leqslant 0, x_2>0\\ h(w_i\cdot x-t), & t>0, x_2>0\end{cases}\qquad (i=1,2)$$

where

$$x=(x_1,x_2), w_1=-w_2=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right), w_1=-w_2=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$$

h is the Heaviside function.

We will consider the singularities of the solution u_3 to (1.2)—(1.5) on the forward half light cone $C_0=\{(t,x_1,x_2)\,|\,t=\sqrt{x_1^2+x_2^2}, x_2>0\}$ as $t>0$. For simplicity, we introduce some notations as follows

$\Sigma_1=\{(t,x_1,x_2)\,|\,x_1+x_2+\sqrt{2}t=0\}$, i.e. the plane $OB'C'$ in Figure 2.

$\Sigma_1^+=\{x_1+x_2+\sqrt{2}t\geqslant 0\}$; $\Sigma_1^-=\{x_1+x_2+\sqrt{2}t<0\}$

$\Sigma_2=\{(t,x_1,x_2)\,|\,-x_1+x_2+\sqrt{2}t=0\}$, i.e. the plane $OA'B'$ in Figure 2.

$\Sigma_2^+=\{-x_1+x_2+\sqrt{2}t\geqslant 0\}$; $\Sigma_2^-=\{-x_1+x_2+\sqrt{2}t<0\}$

$\Sigma_3=\{(t,x_1,x_2)\,|\,x_1-x_3+\sqrt{2}t=0\}$, i.e. the reflection plane OBC of $OB'C'$ about $\{x_2=0\}$ in Figure 2.

$\Sigma_3^+=\{x_1-x_3+\sqrt{2}t\geqslant 0\}$; $\Sigma_3^-=\{x_1-x_3+\sqrt{2}t<0\}$

$\Sigma_4=\{(t,x_1,x_2)\,|\,-x_1-x_2+\sqrt{2}t=0\}$, i.e. the reflection plane OAB of $OA'B'$ about $\{x_2=0\}$ in Figure 2.

$\Sigma_4^+=\{-x_1-x_2+\sqrt{2}t\geqslant 0\}$; $\Sigma_4^-=\{-x_1-x_2+\sqrt{2}t<0\}$

$\Sigma_5=\{(t,x_1,x_2)\,|\,x_1=0\}$, i.e. the plane OBB'.

$\Sigma_5^+=\{x_1\geqslant 0\}$; $\Sigma_5^-=\{x_1<0\}$

$\Sigma_6=\{(t,x_1,x_2)\,|\,t=0\}$, i.e. the plane OMN.

$\Sigma_6^+=\{t\geqslant 0\}$; $\Sigma_6^-=\{t<0\}$

$\mathscr{A}=\Sigma_1^+\cap\Sigma_2^+\cap\Sigma_3^-\cap\Sigma_4^-$, i.e. the pyramid $O\text{-}BMB'N$ in Figure 2.

$\mathscr{B}$=the symmetric region of $\mathscr{A}$ about $\{x_2=0\}$, i.e. the pyramid $O\text{-}B_1M_1B_1'N_1$, where B_1, M_1, B_1', N_1 are on the stretched line of $\overline{OB}, \overline{OM}, \overline{OB'}, \overline{ON}$ respectively.

Obviously, u_3 can be considered as the solution to the following linear problem

$$\begin{cases}\square u_3=\chi_{\mathscr{A}}-\chi_{\mathscr{B}} & (1.6)\\ u_3=0, \quad t<0 & (1.7)\end{cases}$$

where $\chi_{\mathscr{A}}$ and $\chi_{\mathscr{B}}$ are the characteristic functions of $\mathscr{A}$ and $\mathscr{B}$ respectively.

By the general expression of the solutions to wave equations we know the solution u_3 to (1.6) and (1.7) is

$$u_3(p)=(E*\chi_{C_p^-\cap\mathscr{A}}-E*\chi_{C_p^-\cap\mathscr{B}})(p) \qquad (1.8)$$

where $p=(t,x_1,x_2), t>0, C_p^-$ is the backward light cone issuing from p, E is the fundamental solution to $\square$.